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Research paper

On a multivalued prescribed mean curvature problem and inclusions defined on dual spaces

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ABSTRACT

This article addresses two main objectives. First, it establishes a functional analytic framework and presents existence results for a quasilinear inclusion describing a prescribed mean curvature problem with homogeneous Dirichlet boundary conditions, involving a multivalued lower order term. The formulation of the problem is done in the space of functions with bounded variation.

The second objective is to introduce a general existence theory for inclusions defined on nonreflexive Banach spaces, which is specifically applicable to the aforementioned prescribed mean curvature problem. This problem can be formulated as a multivalued variational inequality in the space of functions with bounded variation, which, under suitable conditions, is equivalent to an inclusion involving a maximal monotone mapping of type (D) and a generalized pseudomonotone mapping. We prove an abstract existence theorem for inclusions of this form, under some coercivity conditions involving both the maximal monotone and the generalized pseudomonotone mappings.

1. Introduction

The goal of this article is twofold. First, we establish a functional analytic framework and derive existence results for the following quasilinear inclusion, which describes a prescribed mean curvature problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + f(x,u) \ni 0 \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where the lower order term $f(x,u)$ is multivalued.

In the case where $f(x,u)$ is a single-valued function, the inclusion in (1.1) reduces to the classical equation:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + f(x,u) = 0, \tag{1.2}$$

which has been studied extensively by various methods.

We refer to [1] and references to the original works by Finn, Bombieri/De Giorgi/Miranda, Jenkins, Serrin, etc. (see e.g. [2–7]) for classical existence theorems related to the prescribed mean curvature problem.

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In this paper, we formulate (1.1) as a problem in the space of functions of bounded variation. This formulation was used, for example, in [2,7–13]. However, all these works primarily focus on problems involving single-valued functions, and there seems to be a lack of systematic investigation into equations or inequalities that incorporate the mean curvature operator with multivalued lower-order terms. One significant challenge in formulating the problem within the space of functions of bounded variation is the non-reflexivity of such a space. Consequently, the application of various methods in nonlinear functional analysis, such as topological or monotonicity methods, becomes more challenging for our problem here. Furthermore, due to the general multivalued nature of problem (1.1), classical variational methods are not directly applicable to it.

Therefore, as a second main objective of this paper, we introduce a general existence theory for inclusions defined on nonreflexive Banach spaces, which specifically applies to problem (1.1). It is known that the space of functions with bounded variation is isomorphic to the topological dual of a Banach space. We demonstrate in this paper that the multivalued boundary value problem (1.1) can be formulated, in the weak form, as a multivalued variational inequality within this space. Under specific suitable conditions, this inequality is equivalent to an inclusion that contains a maximal monotone mapping of type (D), as defined by Gossez (cf. [14]), and a nonreflexive version of a (multivalued) generalized pseudomonotone mapping, as defined by Browder–Hess (cf. [15]). Consequently, we establish an abstract existence theorem for inclusions of the following form:

$$(A + B)(u) \ni f_0, \tag{1.3}$$

where $A, B : X^* \to X^{**}$, X^* and X^{**} are the dual and bidual of a Banach space X, $f_0 \in X^{**}$, A is a maximal monotone mapping of type (D), and B is a generalized pseudomonotone mapping in an appropriate sense. We extend an existence and range theorem in [16] for reflexive Banach spaces to duals of nonreflexive Banach spaces. In [16], it was established that, subject to specific coercivity conditions, which involve the sum of a maximal monotone mapping A and a generalized pseudomonotone mapping B from a reflexive Banach space X to its dual space X^*, f_0 belongs to the range of $A + B$. We show here that for the dual X^* of a nonreflexive Banach space, a similar theorem is obtained for the sum of a maximal monotone mappings of type (D) and a generalized pseudomonotone both defined on X^* and take values in X^{**}, subject to a coercivity condition.

The concept of maximal monotone mappings of type (D) was introduced by Gossez in [14] to extend some useful properties of maximal monotone mappings defined on reflexive Banach spaces to nonreflexive spaces. Note that the concept of maximal monotonicity of type (D) is not symmetric between X and X^*, since the closure of the corresponding graph is taken in the weak topology in X and the norm topology in X^*. As a result, the recently developed existence theory for inclusions in nonreflexive Banach space X in [17] does not directly apply to our current case.

Furthermore, it is worth noting that the coercivity condition in this existence theorem applies to both A and B, unlike the usual existence theorems of this type where it solely applies to the generalized pseudomonotone B. This extension expands the applicability of the existence result to a broader range of problems. Additionally, the abstract existence theorem is applicable to inclusions, encompassing equations and variational inequalities in dual spaces, such as boundary value problems and variational inequalities in Orlicz-Sobolev and Musielak-Orlicz-Sobolev spaces of functions with very slow growth rates.

The paper is organized as follows. In Section 2, we introduce the abstract existence theory for inclusions of the form (1.3) in a dual Banach space, X^*. After presenting some basic definitions, assumptions, and auxiliary results in Section 2.1, we prove our main abstract existence result in Section 2.2. An application to multivalued variational inequalities is discussed in Section 2.3. In Section 3, we focus on the existence and certain properties of weak solutions for the multivalued boundary value problem (1.1). The problem is set up as a multivalued variational inequality in a BV space, as described in Section 3.1. We demonstrate that, under certain appropriate and reasonable conditions on the lower-order term f, the operators involved in this variational inequality satisfy the abstract conditions of the general existence theory developed in Section 2. Basic existence results with coercive conditions are presented in Section 3.2. Finally, in Section 3.3, we present a sub-supersolution method for (1.1) in the noncoercive case.

2. Abstract existence result in dual spaces

2.1. Definitions and assumptions - Auxiliary results

In this section, we present the concepts of maximal monotonicity of type (D) and of generalized pseudomonotonicity for mappings defined on dual Banach spaces, together with their properties necessary for later developments.

2.1.1. Basic definitions

Let X be a real Banach space with norm $\|\cdot\| = \|\cdot\|_X$, dual X^*, dual pairing $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{X^*, X}$, and bidual X^{**}. We identify X with a closed subspace of X^{**} by the canonical isometric identification. For simplicity and without confusion, the norms on X, X^*, and X^{**} are denoted by $\|\cdot\|$, with subscripts used when clarification is needed. Similarly, $\langle \cdot, \cdot \rangle$ denotes both the pairing between X and X^* and that between X^* and X^{**}, i.e., for $x \in X$, $x^* \in X^*$, $x^{**} \in X^{**}$, $\langle x^*, x \rangle = x^*(x)$, $\langle x^{**}, x^* \rangle = x^{**}(x^*)$. Since $X \subset X^{**}$, we have $\langle x, x^* \rangle = \langle x^*, x \rangle$. Hence, the order in this pairing notation is not essential in our situation here. We denote by $\sigma(X, X^*)$ and $\sigma(X^*, X^{**})$ the weak topologies on X and X^*, respectively, and by $\sigma(X^*, X)$ and $\sigma(X^{**}, X^*)$ the weak* topologies on X^* and X^{**}, respectively.

Let U, V be nonempty sets and $A : U \to 2^V$. We use the notation $D(A) = \{u \in U : A(u) \neq \emptyset\}$ for the domain of A, $R(A) = D(A^{-1}) = \{v \in V : v \in A(u) \text{ for some } u \in U\}$ for the range of A, and $\mathrm{Gr}(A) = \{(u, v) \in U \times V : v \in A(u)\}$ for the graph of A. We can identify A with its graph $\mathrm{Gr}(A)$ in some places where this identification is convenient and causes no confusion. Let us first present some definitions.

Definition 2.1. (a) (Section 2, [14]) A mapping $A : X^* \to 2^X$ is called a maximal monotone mapping of type (D) from X^* to X (or with respect to (X^*, X)) if A is maximal monotone, and its monotone closure $\widetilde{A} : X^* \to 2^{X^{**}}$ defined by

$$\mathrm{Gr}(\widetilde{A}) = \{(x^*, x^{**}) \in X^* \times X^{**} : \langle x^* - y^*, x^{**} - y\rangle \geqslant 0, \forall (y^*, y) \in \mathrm{Gr}(A)\},$$

is maximal monotone, and for all $(x^*, x^{**}) \in \mathrm{Gr}(\widetilde{A})$, there exists a bounded net $\{(x_i^*, x_i)_{i\in I}\}$ in $X^* \times X$ such that

$$(x_i^*, x_i) \in \mathrm{Gr}(A),\ \forall i \in I,$$

$$x_i \to x^{**} \text{ in } (X^{**}, \sigma(X^{**}, X^*)) \text{ and } \|x_i\| \to \|x^{**}\|,$$

and

$$\|x_i^* - x^*\|_{X^*} \to 0.$$

(b) Let $A : X^* \to 2^{X^{**}}$. For convenience of terminology, we say that A is a maximal monotone mapping of type (D) from X^* to X^{**} (or with respect to (X^*, X^{**})) if the mapping $A_0 : X^* \to 2^X$ defined by $\mathrm{Gr}\,A_0 = \mathrm{Gr}\,A \cap (X^* \times X)$ is a maximal monotone mapping of type (D) from X^* to X, as defined in (a), and $A = \widetilde{A_0}$.

(c) A mapping $A : X^* \to 2^{X^{**}}$ is called generalized pseudomonotone if for any bounded net $\{(x_i^*, x_i)\}_{i\in I} \subset X^* \times X^{**}$ satisfying

$$\text{(i)} \quad x_i \in A(x_i^*),\ \forall i \in I, \tag{2.1}$$

$$\text{(ii)} \quad x_i \to x_0 \text{ in } (X^{**}, \sigma(X^{**}, X^*)), x_i^* \to x_0^* \text{ in } (X^*, \sigma(X^*, X)), \tag{2.2}$$

and

$$\text{(iii)} \quad \limsup \langle x_i^*, x_i\rangle \leqslant \langle x_0^*, x_0\rangle, \tag{2.3}$$

we have

$$x_0 \in A(x_0^*), \tag{2.4}$$

and

$$\lim \langle x_i^*, x_i\rangle = \langle x_0^*, x_0\rangle. \tag{2.5}$$

Remark 2.2. A mapping $A : X^* \to 2^{X^{**}}$ is generalized pseudomonotone in the sense of Definition 2.1(c) if and only its inverse $A^{-1} : X^{**} \to 2^{X^*}$ is generalized pseudomonotone in the sense of Definition 2.1 (e) in [17].

Similarly, a mapping $A : X^* \to 2^{X^{**}}$ is maximal monotone of type (D) from X^* to X^{**} in the sense of Definition 2.1(b) if and only if its inverse $A^{-1} : X^{**} \to 2^{X^*}$ is maximal monotone of type (D) (from X^{**} to X^*) in the sense of Definition 1 (e), [17].

In fact, suppose $A : X^* \to 2^{X^{**}}$ is maximal monotone of type (D) from X^* to X^{**} and $A = \widetilde{A_0}$ where $A_0 : X^* \to 2^X$ is a maximal monotone mapping of type (D) from X^* to X with $\mathrm{Gr}\,A_0 = \mathrm{Gr}\,A \cap (X^* \times X)$. We have the following equivalences

$$\begin{aligned}
(x, x^*) \in \mathrm{Gr}\,A^{-1} \quad &\Longleftrightarrow \quad (x^*, x) \in \mathrm{Gr}(\widetilde{A_0}) \\
&\Longleftrightarrow \quad \langle x^* - y^*, x - y\rangle \geqslant 0,\ \forall (y^*, y) \in \mathrm{Gr}(A_0) \\
&\Longleftrightarrow \quad \langle x - y, x^* - y^*\rangle \geqslant 0,\ \forall (y, y^*) \in \mathrm{Gr}(A_0^{-1}) \\
&\Longleftrightarrow \quad (x, x^*) \in \mathrm{Gr}(\widetilde{A_0^{-1}}).
\end{aligned}$$

Since A is maximal monotone mapping of type (D) from X^* to X^{**} with $A = \widetilde{A_0}$, for all $(x^*, x) \in \mathrm{Gr}\,A$, there exists a net $\{(x_i^*, x_i)\}$ in $\mathrm{Gr}\,A_0$ such that $\{x_i\}$ is a bounded net, $\|x_i\| \to \|x\|$, $x_i^* \to x^*$ in X^* strongly, and $x_i \to x$ in $(X^{**}, \sigma(X^{**}, X^*))$. By the definition of maximal monotone mappings of type (D) from X to X^* in [14] (see also [17]), we see that $A_0^{-1} : X \to 2^{X^*}$ is a maximal monotone mapping of type (D) with respect to (X, X^*) and $A^{-1} = \widetilde{A_0^{-1}}$ is a maximal monotone mapping of type (D) with respect to (X^{**}, X^*). In particular,

$$A^{-1} = (\widetilde{A_0})^{-1} = \widetilde{A_0^{-1}}. \tag{2.6}$$

2.1.2. Some auxiliary results

It follows from Definition 2.1 and Remark 2.2 that some results concerning maximal monotone mappings of type (D) and generalized pseudomonotone mappings in [17] can be directly extended to our case here.

For example, Proposition 3.1 in [17] implies the following relation between maximal monotone mappings of type (D) from X^* to X^{**} and generalized pseudomonotone mappings, which is another extension to the nonreflexive case of the classical relation between maximal monotone mappings and generalized pseudomonotone mappings in reflexive Banach spaces, given in Proposition 2 of [15].

Proposition 2.3. *If a mapping $A : X^* \to 2^{X^{**}}$ is maximal monotone of type (D) from X^* to X^{**} in the sense of Definition 2.1(b), then it is generalized pseudomonotone in the sense of Definition 2.1(c).*

As in the reflexive case, we have the following invariance property of generalized pseudomonotone mappings under addition.

Proposition 2.4. *If F and G are generalized pseudomonotone from X^* to $2^{X^{**}}$ and F is bounded, then $F+G$ is also generalized pseudomonotone from X^* to $2^{X^{**}}$.*

The proof of this result is similar to that of Proposition 3.4 in [17] and is therefore omitted. Next, let us consider a convergence type for multivalued mappings from X^* to X^{**}.

2.1.3. Regularization of Moreau-Yosida type

Next, we consider a regularization of Moreau–Yosida type for maximal monotone mapping of type (D) defined on a dual space. Let us start with a concept of convergence for multivalued mappings.

Definition 2.5. Let $\{A_i\}_{i\in I}$ be a net of mappings $A_i : X^* \to 2^{X^{**}}$, and $A_0 : X^* \to 2^{X^{**}}$.

We say that A_i converges to A_0 in the generalized pseudomonotone sense (denoted by $A_i \xrightarrow{(gpm)} A_0$) if and only if for any subnet $\{A_{i(k)}\}_{k\in K}$ of $\{A_i\}_{i\in I}$, any bounded nets $\{x_k^*\}_{k\in K}$ in X^* and $\{x_k\}_{k\in K}$ in X^{**} such that

$$x_k \in A_{i(k)},\ \forall k \in K, \tag{2.7}$$

$$x_k^* \to x_0^* \text{ in } (X^*, \sigma(X^*, X)), x_k \to x_0 \text{ in } (X^{**}, \sigma(X^{**}, X^*)), \tag{2.8}$$

and

$$\limsup \langle x_k^*, x_k\rangle \leqslant \langle x_0^*, x_0\rangle, \tag{2.9}$$

we have

$$x_0 \in A_0(x_0^*), \tag{2.10}$$

and

$$\lim \langle x_k^*, x_k\rangle = \langle x_0^*, x_0\rangle. \tag{2.11}$$

We consider next a regularization of Moreau–Yosida type for maximal monotone mapping of type (D) defined on a dual space. Let $A_0 : X^* \to 2^X$ be a maximal monotone mapping of type (D) from X^* to X and $A = \widetilde{A_0}$ be its corresponding monotone closure, which is a maximal monotone mapping of type (D) from X^* to X^{**}.

Let $J = J_X : X \to 2^{X^*}$ and $J_* = J_{X^*} : X^* \to 2^{X^{**}}$ be the duality mappings on X and X^*, respectively,

$$J = \frac{1}{2}\partial \|\cdot\|_X^2 \text{ and } J_* = \frac{1}{2}\partial \|\cdot\|_{X^*}^2,$$

where ∂ stands for the subdifferential in the sense of Convex Analysis. It is known (see e.g. [17–19]) that J is a maximal monotone mapping of type (D) from X to X^* where its monotone extension $\widetilde{J} : X^{**} \to 2^{X^*}$, which is a maximal monotone mapping of type (D) from X^{**} to X^*, satisfies the relation

$$\widetilde{J} = J_*^{-1}. \tag{2.12}$$

From (2.12), we see that $J^{-1} : X^* \to 2^X$ is maximal monotone of type (D) from X^* to X, and its maximal monotone extension, $\widetilde{J^{-1}} = J_*$, is maximal monotone of type (D) from X^* to X^{**} in the sense of Definition 2.1.

For $\lambda > 0$, let us define $A_\lambda : X^* \to 2^{X^{**}}$ by

$$A_\lambda := (A^{-1} + \lambda\widetilde{J})^{-1} = (\widetilde{A_0^{-1}} + \lambda\widetilde{J})^{-1}. \tag{2.13}$$

The second equality follows from (2.6).

Some properties of A_λ are given in the following result, whose proof follows the same line as that of Proposition 3.2 of [17] with appropriate adaptations, and is therefore omitted.

Theorem 2.1. *(a) A_λ is monotone and generalized pseudomonotone from X^* to $2^{X^{**}}$ with $D(A_\lambda) = X^*$ and for all $x^* \in X^*$, $A_\lambda(x^*)$ is convex and $\sigma(X^{**}, X^*)$-closed.*

(b) A_λ is a bounded mapping.

(c) For all $\lambda > 0$, $x^* \in D(A)$, $x \in A(x^*)$, and $x_\lambda \in A_\lambda(x^*)$, we have

$$\|x_\lambda\| \leqslant \|x\|. \tag{2.14}$$

(d) If F is a finite dimensional subspace of X^*, then the restriction $A_\lambda|_F$ is upper semicontinuous from F to $2^{X^{**}}$ with X^{**} endowed with the weak topology $\sigma(X^{**}, X^*)$.*

We have the following approximation property of A_λ to A, when A is maximal monotone mapping of type (D).

Proposition 2.6. *Let $A : X^* \to 2^{X^{**}}$ be a maximal monotone mapping of type (D) from X^* to X^{**}. Let $\{\lambda_i\}_{i\in I}$ be a net in $(0,\infty)$ such that $\lambda_i \to 0^+$. We have*

$$A_{\lambda_i} \xrightarrow{(gpm)} A \text{ in the sense of Definition 2.5.}$$

Proof. For simplicity of notation and without loss of generality, we assume in the sequel that the subnet $A_{\lambda_{i(k)}}$ of $\{A_{\lambda_i}\}$ as in Definition 2.5 about generalized pseudomonotone convergence is the net $\{A_{\lambda_i}\}$ itself. Suppose that $\{(x_i^*, x_i)\}$ is a bounded net in $X^* \times X^{**}$ satisfying the following conditions:

$$x_i \in A_{\lambda_i}(x_i^*),\ \forall i \in I, \tag{2.15}$$

$$\begin{aligned} &x_i^* \to x_0^* \text{ in } X^* \text{ with respect to the weak}^* \text{ topology } \sigma(X^*, X), \\ &x_i \to x_0 \text{ in } X^{**} \text{ with respect to the norm topology,} \end{aligned} \tag{2.16}$$

and

$$\limsup \langle x_i, x_i^* \rangle \leqslant \langle x_0, x_0^* \rangle. \tag{2.17}$$

It follows from (2.13), (2.15), and the definition of A_λ, that for each $i \in I$, there is $a_i^* \in A^{-1}(x_i)$ such that

$$x_i^* - a_i^* \in \lambda_i \widetilde{J}(x_i) = J_{X^*}^{-1}(\lambda_i x_i).$$

Hence $\lambda_i x_i \in J_{X^*}(x_i^* - a_i^*)$. Since $\lambda_i \to 0$ and $\{x_i\}$ is a bounded net in X^{**}, we have $\|x_i^* - a_i^*\| = \lambda_i \|x_i\| \to 0$. In particular, $\{a_i^*\}$ is a bounded net in X^* and as a consequence of (2.16),

$$a_i^* \to x_0^* \text{ in } (X^*, \sigma(X^*, X)). \tag{2.18}$$

Moreover, since

$$\lim \langle x_i, x_i^* - a_i^* \rangle = 0, \tag{2.19}$$

we obtain

$$\limsup \langle x_i, a_i^* \rangle = \limsup \langle x_i, x_i^* \rangle \leqslant \langle x_0, x_o^* \rangle. \tag{2.20}$$

Since $x_i \in A(a_i^*)$, it follows from (2.16), (2.18), (2.20), and Proposition 2.3, that $x_0 \in A(x_0^*)$ and $\lim \langle x_i, a_i^* \rangle = \langle x_0, x_0^* \rangle$.
Using again (2.19), we see that $\lim \langle x_i, x_i^* \rangle = \langle x_0, x_0^* \rangle$. □

2.1.4. Assumptions on the generalized pseudomonotone term

Let B be a mapping from X^* to $2^{X^{**}}$ and $u_0 \in X^*$. Suppose that B and u_0 satisfy the following conditions.

(B1) $D(B) = X^*$ and for all $x \in X^*$, $B(x)$ is a convex, $\sigma(X^{**}, X^*)$-compact subset of X^{**}.

(B2) If E is a finite dimensional subspace of X^*, then $B|_E$ is upper semicontinuous from E to $2^{X^{**}}$ with X^{**} equipped with the weak* topology $\sigma(X^{**}, X^*)$.

(B3) B is generalized pseudomonotone from X^* to X^{**}.

(B4) B is strongly quasibounded with respect to u_0 in the following sense: For each $M > 0$, there exists $K = K(M) > 0$ such that if $x \in D(B)$, $b^* \in B(x)$, and

$$\|x\| \leqslant M \text{ and } \langle b^*, x - u_0 \rangle_{X^{**}, X^*} \leqslant M, \tag{2.21}$$

then $\|b^*\|_{X^{**}} \leqslant K$.

2.2. Abstract existence theorem for inclusions in dual spaces

In this section, we prove an existence theorem for inclusions of the form (1.3) in the dual space X^*.

2.2.1. For $R > 0$, let $Q_R = \overline{B_{X^*}}(0, R) = \{u \in X^* : \|u\|_{X^*} \leqslant R\}$ be the closed ball in X^* with radius R, centered at 0, and $S_R = \partial Q_R = \{u \in X^* : \|u\|_{X^*} = R\}$ be the sphere in X^* with radius R, centered at 0.

Lemma 2.7. *Let A be a maximal monotone mapping of type (D) from X^* to $2^{X^{**}}$ and let B and $u_0 \in D(A)$ satisfy (B1)–(B4). For each $R > \|u_0\|$, and each $f_0 \in X^{**}$, there exist $u \in Q_R$ and $a^* \in A(u)$, $b^* \in B(u)$, such that*

$$\langle a^* + b^* - f_0, v - u \rangle \geqslant 0, \forall v \in Q_R. \tag{2.22}$$

Proof. Consider the complementary system (cf. e.g. [18,20])

$$(Y, Y_0; Z, Z_0) = (X^*, X^*; X^{**}, X),$$

(cf. e.g. [18,20]), and for each $\lambda > 0$, consider the mapping $T = A_\lambda + B - f_0$ from $Y = X^*$ into $2^Z = 2^{X^{**}}$. It follows from Theorem 2.1, Proposition 2.4, and the assumptions (B1)–(B4), that T satisfies conditions (T1)–(T3) in [20] with $D(T) = Y = X^*$. Moreover, from (B4) and Theorem 2.1, T is strongly quasi-bounded with respect to u_0 in the sense of condition (B′) and Proposition 2.2 of [20].

Let $\phi = I_{Q_R}$ be the indicator functional of Q_R. Since Q_R is compact with respect to the weak* topology $\sigma(X^*, X)$, ϕ is a convex functional which is lower semicontinuous with respect to the same topology $\sigma(X^*, X)$, that is, ϕ satisfies condition (P1) in [20]. Since $Y = Y_0 = X^*$ and Q_R is bounded, condition (P2) and the coercivity condition (C) of [20] are obviously fulfilled. We see, by Theorem 1.3 of [20] with $f = 0$ and T and ϕ defined as above, that for each $\lambda > 0$, there are $u_\lambda \in Q_R$, $a^*_\lambda \in A_\lambda(u_\lambda)$, $b^*_\lambda \in B(u_\lambda)$, such that

$$\langle a^*_\lambda + b^*_\lambda - f_0, v - u_\lambda\rangle \geqslant 0,\ \forall v \in Q_R. \tag{2.23}$$

For $n \in \mathbb{N}$, let $\lambda_n = 1/n$, and let us put, for simplicity of notation, $u_n = u_{\lambda_n}$, $a^*_n = a^*_{\lambda_n}$, $b^*_n = b^*_{\lambda_n}$. Inequality (2.23) gives, in this particular case,

$$\langle a^*_n + b^*_n - f_0, v - u_n\rangle \geqslant 0,\ \forall v \in Q_R, \tag{2.24}$$

with $u_n \in Q_R$, $a^*_n \in A_{1/n}(u_n)$, $b^*_n \in B(u_n)$. Since $u_0 \in D(A) \cap Q_R$, $A(u_0) \neq \emptyset$ and we can choose a fixed element a^*_0 of $A(u_0)$. On the other hand, since $D(A_\lambda) = X^*$ for all $\lambda > 0$, we can also fix, for each $n \in \mathbb{N}$, an element a^*_{0n} of $A_{\lambda_n}(u_0)$. Moreover, from Theorem 2.1,

$$\|a^*_{0n}\| \leqslant \|a^*_0\|,\ \forall n \in \mathbb{N}. \tag{2.25}$$

Let us prove next that $\{b^*_n\}$ is a bounded sequence in X^{**}. In fact, let $v = u_0$ in (2.24) yields

$$0 \leqslant \langle a^*_n + b^*_n - f_0, u_0 - u_n\rangle. \tag{2.26}$$

Hence,

$$\begin{aligned}
& \langle b^*_n, u_0 - u_n\rangle \\
\leqslant\ & \|f_0\|\|u_0 - u_n\| + \langle a^*_n - a^*_{0n}, u_0 - u_n\rangle + \langle a^*_{0n}, u_0 - u_n\rangle \\
\leqslant\ & \|f_0\|\|u_0 - u_n\| + \langle a^*_{0n}, u_0 - u_n\rangle \\
\leqslant\ & (\|f_0\| + \|a^*_{0n}\|)(\|u_0\| + \|u_n\|) \\
\leqslant\ & (\|f_0\| + \|a^*_0\|)(\|u_0\| + \|u_n\|) \\
\leqslant\ & 2R(\|f_0\| + \|a^*_0\|).
\end{aligned}$$

Since $\|u_n\| \leqslant R$ for all n, we have from condition (B4) and this estimate that

$$\|b^*_n\| \leqslant C_1,\ \forall n \in \mathbb{N}, \tag{2.27}$$

for some constant $C_1 > 0$, i.e., $\{b^*_n\}$ is a bounded sequence in X^{**}.

We prove next that the sequence $\{a^*_n\}$ is also bounded in X^{**}. In fact, since $\|u_0\| < R$, $\delta := \frac{1}{2}(R - \|u_0\|) > 0$. Moreover, if $w \in X^{**}$ and $\|w\| < \delta$, then $\|u_0 - w\| < \|u_0\| + \frac{1}{2}(R - \|u_0\|) = \frac{1}{2}(R + \|u_0\|) < R$, i.e., $v = u_0 - w \in Q_R$.

Letting $v = u_0 - w$ in (2.24) yields $\langle a^*_n + b^*_n - f_0, u_0 - u_n - w\rangle \geqslant 0$. Hence, from (2.25) and (2.27),

$$\begin{aligned}
\langle a^*_n, w\rangle &\leqslant \langle a^*_n, u_0 - u_n\rangle + \langle b^*_n - f_0, (u_0 - w) - u_n\rangle \\
&\leqslant \langle a^*_{0n}, u_0 - u_n\rangle + \langle b^*_n - f_0, (u_0 - w) - u_n\rangle \\
&\leqslant (\|u_0\| + \|u_n\|)\|a^*_{0n}\| + (\|u_0 - w\| + \|u_n\|)(\|b^*_n\| + \|f_0\|) \\
&\leqslant C_2 := 2R\|a^*_{0n}\| + 2R(C_1 + \|f_0\|).
\end{aligned}$$

Since this is true for all $w \in X^*$ with $\|w\| \leqslant \delta$, we get, for all $n \in \mathbb{N}$,

$$\|a^*_n\| = \frac{1}{\delta}\sup\{\langle a^*_n, w\rangle : w \in X^*, \|w\| \leqslant \delta\} \leqslant C_3 := \frac{C_2}{\delta},$$

which shows that the sequence $\{a^*_n\}$ is bounded in X^{**}.

Since the sequences $\{u_n\}$, $\{a^*_n\}$, and $\{b^*_n\}$ are bounded, by passing to subsequences if necessary, we can assume that

$$u_n \to u \text{ in } (X^*, \sigma(X^*, X)), \tag{2.28}$$

$$a^*_n \to a^*, b^*_n \to b^* \text{ in } (X^{**}, \sigma(X^{**}, X^*)), \tag{2.29}$$

and

$$\langle a^*_n, u_n\rangle \to p_1 \in \mathbb{R}, \langle b^*_n, u_n\rangle \to p_2 \in \mathbb{R}. \tag{2.30}$$

We have from (2.24) that

$$\langle a^*_n + b^*_n, u_n\rangle \leqslant \langle f_0, v - u_n\rangle + \langle a^*_n + b^*_n, v\rangle,\ \forall v \in Q_R, \forall n \in \mathbb{N}. \tag{2.31}$$

Letting $n \to \infty$ in this inequality and noting (2.28)–(2.30), we get

$$p_1 + p_2 \leqslant \langle f_0, v-u\rangle + \langle a^* + b^*, v\rangle,\ \forall v \in Q_R. \tag{2.32}$$

Since Q_R is weak* compact in X^*, we have from (2.28) that $u \in Q_R$. Letting $v = u$ in (2.32) yields

$$p_1 + p_2 \leqslant \langle a^*, u\rangle + \langle b^*, u\rangle, \tag{2.33}$$

which implies that either

$$p_1 \leqslant \langle a^*, u\rangle, \tag{2.34}$$

or

$$p_2 \leqslant \langle b^*, u\rangle. \tag{2.35}$$

Assume (2.34). It follows from (2.28), (2.30), (2.34), and Proposition 2.6 that

$$a^* \in A(u) \text{ and } p_1 = \langle a^*, u\rangle. \tag{2.36}$$

We see from (2.33) and (2.36) that (2.35) also holds true. Since B is generalized pseudomonotone, it follows from (2.28)–(2.30) and (2.35) that

$$b^* \in B(u) \text{ and } p_2 = \langle b^*, u\rangle. \tag{2.37}$$

Similarly, if (2.35) holds then by using the generalized pseudomonotonicity of B combined with (2.28)–(2.30), we get (2.37), which together with (2.33), implies (2.34). This shows that (2.33) implies, in all cases, both (2.36) and (2.37).

For any $v \in Q_R$, we have from (2.32), (2.36), and (2.37), that $u \in Q_R$ satisfies

$$\langle a^* + b^*, u\rangle \leqslant \langle f_0, v-u\rangle + \langle a^* + b^*, v\rangle,$$

that is, u is a solution of (2.22). □

2.2.2. We are now ready to prove the main abstract existence theorem of this section.

Theorem 2.2. *Let $A : X^* \to 2^{X^{**}}$ be a maximal monotone mapping of type (D) from X^* to X^{**}, $B : X^* \to 2^{X^{**}}$ be a mapping that satisfies conditions (B1)–(B4) with some $u_0 \in D(A)$, and $f_0 \in X^{**}$.*

(a) Under the following coercivity condition:

(C) There exists $R > \|u_0\|$ such that

$$\inf_{a^* \in A(w), b^* \in B(w)} \langle a^* + b^* - f_0, w - u_0\rangle > 0,\ \forall w \in D(A) \cap S_R, \tag{2.38}$$

there exists $u \in D(A)$ such that

$$\|u\| < R, \tag{2.39}$$

and

$$f_0 \in (A+B)(u), \tag{2.40}$$

that is, there exist $u \in D(A)$, $a^* \in A(u)$, and $b^* \in B(u)$ such that u satisfies (2.39) and $a^* + b^* = f_0$.

(b) If the strict inequality "$>$" in (2.38) is replaced by the nonstrict inequality "$\geqslant$", then the conclusion in (a) still holds with the strict inequality "$<$" in (2.39) replaced by the nonstrict inequality "$\leqslant$".

Proof. (a) For R as in (2.38), it follows from Lemma 2.7, that there are $u \in Q_R$, $a^* \in A(u)$, and $b^* \in B(u)$ satisfying (2.22).

Let us prove that $\|u\| < R$. In fact, if $\|u\| = R$ then $u \in S_R \cap D(A) \cap D(B)$, and (2.22) yields $\langle a^* + b^* - f_0, u_0 - u\rangle \geqslant 0$, contradicting (2.38). This contradiction proves that $\|u\| < R$. Let $w \in X^*$. Then there exists $t > 0$ such that $\|u \pm tw\| \leqslant \|u\| + t\|w\| \leqslant R$, i.e., $v = u \pm tw \in Q_R$. Letting these values of v in (2.22) gives $\pm t\langle a^* + b^* - f_0, w\rangle \geqslant 0$, which implies that $\langle a^* + b^* - f_0, w\rangle = 0$. Consequently, $a^* + b^* - f_0 = 0$, that is, $f_0 \in (A+B)(u)$.

(b) We use in this proof the duality mappings $J = J_X$ and $J_* = J_{X^*}$ defined in Section 2.1. Since J_* is bounded and generalized pseudomonotone from X^* to X^{**}, it follows from Proposition 2.4 that for all $\varepsilon > 0$, the mapping $B_\varepsilon := B + \varepsilon J_*$ is also generalized pseudomonotone from X^* to X^{**}, i.e., it satisfies condition (B3). Moreover, straightforward arguments show that B_ε satisfies conditions (B1), (B2), and (B4) since B satisfies these conditions.

Let us verify inequality (2.38) in the coercivity condition (C) holds for A and B_ε. In fact, let $w \in D(A) \cap S_R$, $a^* \in A(w)$, and $b^*_\varepsilon = b^* + \varepsilon g^* \in B_\varepsilon(w)$ where $b^* \in B(w)$ and $g^* \in J_*(w)$. We have

$$\begin{aligned} \langle a^* + b^* + \varepsilon g^* - f_0, w - u_0\rangle &\geqslant \varepsilon\langle g^*, w - u_0\rangle \\ &\geqslant \varepsilon\|g^*\|(\|w\| - \|u_0\|) = \varepsilon R(R - \|u_0\|) > 0. \end{aligned}$$

Therefore, A and B_ε satisfy all conditions in part (a). Hence from part (a), for each $\varepsilon > 0$, there are $u_\varepsilon \in D(A)$, $a^*_\varepsilon \in A(u_\varepsilon)$, $b^*_\varepsilon \in B(u_\varepsilon)$, and $g^*_\varepsilon \in J_*(u_\varepsilon)$, such that

$$\|u_\varepsilon\| < R, \tag{2.41}$$

and

$$a^*_\varepsilon + b^*_\varepsilon + \varepsilon g^*_\varepsilon = f_0. \tag{2.42}$$

We have $\{u_\varepsilon\} \subset Q_R$ and since $\|g^*_\varepsilon\| = \|u_\varepsilon\|$, the collection $\{g^*_\varepsilon\}$ is bounded. Let a^*_0 be an element of $A(u_0)$. We have

$$\langle a^*_\varepsilon, u_\varepsilon - u_0\rangle + \langle b^*_\varepsilon, u_\varepsilon - u_0\rangle + \varepsilon\langle g^*_\varepsilon, u_\varepsilon - u_0\rangle = \langle f_0, u_\varepsilon - u_0\rangle.$$

Hence, from the monotonicity of A, there is a constant $C_0 > 0$ such that for all $\varepsilon > 0$,

$$\langle b^*_\varepsilon, u_\varepsilon - u_0\rangle \leqslant \langle f_0, u_\varepsilon - u_0\rangle - \langle a^*_0, u_\varepsilon - u_0\rangle - \varepsilon\langle g^*_\varepsilon, u_\varepsilon - u_0\rangle \leqslant C_0.$$

It follows from the quasiboundedness of (B4) that $\{b^*_\varepsilon\}$ is bounded, which together with (2.42), implies that $\{a^*_\varepsilon\}$ is also bounded. Therefore, there is a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$ as $n \to \infty$, such that

$$\begin{aligned} u_{\varepsilon_n} &\to u \text{ in } (X^*, \sigma(X^*, X)),\\ a^*_{\varepsilon_n} \to a^*, b^*_{\varepsilon_n} &\to b^* \text{ in } (X^{**}, \sigma(X^{**}, X^*))\\ \langle a^*_{\varepsilon_n}, u_{\varepsilon_n}\rangle \to r_1, \langle b^*_{\varepsilon_n}, u_{\varepsilon_n}\rangle &\to r_2. \end{aligned}$$

Arguing as in the proof of Lemma 2.7, we see that $u \in Q_R$, $a^* \in A(u)$, $b^* \in B(u)$, $r_1 = \langle a^*, u\rangle$, and $r_2 = \langle b^*, u\rangle$. Passing to the limit in (2.42) with $\varepsilon = \varepsilon_n$, we obtain $a^* + b^* = f_0$, that is, u is a solution of (2.40) with $\|u\| \leqslant R$. □

2.3. Application to multivalued variational inequalities defined in dual spaces

We present in this section an example of maximal monotone mapping of type (D) and an application of Theorem 2.2 to variational inequalities. The following result is a dual version of Theorem 3.1 in [14].

Proposition 2.8. *Let $j : X^* \to \mathbb{R} \cup \{\infty\}$ be a convex, proper, and lower semicontinuous with respect to the weak* topology $\sigma(X^*, X)$ on X^*. Then $\partial j : X^* \to X^{**}$ is a maximal monotone mapping of type (D) from X^* to X^{**} in the sense of Definition 2.1.*

Proof. Let $j^* : X^{**} \to \mathbb{R}\cup\{\infty\}$ be the conjugate of j and $j_* = j^*|_X$ be the restriction of j^* on X, as defined in [21] (where j_* is called the weak* Legendre–Fenchel conjugate of j). We observe that j^* and j_* are proper, convex, and lower semicontinuous functionals (with respect to the norm topology) on X^{**} and X, respectively. Moreover, from Proposition 2.2 in [21], we have $j = (j_*)^*$.

On the other hand, from Theorem 3.1 in [14], we know that $\partial j_* : X \to 2^{X^*}$ is a maximal monotone mapping of type (D) from X to X^*, in the sense of the definition in Section 2 of [14] or in Definition 2.1 of [17]. Moreover, the monotone closure $\widetilde{(\partial j_*)}$ of ∂j_*, which is a mapping from X^{**} to 2^{X^*}, is characterized by

$$\widetilde{\partial j_*} = \left[\partial (j_*)^*\right]^{-1} = (\partial j)^{-1}. \tag{2.43}$$

Next, we show that $\partial j = \left(\widetilde{\partial j_*}\right)^{-1}$ is maximal monotone of type (D) from X^* to $2^{X^{**}}$ in the sense of Definition 2.1(b). In fact, we first note that $A_0 = (\partial j_*)^{-1}$ is maximal monotone of type (D) from X^* to 2^X by definition and, by (2.43), its monotone closure in $X^* \times X^{**}$ satisfies

$$\widetilde{A_0} = \widetilde{(\partial j_*)^{-1}} = \left(\widetilde{\partial j_*}\right)^{-1} = [(\partial j)^{-1}]^{-1} = \partial j.$$

This means that ∂j is maximal monotone of type (D) from X^* to $2^{X^{**}}$ in the sense of Definition 2.1(b). □

Let us consider an application of the above results to variational inequalities in dual spaces. Let $f_0 \in X^{**}$, $B : X^* \to 2^{X^{**}}$ satisfy conditions (B1)–(B4), and $\phi : X^* \to \mathbb{R}\cup\{\infty\}$ be a convex and proper functional which is lower continuous with respect to the weak* topology $\sigma(X^*, X)$. We shall use $D(\phi)$ to denote the effective domain of ϕ, $D(\phi) = \{u \in X^* : \phi(u) < \infty\}$.

Let us consider the following variational inequality: Find $u \in D(\phi)$ and $b^* \in B(u)$ such that

$$\langle b^*, v - u\rangle + \phi(v) - \phi(u) \geqslant \langle f_0, v - u\rangle, \ \forall v \in X^*. \tag{2.44}$$

Let $\partial\phi : X^* \to 2^{X^{**}}$ be the subdifferential of ϕ. Note that u and b^* satisfy (2.44) if and only if $u \in D(\partial\phi)$ and $f_0 - b^* \in \partial\phi(u)$. Hence (2.44) is equivalent to the inclusion

$$f_0 \in (\partial\phi + B)(u). \tag{2.45}$$

Let u_0 be an element of $D(\partial\phi)$. Thanks to Proposition 2.8, we see that the inclusion (2.45) is of the form (2.40) with $A = \partial\phi$. We have the following corollary of Theorem 2.2 in this particular case.

Corollary 2.9. *Let $\phi : X^* \to \mathbb{R} \cup \{\infty\}$ be a convex, proper, lower continuous with respect to the weak* topology on X^* and let $B : X^* \to 2^{X^{**}}$ satisfy conditions (B1)-(B4) with $u_0 \in D(\partial\phi)$ and $f_0 \in X^{**}$.*

Assume there exists $R > \|u_0\|$ such that for all $w \in S_R (:= \{v \in X^* : \|v\| = R\})$,

$$\inf_{b^* \in B(w)} \langle b^* - f_0, w - u_0 \rangle + \phi(w) > \phi(u_0), \tag{2.46}$$

(resp. $\inf_{b^* \in B(w)} \langle b^* - f_0, w - u_0 \rangle + \phi(w) \geqslant \phi(u_0)$).

Then the variational inequality (2.44) has a solution u with $\|u\| < R$ (resp. $\|u\| \leqslant R$).

Proof. For $w \in S_R \cap D(\partial\phi)$, $a^* \in \partial\phi(w)$, and $b^* \in B(w)$, we have, by the definition of $\partial\phi$, that

$$\phi(w) - \phi(u_0) \geqslant \langle a^*, w - u_0 \rangle,$$

and thus

$$\langle a^* + b^* - f_0, w - u_0 \rangle \geqslant \langle b^* - f_0, w - u_0 \rangle + \phi(w) - \phi(u_0).$$

This inequality and (2.46) imply the coercivity condition (2.38). Our result now follows directly from Theorem 2.2. □

3. Multivalued variational inequalities with the minimal surface operator

In this section, we consider an application of the above abstract theory to the boundary value problem (1.1) with the nonparametric minimal surface operator and multivalued lower order term.

3.1. Problem setting – Auxiliary results

In this section, we present a formulation of (1.1) as a multivalued variational inequality in a space of functions of bounded variation, and present some properties of functions of bounded variation that are useful in the sequel.

3.1.1. Problem setting

Let Ω be an open bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with Lipschitz boundary $\partial\Omega$, and let $\mathcal{U}$ be an open ball in $\mathbb{R}^N$ that contains $\overline{\Omega}$.

In the case where f is a single-valued function, the inclusion in (1.1) reduces to an equation, and a weak formulation for (1.1) in this case is given in [10] as the following variational inequality:

$$\begin{cases} J(v) - J(u) + \int_\Omega f(x,u)(v-u)dx \geqslant 0, \ \forall v \in W \\ u \in W, \end{cases} \tag{3.1}$$

where

$$W = \{u \in BV(\mathcal{U}) : u = 0 \ \text{ a.e. in } \ \mathcal{U} \setminus \Omega\}$$

is a closed subspace of $BV(\mathcal{U})$. The space W is a Banach space with the norm:

$$\|u\| = \|u\|_W = \int_{\mathcal{U}} |\nabla u|, \ \forall u \in W.$$

Here,

$$\int_{\mathcal{U}} |\nabla u| := \sup\left\{ \int_{\mathcal{U}} u \operatorname{div} g\, dx : g = (g_1, \dots, g_N) \in C_0^1(\mathcal{U}, \mathbb{R}^N) \text{ and } \max_{x\in\mathcal{U}} |g(x)| \leqslant 1 \right\},$$

($\operatorname{div} g = \sum_{i=1}^N \partial_i g_i$). Note that on W, the norm $\|\cdot\|_W$ is equivalent to the usual norm on $BV(\mathcal{U})$, defined by

$$\|u\|_{BV(\mathcal{U})} = \int_{\mathcal{U}} |u| dx + \int_{\mathcal{U}} |\nabla u|, \ u \in BV(\mathcal{U}),$$

thanks to the Poincaré inequality in $BV(\mathcal{U})$ (cf. Proposition 3.3).

The functional $J : BV(\mathcal{U}) \to \mathbb{R}$ in (3.1) is given by

$$J(v) = \int_{\mathcal{U}} \left[\sqrt{1 + |\nabla v|^2} - 1\right] = \int_{\mathcal{U}} \sqrt{1 + |\nabla v|^2} - |\mathcal{U}|, \tag{3.2}$$

where (cf. e.g. [6,22])

$$\int_{\mathcal{U}} \sqrt{1 + |\nabla u|^2} = \sup\left\{ \int_{\mathcal{U}} (g_{n+1} + u \operatorname{div} g) dx : g = (g_1, g_2, \dots, g_{n+1}) \in C_0^1(\mathcal{U}, \mathbb{R}^{n+1}), \max_{x\in\mathcal{U}} |g(x)| \leqslant 1 \right\}.$$

It is known that J is convex on $BV(\mathcal{U})$ and lower semicontinuous with respect to the $L^1(\mathcal{U})$-topology (cf. [10]).

In the case where f is a multivalued function that we are interested here, the inclusion (1.1) can be rewritten as:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+\eta=0 \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega, \\ \eta(x)\in f(x,u(x)) \text{ for } x\in\Omega. \end{cases} \tag{3.3}$$

This is thus formulated in the weak form as the following variational inequality:

$$\begin{cases} J(v)-J(u)+\int_\Omega \eta(v-u)dx \geqslant 0,\ \forall v\in W \\ u\in W, \end{cases} \tag{3.4}$$

with

$$\eta(x)\in f(x,u(x)) \text{ for a.e. } x\in\Omega. \tag{3.5}$$

3.1.2. Auxiliary results

Let us list some auxiliary results about functions of bounded variation that will be needed in the sequel.

Proposition 3.1. *The space $Z=BV(\mathcal{U})$ is the dual space of a separable Banach space Y.*

The separable Banach space Y can be constructed as follows (cf. Section 3.1, [23]). Let $C_0(\mathcal{U})$ be the Banach space of all (real valued) continuous functions on $\mathcal{U}$ that vanish at the boundary of $\mathcal{U}$, equipped with the sup-norm, $C_c^\infty(\mathcal{U})$ be the set of all (real valued) infinitely differentiable functions on $\mathcal{U}$ with compact support in $\mathcal{U}$.

Let $E_0\subset[C_0(\mathcal{U})]^{N+1}$ be the closure in $[C_0(\mathcal{U})]^{N+1}$ of the subspace

$$\begin{aligned} E_1 \quad = \quad & \Big\{\ \psi=(\psi_0,\psi_0,\dots,\psi_N)\in[C_0(\mathcal{U})]^{N+1} : (\psi_1,\dots.\psi_N)\in\left[C_c^\infty(\mathcal{U})\right]^N, \\ & \psi_0=\mathrm{div}(\psi_1,\dots,\psi_N)\ \Big\}. \end{aligned}$$

Since the dual $([C_0(\mathcal{U})]^{N+1})^*$ of $[C_0(\mathcal{U})]^{N+1}$ consists of all finite $\mathbf{R}^{N+1}$-valued Radon vector measures $\mu=\left(\mu_0\dots,\mu_N\right)$ in $\mathcal{U}$, we can associate with any $u\in BV(\mathcal{U})$ a measure $\mu=Tu\in([C_0(\mathcal{U})]^{N+1})^*$ by setting

$$Tu=(u\mathcal{L},\nabla u),$$

where $u\mathcal{L}$ is the measure on $\mathcal{U}$ defined by the Lebesgue integral with respect to u.

By the definition of the BV norm, we see that $\|u\|_{BV}\leqslant 2\|Tu\|\leqslant 2\|u\|_{BV}$ for every $u\in BV(\mathcal{U})$. Moreover, $\langle Tu,\psi\rangle=0$ if $\psi\in E_1$. Consequently, $\mathrm{Ker}(Tu)\supset E_1$. On the other hand, any $\mu([C_0(\mathcal{U})]^{N+1})^*$ whose kernel contains E_1 is equal to Tu for some $u\in BV(\mathcal{U})$. For $Y=[C_0(\mathcal{U})]^{N+1}/E_0$, this implies that T is an isomorphism between $Z=BV(\mathcal{U})$ and $([C_0(\mathcal{U})]^{N+1}/E_0)^*=Y^*$.

The following theorems are useful in our proofs in the sequel. A simple criterion for weak* convergence on $BV(\mathcal{U})$ is given by the following result.

Proposition 3.2 (*Proposition 3.13, [23]*). *Let $\{u_h\}\subset BV(\mathcal{U})$. Then u_h weakly* converges to u in $BV(\mathcal{U})$ if and only if $\{u_h\}$ is bounded in $BV(\mathcal{U})$ and converges to u in $L^1(\mathcal{U})$.*

We also need the following embedding results for $BV(\mathcal{U})$ and Poincaré's inequality for W, which are given e.g. in Theorems 10.1.3, 10.1.4 in [24], Section 1.19 and Theorem 1.28 in [6], and Theorem 3.1 in [10].

Proposition 3.3. *(a) The embedding $i_q : BV(\mathcal{U})\hookrightarrow L^p(\mathcal{U})$, $u\mapsto u$ is continuous if $1\leqslant p\leqslant \frac{N}{N-1}$ and compact if $1\leqslant p<\frac{N}{N-1}$.*

(b) For each $p\in\left[1,\frac{N}{N-1}\right]$, there exists $\alpha_p>0$ such that

$$\left(\int_{\mathcal{U}}|u|^p dx\right)^{\frac{1}{p}}\leqslant \alpha_p\int_{\mathcal{U}}|\nabla u|,\ \forall u\in W. \tag{3.6}$$

We also have the following estimate for $\sqrt{1+|\nabla u|^2}$ for $u\in BV(\mathcal{U})$.

Proposition 3.4 (*Lemma 3.1, [10]*). *We have*

$$\int_{\mathcal{U}}\sqrt{1+|\nabla u|^2}\geqslant\sqrt{|\mathcal{U}|^2+\left(\int_{\mathcal{U}}|\nabla u|\right)^2},\ \forall u\in BV(\mathcal{U}). \tag{3.7}$$

As consequences of this estimate, we see that J given by (3.2) satisfies $J(u)\geqslant 0$ and $J(u)\geqslant\int_{\mathcal{U}}|\nabla u|-|\mathcal{U}|$ for all $u\in BV(\mathcal{U})$.

3.2. Existence theorems - Coercive case

We show in this section existence theorems for (3.4)–(3.5) under coercivity conditions.

3.2.1. Assumptions

Let us first consider the conditions imposed on the lower order term f. Together with the standard notation for Lebesgue and Sobolev spaces, we use $L^0(\mathcal{U})$ for the set of all (equivalent classes of) real valued measurable functions defined on $\mathcal{U}$. For a normed vector space S, we use the notation

$$\mathcal{K}(S) = \{A \subset S : A \neq \emptyset, A \text{ is closed and convex}\},$$

Let f be a function from $\mathcal{U} \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ that has the following properties.

(F1) f is superpositionally measurable, that is, if u is a measurable function on $\mathcal{U}$ then the (multivalued) function $f(\cdot, u(\cdot))$, $x \mapsto f(x, u(x))$ is measurable on $\mathcal{U}$.

Note that if f is graph measurable on $\mathcal{U} \times \mathbb{R}$, that is, $\mathrm{Gr}(f) = \{(x, u, \xi) \in \mathcal{U} \times \mathbb{R} \times \mathbb{R} : \xi \in f(x, u)\}$ belongs to $[\mathcal{L}(\mathcal{U}) \times \mathcal{B}(\mathbb{R})] \times \mathcal{B}(\mathbb{R})$ ($\mathcal{L}(\mathcal{U})$ is the family of Lebesgue measurable subsets of $\mathcal{U}$ and $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$), then f is superpositionally measurable. Also, if f is measurable from $\mathcal{U} \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ in the usual sense, that is $f^-(W) := \{(x, u) \in \mathcal{U} \times \mathbb{R} : f(x, u) \cap W \neq \emptyset\} \in \mathcal{L}(\mathcal{U}) \times \mathcal{U}(\mathbb{R})$ for all $W \subset \mathbb{R}$ open, then f is graph measurable on $\mathcal{U} \times \mathbb{R}$, and thus superpositionally measurable.

(F2) For a.e. $x \in \mathcal{U}$, the function $f(x, \cdot) : \mathbb{R} \to \mathcal{K}(\mathbb{R})$ is upper semicontinuous, that is, for each $u \in \mathbb{R}$ and each open $U \subset \mathbb{R}$ such that $f(x, u) \subset U$, there exists $\delta > 0$ such that if $|v - u| < \delta$ then $f(x, v) \subset U$.

Note that since $f(x, u)$ is a compact interval in $\mathbb{R}$, condition (F2) is equivalent to the Hausdorff upper semicontinuity (h-u.s.c.) of $f(x, \cdot)$ for a.e. $x \in \mathcal{U}$ (cf. Theorem 2.68, Chap. 1, [25]). Moreover, since we consider only functions in W, we also assume that $f(x, u) = \{0\}$ for a.e. $x \in \mathcal{U} \setminus \Omega$, all $u \in \mathbb{R}$.

In several places in what follows, we also need the following growth condition on f:

(F3) There exist $1 < q < N/(N-1)$ and $a_3 \in L^{q'}(\mathcal{U})$ (q' is the Hölder conjugate of q), $b_3 \geqslant 0$ such that

$$\sup\{|\xi| : \xi \in f(x, u)\} \leqslant a_3(x) + b_3|u|^{q-1}, \tag{3.8}$$

for a.e. $x \in \mathcal{U}$, all $u \in \mathbb{R}$.

Solutions of (3.4)–(3.5) are defined as follows.

Definition 3.5. A function $u \in W$ is a solution of (3.4)–(3.5) if there exists a function $\eta \in L^0(\mathcal{U})$ such that

$$\eta(x) \in f(x, u(x)) \text{ for a.e. } x \in \mathcal{U}, \tag{3.9}$$

$$\eta v \in L^1(\mathcal{U}) \text{ for all } v \in Z, \tag{3.10}$$

and

$$J(v) - J(u) + \int_{\mathcal{U}} \eta(v - u)dx \geqslant 0, \ \forall v \in W, \tag{3.11}$$

or equivalently,

$$(J + I_W)(v) - (J + I_W)(u) + \int_{\mathcal{U}} \eta(v - u)dx \geqslant 0, \ \forall v \in Z, \tag{3.12}$$

where I_W is the indicator function of W: $I_W(v) = 0$ if $v \in W$ and $I_W(v) = \infty$ if $v \notin W$.

Let u be any measurable function on $\mathcal{U}$. From (F1), the function $f(\cdot, u(\cdot))$, $x \mapsto f(x, u(x))$, is also a measurable function from $\mathcal{U}$ to $\mathcal{K}(\mathbb{R})$. Let $\widetilde{f}(u)$ be the set of all measurable selections of $f(\cdot, u(\cdot))$, that is,

$$\widetilde{f}(u) = \{\eta : \mathcal{U} \to \mathbb{R} : \eta \text{ is measurable on } \mathcal{U} \text{ and } \eta(x) \in f(x, u(x)) \text{ for a.e. } x \in \mathcal{U}\}. \tag{3.13}$$

We know that $\widetilde{f}(u) \neq \emptyset$ whenever u is measurable on $\mathcal{U}$ since $f(\cdot, u(\cdot))$ is measurable.

For $1 \leqslant q < N/(N-1)$, we have from Proposition 3.3 that i_q is compact. Therefore its adjoint i_q^*, which is the projection from $L^{q'}(\mathcal{U}) \equiv [L^q(\mathcal{U})]^*$ to $Z^* = [BV(\mathcal{U})]^* = Y^{**}$, is also compact. Note that $i_q(u) = u$ for $u \in Z$, that is, $i_q(u)(x) = u(x)$ for a.e. $x \in \mathcal{U}$. Thus, to simplify the notation in the sequel, we shall use in many places u instead of $i_q(u)$. Similarly, i_q^* is the restriction of elements in $L^{q'}(\mathcal{U}) \equiv [L^q(\mathcal{U})]^*$ on the functions in $BV(\mathcal{U})$, i.e., for $\eta \in L^{q'}(\mathcal{U})$, $i_q^*(\eta) = \eta|_Z$,

$$\langle i_q^*(\eta), v\rangle_{Z^*,Z} = \langle \eta, i_q(v)\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} = \int_{\mathcal{U}} \eta v dx, \forall v \in Z. \tag{3.14}$$

Therefore, if f satisfies the growth condition (3.8) in (F3), then for any η satisfying (3.9), we have $\eta \in L^{q'}(\mathcal{U})$, which implies (3.10).

Moreover, if the growth condition (3.8) is fulfilled then $\widetilde{f}(u) \subset L^{q'}(\mathcal{U})$ whenever $u \in L^q(\mathcal{U})$. In this case, problem (3.9)–(3.10)–(3.11) can also be stated equivalently as follows: Find $u \in W$ and $\eta \in (\widetilde{f} i_q)(u)$ such that

$$(J + I_W)(v) - (J + I_W)(u) + \langle \eta, v - u\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} \geqslant 0, \ \forall v \in Z. \tag{3.15}$$

Letting $\eta^* = i_q^*(\eta)$, we can reformulate problem (3.9)–(3.10)–(3.11) as follows:

(P) Find $u \in W$ and $\eta^* \in (i_q^* \widetilde{f} i_q)(u)$ such that

$$J(v) - J(u) + \langle \eta^*, v - u \rangle_{Z^*,Z} \geqslant 0,\ \forall v \in W, \tag{3.16}$$

or equivalently,

$$(J + I_W)(v) - (J + I_W)(u) + \langle \eta^*, v - u \rangle_{Z^*,Z} \geqslant 0,\ \forall v \in Z. \tag{3.17}$$

3.2.2. We study in this section properties of the mappings involved in problem (P), and show that problem (P) fits the abstract framework set up in the previous sections. Let us start with some properties of the mappings $u \mapsto \widetilde{f}(u)$ and $u \mapsto (i_q^* \widetilde{f} i_q)(u)$.

Lemma 3.6. *Suppose conditions (F1)-(F2)-(F3).*

(a) If $u \in L^q(\mathcal{U})$ then, $\widetilde{f}(u)$ is a bounded, closed, and convex subset of $L^{q'}(\mathcal{U})$; in particular, $\widetilde{f}(u) \in \mathcal{K}(L^{q'}(\mathcal{U}))$. Moreover, $\widetilde{f} : u \mapsto \widetilde{f}(u)$ is a bounded mapping from $L^q(\mathcal{U})$ to $\mathcal{K}(L^{q'}(\mathcal{U}))$.

(b) If $u \in Z$ then $\mathcal{F}(u) := i_q^* \widetilde{f} i_q(u)$ is a convex and weak-compact subset of Z^*. Moreover, $\mathcal{F} : u \mapsto \mathcal{F}(u)$ is a bounded mapping from Z to 2^{Z^*}.*

Proof. (a) The convexity of $\widetilde{f}(u)$ follows directly from the fact that $f(x,u)$ is a closed interval. Let $u \in L^q(\mathcal{U})$ and $\eta \in \widetilde{f}(u)$. From (3.8),

$$|\eta(x)| \leqslant a_3(x) + b_3|u(x)|^{q-1}, \quad \text{a.e. } x \in \mathcal{U}. \tag{3.18}$$

Since $|u|^{q-1} \in L^{q'}(\mathcal{U})$ due to $u \in L^q(\mathcal{U})$, we have the boundedness of $\widetilde{f}(u)$ in $L^{q'}(\mathcal{U})$. Inequality (3.18) also proves that if W is a bounded set in $L^q(\mathcal{U})$ then $\widetilde{f}(W) = \bigcup_{u\in W} \widetilde{f}(u)$ is a bounded set in $L^{q'}(\mathcal{U})$, that is, $\widetilde{f}$ is a bounded mapping from $L^q(\mathcal{U})$ to $2^{L^{q'}(\mathcal{U})}$.

To verify that $\widetilde{f}(u)$ is closed in $L^{q'}(\mathcal{U})$, let $\{\eta_n\}$ be a sequence in $\widetilde{f}(u)$ such that $\eta_n \to \eta$ in $L^{q'}(\mathcal{U})$. By passing to a subsequence if necessary, we can assume that $\eta_n(x) \to \eta(x)$ for a.e. $x \in \mathcal{U}$. Since $\eta_n(x) \in f(x,u(x))$ for a.e. $x \in \mathcal{U}$, all $n \in \mathbb{N}$, and $f(x,u(x))$ is closed in $\mathbb{R}$, we have $\eta(x) \in f(x,u(x))$. Thus $\eta \in \widetilde{f}(u)$, which proves the closedness of $\widetilde{f}(u)$ in $L^{q'}(\mathcal{U})$.

(b) Since $L^{q'}(\mathcal{U})$ is reflexive, we see from (a) that for any $u \in L^q(\mathcal{U})$, $\widetilde{f}(u)$ is a convex and weak and thus weak* compact subset of $L^{q'}(\mathcal{U})$. On the other hand, the mapping i_q^* is continuous from $L^{q'}(\mathcal{U}) = [L^q(\mathcal{U})]^*$ to Z^* both equipped with the norm topologies. Therefore, i_q^* is also continuous with both $L^{q'}(\mathcal{U})$ and Z^* both equipped with the weak* topologies. Let $u \in Z$. Since the set $\widetilde{f}(u) = \widetilde{f}(i_q u)$ is convex and weak* compact in $L^{q'}(\mathcal{U})$, it follows that the set $\mathcal{F}(u) = (i_q^* \widetilde{f} i_q)(u) = i_q^*[\widetilde{f}(u)]$ is convex and weak* compact in Z^*. Lastly, the boundedness of $\mathcal{F}$ follows directly from that of $\widetilde{f}$. □

We also need the following property of $\widetilde{f}(u)$.

Lemma 3.7 (*Theorem 7.26, [25]*).

Under assumptions (F1)-(F2)-(F4), $\widetilde{f}$ is Hausdorff upper semicontinuous (h-u.s.c.) from $L^q(\mathcal{U})$ to $\mathcal{K}(L^{q'}(\mathcal{U}))$, that is, for each $u_0 \in L^q(\mathcal{U})$, the function

$$u \mapsto h^*_{L^{q'}(\mathcal{U})}(\widetilde{f}(u), \widetilde{f}(u_0)) \tag{3.19}$$

is continuous at u_0, where

$$h^*_{L^{q'}(\mathcal{U})}(A, B) = \sup_{u\in A} \left(\inf_{v\in B} \|u - v\|_{L^{q'}(\mathcal{U})} \right), \tag{3.20}$$

for $A, B \subset L^{q'}(\mathcal{U})$.

The following result on a property of $\mathcal{F}$ is essential for later developments. As above, we use the notation $Z = BV(\mathcal{U}) = Y^*$ and thus $Z^* = [BV(\mathcal{U})]^* = Y^{**}$.

Lemma 3.8. *If f satisfies condition (F1)-(F2)-(F3) then the mapping $\mathcal{F} = i_q^* \widetilde{f} i_q$ is weakly* closed from Z to 2^{Z^*} in the following sense.*

Let $\{(u_i, u_i^*)\}_{i\in I}$ be a net in $Z \times Z^* = Y^* \times Y^{**}$ that satisfies the following conditions:

$$u_i^* \in \mathcal{F}(u_i),\ \forall i \in I, \tag{3.21}$$

$$u_i \to u_0 \ \text{ in } Z \text{ with respect to the weak* topology } \sigma(Z,Y) = \sigma(Y^*,Y), \tag{3.22}$$

and

$$u_i^* \to u_0^* \ \text{ in } Z^* \text{ with respect to the weak* topology } \sigma(Z^*,Z) = \sigma(Y^{**},Y^*). \tag{3.23}$$

Then

$$u_0^* \in \mathcal{F}(u_0), \tag{3.24}$$

and

$$\langle u_i^*, u_i \rangle_{Z^*,Z} \to \langle u_0^*, u_0 \rangle_{Z^*,Z}. \tag{3.25}$$

Proof. Assume (3.21)–(3.22). As noted in (3.14), $i_q(z) = z$ for $z \in Z$ and $i_q^*(\eta) = \eta|_{BV(\mathcal{U})}$ for $\eta \in L^{q'}(\mathcal{U})$. From (3.21), for each i, there exists $\eta_i \in \widetilde{f} i_q(u_i) = \widetilde{f}(u_i) \subset L^{q'}(\mathcal{U})$ such that $u_i^* = i_q^*(\eta_i) = \eta_i|_{BV(\mathcal{U})}$. From (3.22) and the compactness of i_q, we have

$$u_i = i_q(u_i) \to i_q(u_0) = u_0 \ \text{ (strongly) in } \ L^q(\mathcal{U}). \tag{3.26}$$

Hence, from the h-upper semicontinuity of $\widetilde{f}$ from $L^q(\mathcal{U})$ to $\mathcal{K}(L^{q'}(\mathcal{U}))$ (cf. Lemma 3.7), we have

$$h^*(\widetilde{f}(u_i), \widetilde{f}(u_0)) \to 0, \tag{3.27}$$

where h^* is given in (3.20). Since $\eta_i \in \widetilde{f}(u_i)$,

$$\inf_{v \in \widetilde{f}(u_0)} \|\eta_i - v\|_{L^{q'}(\mathcal{U})} \leqslant h^*(\widetilde{f}(u_i), \widetilde{f}(u_0)).$$

Hence, $\inf_{v\in \widetilde{f}(u_0)} \|\eta_i - v\|_{L^{q'}(\mathcal{U})} \to 0$, and there exists a net $\{\widetilde{\eta}_i\} \subset \widetilde{f}(u_0)$ such that

$$\lim \|\eta_i - \widetilde{\eta}_i\|_{L^{q'}(\mathcal{U})} = 0. \tag{3.28}$$

This implies that

$$\lim \|u_i^* - i_q^*(\widetilde{\eta}_i)\|_{Z^*} = \lim \|i_q^*(\eta_i) - i_q^*(\widetilde{\eta}_i)\|_{Z^*} = 0. \tag{3.29}$$

On the other hand, we have $\{\widetilde{\eta}_i\}$ is a net in $\widetilde{f}(u_0)$ and thus $\{i_q^*(\widetilde{\eta}_i)\}$ is a net in $i_q^* \widetilde{f}(u_0) = \mathcal{F}(u_0)$. From the continuity of i_q^* from $(L^{q'}(\mathcal{U}), \sigma(L^{q'}(\mathcal{U}), L^q(\mathcal{U})))$ to $(Z^*, \sigma(Z^*, Z))$, the $\sigma(L^{q'}(\mathcal{U}), L^q(\mathcal{U}))$-compactness of $\widetilde{f}(u_0)$ in $L^{q'}(\mathcal{U})$, and the $\sigma(Z^*, Z)$-compactness of $\mathcal{F}(u_0)$ in Z^* (cf. Lemma 3.6), by passing to a subnet if necessary, we can assume that

$$\widetilde{\eta}_i \to \widetilde{\eta}_0 \ \text{ in } (L^{q'}(\mathcal{U}), \sigma(L^{q'}(\mathcal{U}), L^q(\mathcal{U}))), \tag{3.30}$$

for some $\widetilde{\eta}_0 \in \widetilde{f}(u_0)$. Thus

$$i_q^*(\widetilde{\eta}_i) \to i_q^*(\widetilde{\eta}_0) \ \text{ in } (Z^*, \sigma(Z^*, Z)), \tag{3.31}$$

with $i_q^*(\widetilde{\eta}_0) \in i_q^*(\widetilde{f}(u_0)) = \mathcal{F}(u_0)$. Combining (3.29) and (3.31) yields

$$u_i^* \to i_q^*(\widetilde{\eta}_0) \ \text{ in } (Z^*, \sigma(Z^*, Z)). \tag{3.32}$$

In view of (3.23), we see from (3.32) that $u_0^* = i_q^*(\widetilde{\eta}_0) \in \mathcal{F}(u_0)$.

To prove (3.25), we note that

$$\begin{aligned} \langle u_i^*, u_i\rangle_{Z^*,Z} &= \langle i_q^*(\eta_i), u_i\rangle_{Z^*,Z} \\ &= \langle \eta_i, i_q(u_i)\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} = \langle \eta_i, u_i\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})}. \end{aligned} \tag{3.33}$$

On the other hand, it follows from (3.28) and (3.30) that

$$\eta_i \to \widetilde{\eta}_0 \ \text{ in } (L^{q'}(\mathcal{U}), \sigma(L^{q'}(\mathcal{U}), L^q(\mathcal{U}))). \tag{3.34}$$

Hence, from (3.33), (3.26) and (3.34),

$$\begin{aligned} \langle u_i^*, u_i\rangle_{Z^*,Z} &= \langle \eta_i, u_i\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} \\ &\to \langle \widetilde{\eta}_0, u_0\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} = \langle \widetilde{\eta}_0, i_q(u_0)\rangle_{L^{q'}(\mathcal{U}),L^q(\mathcal{U})} \\ &= \langle i_q^*(\widetilde{\eta}_0), u_0\rangle_{Z^*,Z} \\ &= \langle u_0^*, u_0\rangle_{Z^*,Z}. \end{aligned}$$

This proves (3.25). □

Combining the above results, we arrive at the following property of $\mathcal{F}$.

Theorem 3.9. *The mapping $\mathcal{F}$ from $Z = Y^*$ to $2^{Z^*} = 2^{Y^{**}}$ satisfies conditions (B1)–(B4).*

Proof. Note that Lemma 3.6 implies conditions (B1) and (B4). Condition (B3) is a direct consequence of Lemma 3.8.

To verify (B2), let E be a finite dimensional subspace of Z and let $u_0 \in E$. Suppose by contradiction that $\mathcal{F}|_E$ is not upper semicontinuous at u_0 with respect to $\sigma(Z^*, Z)$-topology on Z^* and there are a $\sigma(Z^*, Z)$-open subset D of Z^* and a net $\{u_i\}_{i\in I}$ in E such that $u_i \to u_0$, $\mathcal{F}(u_0) \subset D$, and for each $i \in I$, there exists u_i^* in $\mathcal{F}(u_i) \setminus D$. Since E is finite dimensional, the net $\{u_i\}$ is eventually bounded. Hence, by considering an eventual subset of the index set I instead of I, we can assume without loss of generality that $\{u_i\}$ is bounded. From the boundedness of $\mathcal{F}$ (cf. Lemma 3.6), we see that $\{u_i^*\}$ is also bounded and by passing to a subnet if necessary, we can assume that

$$u_i^* \to u_0^* \ \text{ in } (Z^*, \sigma(Z^*, Z)), \tag{3.35}$$

for some $u_0^* \in Z^*$. As a consequence of Lemma 3.6, $u_0^* \in \mathcal{F}(u_0) \subset D$. The limit in (3.35) implies that $u_i^* \in D$ eventually, contradicting the choice of u_i^*. This contradiction shows the upper semicontinuity of $\mathcal{F}|_E$ at u_0. □

Regarding the convex functional in (3.17), we have the following result.

Theorem 3.10. *The functional $\phi = J + I_W$ is lower semicontinuous from Z with the weak* topology $\sigma(Z,Y)$ to $\mathbb{R}\cup\{\infty\}$ with $D(\phi) = W$.*

Proof. It follows directly from the characterization of the weak* convergence in Z in Proposition 3.3 and from its definition that W is a weak* closed subspace of Z.

On the other hand, J is lower semicontinuous on Z with respect to the L^1-topology and therefore with respect to the weak* topology $\sigma(Z,Y)$ on Z (cf. Theorem 14.2, [6]) with $D(J) = Z$. Combining these facts, we immediately obtain the lower semicontinuity of $\phi = J + I_W$ on Z with respect to the weak* topology $\sigma(Z,Y)$, with $D(\phi) = W$. □

3.2.3. Existence theorems

As a result of the above discussion, we have the following basic existence result for Problem (P).

Theorem 3.11. *Let f satisfy conditions (F1)-(F3). Suppose there exists $R > 0$ such that for all $w \in S_R \cap W (:= \{v \in W : \|v\| = R\})$,*

$$\inf_{w^*\in\mathcal{F}(w)} \langle w^*, w\rangle + J(w) > 0, \tag{3.36}$$

(resp. $\inf_{w^*\in\mathcal{F}(w)} \langle w^*, w\rangle + J(w) \geqslant 0$).

Then Problem (P) has a solution u with $\|u\| < R$ (resp. $\|u\| \leqslant R$).

Proof. Theorems 3.9 and 3.10 show that the variational inequality (3.17) of Problem (P) is of the form (2.44) with $f_0 = 0$. Note that $0 \in D(\partial\phi)$ since $J \geqslant 0$ and $J(0) = 0$ and thus $0 \in \partial\phi(0)$. Therefore, Theorem 3.11 follows directly from Corollary 2.9 with the choice $u_0 = 0$. □

If some sublinear growth and/or boundedness conditions are imposed on $f(x,u)$ then the above coercivity condition holds. In fact, we have the following existence result.

Theorem 3.12. *Suppose $f = f_1 + f_2$ where f_1 and f_2 satisfy (F1)-(F2). Assume there exist $\alpha \in (0,1)$ and $A_0 > 0$ such that*

$$\sup\{|\xi| : \xi \in f_1(x,u)\} \leqslant \frac{A_0}{1+|u|^\alpha} \tag{3.37}$$

for a.e. $x \in \mathcal{U}$, all $u \in \mathbb{R}$, and f_2 satisfies (F3) and is bounded below in the following sense:

$$\inf\{\xi u : \xi \in f_2(x,u)\} \geqslant B_0, \tag{3.38}$$

for a.e. $x \in \mathcal{U}$, all $u \in \mathbb{R}$, for some $B_0 \in \mathbb{R}$. Then Problem (P) has a solution.

Proof. First, note that (3.37) implies that $\sup\{|\xi| : \xi \in f_1(x,u)\} \leqslant A_0$ for a.e. $x \in \mathcal{U}$, all $u \in \mathbb{R}$. Hence f_1 satisfies (F3) for any $1 < q < N/(N-1)$. Therefore, $f = f_1 + f_2$ also satisfies (F3). Moreover, it follows from (3.37) that there is $A_1 > 0$ such that for a.e. $x \in \mathcal{U}$, all $u \in \mathbb{R}$, all $\eta \in f_1(x,u)$,

$$|\eta u| \leqslant A_1(1 + |u|^\beta), \tag{3.39}$$

with $\beta = 1-\alpha \in (0,1)$. Hence, for $w \in Z$ and $w^* = i_q^*\widetilde{w} \in \mathcal{F}(u)$, we have $w^* = w_1^* + w_2^* = i_q^*(\widetilde{w_1} + \widetilde{w_2})$, where $\widetilde{w_j} \in \widetilde{f_j}(w)$ $(j = 1,2)$, that is, $\widetilde{w_j}(x) \in f_j(x,w(x))$ for a.e. $x \in \mathcal{U}$ $(j = 1,2)$. It follows from (3.39) that

$$\begin{aligned} |\langle w_1^*, w\rangle| &= \left|\int_{\mathcal{U}} \widetilde{w_1} w\,dx\right| \\ &\leqslant \int_{\mathcal{U}} |\widetilde{w_1} w|\,dx \\ &\leqslant A_1\left(|\mathcal{U}| + \int_{\mathcal{U}} |w|^\beta dx\right) \\ &\leqslant A_2(1 + \|w\|^\beta_{L^1(\mathcal{U})}) \\ &\leqslant A_3(1 + \|w\|^\beta_Z), \end{aligned}$$

for some constant $A_3 > 0$. Moreover, we have from (3.38) that

$$\langle w_2^*, w\rangle = \int_{\mathcal{U}} \widetilde{w_2} w\,dx \geqslant B_0|\mathcal{U}|.$$

We also note from Propositions 3.4 and 3.3 that there is $B_1 > 0$ such that

$$J(w) \geqslant \int_{\mathcal{U}} |\nabla w| - |\mathcal{U}| \geqslant B_1\|w\|_Z - |\mathcal{U}|, \ \forall w \in W.$$

Combining these estimates, we get, for $w \in W$ and $w^* \in \mathcal{F}(w)$,

$$\langle w^*, w\rangle + J(w) \geqslant \|w\|_Z(B_1 - A_3\|u\|_Z^{\beta-1}) - (|\mathcal{U}| + A_3 + |B_0||\mathcal{U}|). \tag{3.40}$$

Since $\beta < 1$, we see that the coercivity condition (3.36) is satisfied for R sufficiently large. Our conclusion is a direct consequence of Theorem 3.11. □

Remark 3.13. Note that if $f_2(x,\cdot)$ is monotone with $0 \in f_2(x,0)$ for a.e. $x \in \mathcal{U}$, then (3.38) immediately holds with the choice $B_0 = 0$.

3.3. Noncoercive case - A sub-supersolution method

As with problems containing single-valued mappings, without a coercivity condition such as (3.36), problem (P) may not have solutions. In the noncoercive case, if sub- and supersolutions of problem (P), in some appropriate sense, exist, then by imposing only a local growth condition on f and applying the existence Theorem 3.11 on an appropriately regularized variational inequality, we obtain once again the solvability of problem (P).

As usual, for $u, v \in L^0(\mathcal{U})$ and $A, B \subset L^0(\mathcal{U})$, we denote

$$\begin{aligned} &u \vee v = \max\{u,v\},\ u \wedge v = \min\{u,v\} \\ &A * B = \{u * v : u \in A, v \in B\},\ u * A = \{u\} * A, \quad \text{with} \ \ * \in \{\vee,\wedge\}. \end{aligned} \tag{3.41}$$

It is known that $BV(\mathcal{U})$ is closed with respect to the operations $\vee$ and $\wedge$, that is,

$$u, v \in BV(\mathcal{U}) \Rightarrow u \wedge v, u \vee v \in BV(\mathcal{U})$$

(cf. e.g. [26,27]). As a consequence, W is also closed with respect to $\vee$ and $\wedge$. We consider on $L^0(\mathcal{U})$ (and thus on $BV(\mathcal{U})$ and W) the usual partial ordering:

$$u \leqslant v \ \text{ if and only if } \ u(x) \leqslant v(x) \ \text{ for a.e. } x \in \mathcal{U}.$$

We propose the following concepts of sub- and supersolutions for the inequality (3.11) of Problem (P).

Definition 3.14. A function $\underline{u} \in Z = BV(\mathcal{U})$ is called a subsolution of (3.4)–(3.5) (or of Problem (P)) if

(i) $\underline{u} \leqslant 0$ a.e. on $\mathcal{U} \setminus \Omega$, (3.42)

and there exists $\underline{\eta} \in \widetilde{f}(\underline{u})$ such that

(ii) $\underline{\eta} v \in L^1(\mathcal{U}),\ \forall v \in Z$, (3.43)

and

(iii) $J(v) - J(\underline{u}) + \int_{\mathcal{U}} \underline{\eta}(v - \underline{u})dx \geqslant 0,\ \forall v \in \underline{u} \wedge W.$ (3.44)

Similarly, a function $\overline{u} \in Z$ is called a supersolution of (3.4)–(3.5) (or of Problem (P)) if

(i) $\overline{u} \geqslant 0$ a.e. on $\mathcal{U} \setminus \Omega$, (3.45)

and there exists $\overline{\eta} \in \widetilde{f}(\overline{u})$ such that

(ii) $\overline{\eta} v \in L^1(\mathcal{U}),\ \forall v \in Z$, (3.46)

and

(iii) $J(v) - J(\overline{u}) + \int_{\mathcal{U}} \overline{\eta}(v - \overline{u})dx \geqslant 0,\ \forall v \in \overline{u} \vee W.$ (3.47)

By combining the existence results and properties of the mapping $\mathcal{F}$ and the functionals J and ϕ established in Section 3.2 with the sub-supersolution approach for inclusions in reflexive Banach spaces, and for inequalities with single-valued mappings in BV spaces (cf. e.g. [11,28]), we have the following existence and enclosure results for solutions of (3.4)–(3.5).

Theorem 3.15. *Assume there is a pair of subsolution $\underline{u}$ and supersolution $\overline{u}$ of (3.4)–(3.5) such that*

$$\underline{u} \leqslant \overline{u} \ \text{ a.e. on } \Omega \tag{3.48}$$

and that f has the following growth condition between $\underline{u}$ and $\overline{u}$: There exists a function $h \in L^{q'}(\mathcal{U})$ such that

$$\sup\{|\eta| : \eta \in f(x,u)\} \leqslant h(x) \tag{3.49}$$

for a.e. $x \in \mathcal{U}$, for all $u \in [\underline{u}(x), \overline{u}(x)]$.

Then, there exists a solution u of (3.4)–(3.5) such that

$$\underline{u} \leqslant u \leqslant \overline{u} \ \text{ on } \ \mathcal{U}. \tag{3.50}$$

Theorem 3.15 can also be generalized to the enclosure of solutions of (3.4)–(3.5) between several subsolutions and supersolutions. In fact, we have the following result:

Theorem 3.16. *Assume $\underline{u}_1, \dots, \underline{u}_k$ are subsolutions and $\bar{u}_1, \dots, \bar{u}_m$ are supersolutions of (3.4)–(3.5) such that*

$$\underline{u} := \max\{\underline{u}_1, \dots, \underline{u}_k\} \leqslant \bar{u} := \min\{\bar{u}_1, \dots, \bar{u}_m\},$$

a.e. on Ω and f has the growth condition (3.49) for a.e. $x \in \mathcal{U}$, all

$$u \in [\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\}, \max\{\bar{u}_1(x), \dots, \bar{u}_m(x)\}].$$

Then, there exists a solution u of (3.4)–(3.5) that satisfies (3.50).

Suppose that (3.4)–(3.5) has a pair of sub- and supersolutions and that the assumptions of Theorem 3.15 are satisfied. Let $\mathcal{S}$ be the set of solutions of (3.4)–(3.5) within the interval $[\underline{u}, \bar{u}]$, where $[\underline{u}, \bar{u}] = \{u \in W : \underline{u} \leqslant u \leqslant \bar{u}\}$. As a consequence of the above results, we can prove the following theorem.

Theorem 3.17. *$\mathcal{S}$ has the greatest element u^* and smallest element u_* with respect to the partial ordering "$\leqslant$" on W, that is, $u^*, u_* \in \mathcal{S}$ and $u_* \leqslant u \leqslant u^*$ for every $u \in \mathcal{S}$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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