01 Jun 2023

Vallée-Poussin Theorem For Equations With Caputo Fractional Derivative

Martin Bohner
Missouri University of Science and Technology, bohner@mst.edu

Alexander Domoshnitsky

Seshadev Padhi

Satyam Narayan Srivastava

Recommended Citation

M. Bohner et al., "Vallée-Poussin Theorem For Equations With Caputo Fractional Derivative," *Mathematica Slovaca*, vol. 73, no. 3, pp. 713 - 728, De Gruyter, Jun 2023.
The definitive version is available at https://doi.org/10.1515/ms-2023-0052

DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2023-0052
Math. Slovaca **73** (2023), No. 3, 713–728

VALLÉE-POUSSIN THEOREM FOR EQUATIONS WITH CAPUTO FRACTIONAL DERIVATIVE

Martin Bohner* — Alexander Domoshnitsky**, c — Seshadev Padhi*** — Satyam Narayan Srivastava**

(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper, the functional differential equation

$$({}^{C}D^{\alpha}_{a+}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = f(t), \quad t \in [a,b],$$

with Caputo fractional derivative ${}^{C}D^{\alpha}_{a+}$ is studied. The operators T_i act from the space of continuous to the space of essentially bounded functions. They can be operators with deviations (delayed and advanced), integral operators and their various linear combinations and superpositions. Such equations could appear in various applications and in the study of systems of, for example, two fractional differential equations, when one of the components can be presented from the first equation and substituted then to another. For two-point problems with this equation, assertions about negativity of Green's functions and their derivatives with respect to t are obtained. Our technique is based on an analog of the Vallée-Poussin theorem for differential inequalities, which is proven in our paper and gives necessary and sufficient conditions of negativity of Green's functions and their derivatives for two-point problems: there exists a positive function v satisfying corresponding boundary conditions and the inequality $({}^{C}D^{\alpha}_{a+}v)(t) + \sum\limits_{i=0}^{m}(T_i v^{(i)})(t) < 0$, $t \in [a,b]$. Choosing the function v, we obtain explicit sufficient tests of sign-constancy of Green's functions and its derivatives. It is demonstrated that these tests cannot be improved in a general case. Influences of delays on these sufficient conditions are analyzed. It is demonstrated that the tests can be essentially improved for "small" deviations.

1. Introduction

In this paper, we consider the fractional functional differential equation

$$({}^{C}D^{\alpha}_{a+}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = f(t), \qquad t \in [a,b], \tag{1.1}$$

where ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative of the order $n-1 < \alpha \le n$ (see Definition 2 below), the operators $T_i \colon C \to L_\infty$ are linear continuous operators acting from the space of the continuous functions $x \colon [a,b] \to \mathbb{R}$ to the space of essentially bounded functions L_∞, $i = 0, \dots, m$, and $f \in L_\infty$.

We consider the boundary value problem consisting of equation (1.1) and the boundary conditions

$$x^{(i)}(a) = x^{(k)}(b) = 0, \qquad 0 \le i \le n-1 \ \text{ and } \ i \ne k, \tag{1.2}$$

2020 Mathematics Subject Classification: 34K10, 34K37, 34K38.
Keywords: Vallée-Poussin theorem, fractional differential equations, Caputo derivative, boundary value problems, positive solutions, sign constancy of Green's function, differential inequality.

c Corresponding author.

where k is an integer which is between 1 and $n-1$. In the case of $k \leq n-1$, we have the sort of focal problems. For our positivity studies below, we assume that $m \leq k$. We consider problem (1.1),(1.2) in the space D of functions $x\colon [a,b] \to \mathbb{R}$ such that $x^{(n-1)}$ is absolutely continuous and $x^{(n)}$ is essentially bounded. The norm in the space D is defined as $\|x\|_D = \sum_{i=0}^{n-1} \max_{a\leq t\leq b} \left|x^{(i)}(t)\right| + \int_a^b |x^{(n)}(t)|\mathrm{d}t$. Considering this space D looks natural when fractional equations with Caputo derivatives and the boundary conditions (1.2) are taken into account. We say that $x \in D$ is a solution of (1.1) if it satisfies this equation for almost every $t \in [a,b]$. If the problem consisting of the homogeneous equation $({}^C D^{\alpha}_{0+}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = 0$ and conditions (1.2) has only the trivial solution, then problem (1.1),(1.2) has a unique solution, which can be represented in the form [8]

$$x(t) = \int_a^b G(t,s)f(s)\mathrm{d}s. \tag{1.3}$$

Fractional differential equations with Caputo derivatives have a wide range of applications in analyzing and modeling real-world phenomena. To note some of the applications in different fields, one can refer to applications in mechanics [21], economics [20, 24], fluid models [17], dynamic systems [23], intelligent control [26] and neuroscience [9, 28]. Let us note some of the works that inspired us to study problem (1.1),(1.2).

Agarwal et al. [3] obtained tests for existence of several positive solutions of the fractional singular boundary value problem

$$\begin{cases} ({}^c D^{\alpha}_{0+}x)(t) + \lambda f(t,x(t)) = 0, & 0 < t < 1, \\ x^{(i)}(0) = x''(1) = 0, & 0 \leq i \leq n-1,\ i \neq 2, \end{cases} \tag{1.4}$$

where $\alpha \in (n-1, n]$, $n \geq 4$, λ is a positive number and $f\colon (0,1)\times(0,+\infty) \to \mathbb{R}$ is continuous such that $\lim_{x\to 0^+} f(t,x) = \infty$ for $0 < t < 1$.

Tian and Liu [25] constructed a special cone and, using approximate method and fixed point index theory, obtained the existence of multiple positive solutions and the nonexistence of a positive solution for boundary value problem (1.4).

Inspired by the work [3] and [25], Sun and Zhang [22] used a fixed point theorem of cone expansion and compression of functional type due to Avery et al. [4] to obtain sufficient conditions for the existence of positive solutions and discussed the nonexistence of positive solutions for the fractional boundary value problem

$$\begin{cases} ({}^c D^{\alpha}_{0+}x)(t) + f(t,x(t)) = 0, & 0 < t < 1, \\ x^{(i)}(0) = x''(1) = 0, & 0 \leq i \leq n-1,\ i \neq 2, \end{cases} \tag{1.5}$$

where $\alpha \in (n-1, n]$, $n \geq 4$ and $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ is continuous.

Cabera et al. [11] studied the similar fractional two-point boundary value problem

$$\begin{cases} ({}^c D^{\alpha}_{a}x)(t) + q(t)x(t) = 0, & a < t < b,\ \alpha \in (n-1,n],\ n \geq 4, \\ x^{(i)}(a) = x''(b) = 0, & 0 \leq i \leq n-1,\ i \neq 2, \end{cases} \tag{1.6}$$

and obtained some Lyapunov-type inequalities and lower bound for the eigenvalues of the corresponding problem.

The work by Goodrich [13], where the author studied existence of a positive solution to a boundary value problem using the cone technique, should also be noted. In [13], the author considered the boundary value problem (1.1) in the case where the derivative is of Riemann–Liouville type, the operators T_i are all identically zero, the forcing term f is actually much more general, and the boundary conditions considered are

$$y^{(i)} = 0, \quad 0 \le i \le n-2, \qquad [D^{\beta}_{0+} y(t)]_{t=1} = 0, \quad 1 \le \beta \le n-2.$$

In some ways, our manuscript generalizes [13], and in other ways, complements [13].

In another work, Agarwal et al. [2] constructed monotone successive approximation for solutions to intial value problems for a scalar nonlinear Caputo differential equation with non-instaneous impulses. In [18], numerical solutions of Caputo fractional differential equations are discussed, and the authors show a new result, where the Riemann–Liouville derivative with the lower limit at infinity is related with a Caputo derivative with the lower limit at a finite real value allowing the infinite memory effect of fractional calculus to be adequately dealt with. Lan and Lin [16] obtained results on existence of multiple positive solutions of a system of Caputo fractional differential equations of the form

$$({}^cD^q_{0+} z_i)(t) = f_i(t, z(t)) \qquad \text{for} \;\; t \in [0,1] \;\; \text{and} \;\; i = 1, \dots, n,$$

subject to some of the general separated boundary conditions

$$\alpha z_i(0) - \beta z_i'(0) = 0, \qquad \gamma z_i(1) + \delta z_i'(1) = 0,$$

where $z(t) = (z_1(t), \dots, z_n(t))$, $q \in (1,2)$. The parameters α, β, γ, δ are positive real numbers.

Recently, Wang and Zhang [27], using a fixed point theorem, obtained the existence and nonexistence of positive solutions for the boundary value problem

$$({}^cD^{\alpha}_a v)(x) + \varphi(x, v(x)) = 0, \qquad 0 \le x \le 1,$$
$$v(0) = \cdots = v^{(n-1)}(0) = v^{(n+1)}(0) = 0,$$
$$pv(1) + qv'(1) = \mu \int_0^{\xi} v(t)\mathrm{d}t,$$

where $n+1 \le \alpha < n+2$, $0 < \mu < n+1$, $n \in \mathbb{N}$, $p \ge 0$, $q \ge 0$, $p+q \ne 0$, $0 \le \xi \le 1$, $0 \le \mu\xi^{\alpha} \le p\alpha + \alpha(\alpha-1)q$, $\varphi\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Lyapunov-type inequalities for fractional equations with Caputo and Riemann–Liouville fractional derivatives can be found in [1]. Motivated by these works, we study the Caputo fractional boundary value problem (1.1),(1.2). Our boundary condition (1.2) is a more general case in comparison to the problem studied by Agarwal et al. [3], Tian and Liu [25], Sun and Zhang [22], Cabera et al. [11]. This allows us to consider also the cases when $k \ne 2$. Our main development is considering fractional functional differential equations with Caputo derivatives and sufficiently general operators $T_i\colon C \to L_\infty$ and obtaining analogs of the Vallée-Poussin theorem for them.

The Vallée-Poussin theorem for differential inequalities [12] plays an important role in the analysis of oscillation/nonoscillation properties of solutions, solvability and positivity of solutions to boundary value problems for ordinary differential equations. The simple formulation of this theorem is as follows: if there exists a positive function v such that $v''(t) + p(t)v(t) \le 0$ for $t \in [a,b]$, then $[a,b)$ is a nonoscillation interval of the equation $x''(t) + p(t)x(t) = 0$, i.e., there are no nontrivial solutions having more than one zero on $[a,b)$. The modern formulation of the Vallée-Poussin theorem for differential inequalities in a form of five equivalences among which nonoscillation, negativity of Green's function of the classical two-point problem, positivity of the Cauchy function and differential inequality for the equation $x''(t) + q(t)x'(t) + p(t)x(t) = 0$ can be found in [5].

The Vallée-Poussin theorem generally speaking is not valid for functional differential equations, for example, delay and advance differential equations. For n-th order functional differential equations, an analog of the Vallée-Poussin theorem and results on sign-constancy of Green's functions on its basis were obtained in [6, 7, 10]. The Vallée-Poussin theorem generally speaking is not valid for fractional functional differential equations. We propose below an analog of the Vallée-Poussin theorem in the form of a theorem about three equivalences, connecting an assertion for a differential inequality of the form $({}^C D^{\alpha}_{a+}v)(t) + \sum_{i=0}^{m}(T_i v^{(i)})(t) < 0$, where $v \in D$ satisfies corresponding boundary conditions, an estimate of the spectral radius of a corresponding operator and sign-constancy of Green's function $G(t,s)$ and its derivatives $\frac{\partial^j}{\partial t^j}G(t,s)$, $j = 1,\ldots,k$ of two-point problems.

On this basis, we propose necessary and sufficient conditions of the negativity of Green's functions and its derivatives in the form of the Vallée-Poussin theorem for differential inequalities and obtain new sufficient tests of these sign properties in the explicit form of simple algebraic inequalities. As far as we know, our results on sign-constancy of the derivatives $\frac{\partial^j}{\partial t^j}G(t,s)$ of Green's function and positivity of solutions' derivatives are the first ones for fractional functional differential equations with Caputo derivatives and such general operators T_i. We demonstrate influence of delays on negativity of Green's function and its derivatives.

2. Preliminaries

Let us introduce several definitions and lemmas.

Definition 1 (see [14,19]). The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of an essentially bounded function $f\colon [0,1] \to \mathbb{R}$ is defined as

$$(I^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \qquad \alpha > 0,\ t \in [a,b]. \tag{2.1}$$

Definition 2 (see [14,19]). The Caputo fractional derivative of order $\alpha \in (n-1,n]$ for the function $f \in D$, $n \geq 1$, is defined by

$$({}^C D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s. \tag{2.2}$$

Lemma 2.1 (see [14,19]). *The general solution of the equation $({}^C D^{\alpha}_{a+}x)(t) = 0$ with $\alpha > 0$ and $n = [\alpha]+1$ is of the form*

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 - \cdots + c_{n-1}(t-a)^{n-1}, \tag{2.3}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

Lemma 2.2 (see [14,19]). *If $\alpha > 0$ and $n = [\alpha]+1$, then*

$$(I^{\alpha}_{a+}({}^C D^{\alpha}_{a+}x))(t) = x(t) + c_0 + c_1(t-a) + c_2(t-a)^2 - \cdots + c_{n-1}(t-a)^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

Lemma 2.3. *Assume that $\alpha \in (n-1,n]$ and $f \in L_\infty$. Then the unique solution of the fractional boundary value problem*

$$\begin{cases} ({}^C D^{\alpha}_{a+}x)(t) = f(t), & a < t < b, \\ x^{(i)}(a) = x^{(k)}(b) = 0, & 0 \leq i \leq n-1 \text{ and } i \neq k, \end{cases} \tag{2.5}$$

where k is an integer satisfying the inequality $\alpha > k+1$, where $k \geq 1$, can be represented by the formula

$$x(t) = \int_a^b G_k(t,s)f(s)\mathrm{d}s, \tag{2.6}$$

where Green's function $G_k(t,s)$ is represented as

$$G_k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -\frac{1}{k!}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^k(b-s)^{\alpha-k-1} + (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ -\frac{1}{k!}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^k(b-s)^{\alpha-k-1}, & a \le t \le s \le b. \end{cases} \tag{2.7}$$

P r o o f. By using several known formulas from [14,19], we find

$$({}^C D^\alpha_{a+}x)(t) = f(t) \quad \Longrightarrow \quad (I^\alpha_{a+}({}^C D^\alpha_{a+}x))(t) = (I^\alpha_{a+}f)(t)$$

$$\Longrightarrow x(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1} = (I^\alpha_{a+}f)(t)$$

$$\Longrightarrow x(t) = b_0 + b_1(t-a) + b_2(t-a)^2 + \cdots + b_{n-1}(t-a)^{n-1} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s,$$

where $b_i = -c_i$ $(i = 0,\dots,n-1)$ and $t \in [a,b]$. From the boundary condition $x^{(i)}(a) = 0$ for $0 \le i \le n-1$ and $i \ne k$, we obtain $b_i = 0$ for $0 \le i \le n-1$, $i \ne k$. Since $x^{(k)}(a) \ne 0$, we have $b_k \ne 0$. Consequently,

$$x(t) = b_k(t-a)^k + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \tag{2.8}$$

$$x'(t) = kb_k(t-a)^{k-1} + \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-2}f(s)\mathrm{d}s, \tag{2.9}$$

$$x''(t) = k(k-1)b_k(t-a)^{k-2} + \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-3}f(s)\mathrm{d}s, \tag{2.10}$$

$$\vdots$$

$$x^{(k)}(b) = k!\; b_k + \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-k-1}f(s)\mathrm{d}s. \tag{2.11}$$

Using the boundary condition $x^{(k)}(b) = 0$, we get

$$b_k = -\frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{k!\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-k-1}f(s)\mathrm{d}s.$$

Putting this value of b_k in (2.8), we get

$$x(t) = -\frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{k!\Gamma(\alpha)}(t-a)^k\int_a^b (b-s)^{\alpha-k-1}f(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s \tag{2.12}$$

and

$$x(t) = -\frac{(\alpha-1)(\alpha-2)\dots(\alpha-k)}{k!\Gamma(\alpha)}(t-a)^k\left[\int_a^t (b-s)^{\alpha-k-1}f(s)\mathrm{d}s + \int_t^b (b-s)^{\alpha-k-1}f(s)\mathrm{d}s\right] + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s. \tag{2.13}$$

This completes the proof. □

Lemma 2.4. *If $\alpha \in (n-1, n]$, $\alpha > k+1$, $k \geq 1$, then Green's function $G_k(t,s)$ of (2.7) satisfies the sign inequalities*

1) $G_k(t,s) < 0$ *for* $(t,s) \in (a,b) \times (a,b)$;
2) $\frac{\partial^j G_k(t,s)}{\partial t^j} < 0$, *for* $(t,s) \in (a,b) \times (a,b)$ *and* $j = 1, \dots, k$.

Proof. 1) It is clear $a < t < s < b$, that

$$G_k(t,s) = -\frac{1}{k!}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^k(b-s)^{\alpha-k-1} < 0,$$

because $\alpha > k+1$. In the case $a < s \leq t < b$, we obtain

$$\begin{aligned} G_k(t,s) &= -\frac{1}{k!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^k(b-s)^{\alpha-k-1} + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ &\leq -\frac{1}{k!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-s)^k(t-s)^{\alpha-k-1} + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ &= -\frac{1}{k!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-s)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Big[-\frac{1}{k!}(\alpha-1)(\alpha-2)\cdots(\alpha-k)+1\Big] \\ &< \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Big[-\frac{1}{k!}k(k-1)\cdots 1+1\Big] = 0, \end{aligned}$$

where we have used the facts that $\alpha > k+1$, $t-a \geq t-s$, $b-s \geq t-s$. Therefore,

$$-(t-a)^k(b-s)^{\alpha-k-1} \leq (t-s)^k(t-s)^{\alpha-k-1}.$$

2) For $a < t < s < b$ and $j \leq k$, we obtain

$$\begin{aligned} \frac{\partial^j G_k(t,s)}{\partial t^j} &= -\frac{k(k-1)(k-2)\cdots(k-j+1)}{k!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^{k-j}(b-s)^{\alpha-k-1} \\ &= -\frac{1}{(k-j)!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^{k-j}(b-s)^{\alpha-k-1} < 0. \end{aligned}$$

For $a < s \leq t < b$,

$$\begin{aligned} \frac{\partial^j G_k(t,s)}{\partial t^j} &= -\frac{k(k-1)(k-2)\cdots(k-j+1)}{k!\Gamma(\alpha)}(\alpha-1)(\alpha-2)\cdots(\alpha-k)(t-a)^{k-j}(b-s)^{\alpha-k-1} \\ &\quad + \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)}(t-s)^{\alpha-j-1} \\ &\leq \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)} \\ &\quad \times \left[-\frac{(\alpha-j-1)(\alpha-j-2)\cdots(\alpha-k)(t-s)^{k-j}(t-s)^{\alpha-k-1}}{(k-j)!} + (t-s)^{\alpha-j-1}\right] \\ &= \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)} \\ &\quad \times \left[-\frac{(\alpha-j-1)(\alpha-j-2)\cdots(\alpha-k)(t-s)^{\alpha-j-1}}{(k-j)!} + (t-s)^{\alpha-j-1}\right] \\ &< \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)}(t-s)^{\alpha-j-1}\left[-\frac{(k-j)(k-j-1)\cdots 1}{(k-j)!}+1\right] = 0, \end{aligned}$$

where we have used that $\alpha > k+1$ and $j \leq k$.

This completes the proof. □

3. Main results

Let us define the operator $K : L_\infty \to L_\infty$ by

$$(Kz)(t) = -\sum_{i=0}^{m} T_i\left[\int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s)z(s)\mathrm{d}s\right](t) = f(t). \tag{3.1}$$

We use here and below the notation $T_i[\gamma(t)]$ meaning that the operator T_i acts on the continuous function γ, i.e., $T_i[\gamma(t)] = (T_i\gamma)(t)$. We assume the positivity of operators in the standard sense, i.e., the operator K is positive if $(Kz)(t) \geq 0$ for $t \in [a,b]$ for every $z \in L_\infty$ such that $z(t) \geq 0$ for $t \in [a,b]$.

Theorem 3.1. *Let $T_i\colon C \to L_\infty$, $i = 0,1,\ldots,m$, be positive operators, $k \geq m$, $n-1 < \alpha \leq n$, $n \geq k+2$. Then the following assertions are equivalent:*

1) *there exist a positive number ε and a function $v \in D$ such that $v(t) > 0$, $v'(t) > 0,\ldots, v^{(k)}(t) > 0$ for $t \in (a,b)$, $v^{(i)}(a) = 0$ for $0 \leq i \leq n-1$, $i \neq k$, and*

$$({}^C D^\alpha_{a+}v)(t) + \sum_{i=0}^{m}(T_i v^{(i)})(t) \equiv \psi(t) \leq -\varepsilon < 0 \quad \textit{for} \quad t \in (a,b); \tag{3.2}$$

2) *the spectral radius $\rho(K)$ of the operator K is less than 1;*
3) *problem (1.1),(1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t,s)$ and the derivatives $\frac{\partial^j G(t,s)}{\partial t^j}$ satisfy the inequalities $G(t,s) < 0$ for $(t,s) \in (a,b)\times(a,b)$ and $\frac{\partial^j G(t,s)}{\partial t^j} < 0$, $j = 1,\ldots,k-1$, for $(t,s) \in (a,b)\times(a,b)$, and $\frac{\partial^k G(t,s)}{\partial t^k} < 0$ for $a < t < s < b$ and $\frac{\partial^k G(t,s)}{\partial t^k} \leq 0$ for $a < s < t < b$.*

P r o o f. 1) $\Rightarrow$ 2). Consider the auxiliary problem

$$\begin{cases} ({}^C D^\alpha_{a+}x)(t) = z(t), \\ x^{(i)}(a) = v^{(i)}(a), \quad x^{(k)}(b) = v^{(k)}(b), \qquad 0 \leq i \leq n-1, \quad i \neq k, \end{cases} \tag{3.3}$$

where z from L_∞ is such that there exists a positive number δ such that $z(t) \leq -\delta$ for $t \in [a,b]$. It is clear that

$$\begin{cases} x(t) = \int_a^b G_k(t,s)z(s)\mathrm{d}s + u_k(t), \\ x'(t) = \int_a^b \frac{\partial}{\partial t}G_k(t,s)z(s)\mathrm{d}s + u_k'(t), \\ x''(t) = \int_a^b \frac{\partial^2}{\partial t^2}G_k(t,s)z(s)\mathrm{d}s + u_k''(t), \\ \vdots \\ x^{(m)}(t) = \int_a^b \frac{\partial^m}{\partial t^m}G_k(t,s)z(s)\mathrm{d}s + u_k^{(m)}(t), \end{cases} \tag{3.4}$$

where u is a solution of the homogeneous equation $({}^C D^\alpha_{a+}u)(t) = 0$ satisfying the conditions $u^{(i)}(a) = v^{(i)}(a)$, $u^{(k)}(b) = v^{(k)}(b)$, $0 \leq i \leq n-1$ and $i \neq k$. Let us substitute this representation instead of v and its derivative into inequality (3.2)

$$z(t) + \sum_{i=0}^{m} T_i \left[\int_a^b \frac{\partial^i}{\partial t^i} G_k(t,s) z(s) \mathrm{d}s \right] + \sum_{i=0}^{m} (T_i u^i)(t) = \psi(t). \tag{3.5}$$

The assumptions that $T_i \colon C \to L_\infty$ are positive operators for $i = 0, 1, \ldots, m$, and Lemma 2.4 imply that the operator $K \colon L_\infty \to L_\infty$ defined by (3.1) is positive. Thus, we have the equation

$$z(t) - (Kz)(t) = \Psi(t), \qquad t \in [a,b], \tag{3.6}$$

where

$$\Psi(t) \equiv \psi(t) - \sum_{i=0}^{m} (T_i u^{(i)})(t). \tag{3.7}$$

Let us consider the function u, it is the solution of $({}^C D^\alpha_{a+} u)(t) = 0$, $u^{(i)}(a) = 0$, $u^{(k)}(b) = c > 0$, $0 \le i \le n-1$, $i \ne k$. By Lemma 2.1, we have $u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}$, $u^{(i)}(a) = 0$ and $u^{(k)}(t) = c_0 + c_1(t-a) + \cdots + c_k k(k-1) \cdots 2 \cdot 1 + c_{k+1}(k+1)k \cdots 2 \cdot 1 (t-a) + c_{n-1}(n-1)(n-2) \cdots (n-1-k)(t-a)^{n-1-k}$, we get $c_i = 0$, $i \ne k$. Thus, $u^{(k)}(b) = c_k k!$, which follows from the fact that if $u^{(k)}(t) = c_k k! > 0$ for some $c_k \in \mathbb{R}$, then $c_k > 0$. Therefore $u(t) = c_k (t-a)^{(k)} > 0$. It is clear that if $u^{(i)}(t) > 0$ for $t \in (a,b)$, then $\Psi(t) \le -\varepsilon < 0$. The function $w = -z$ satisfies the inequality $w(t) - (Kw)(t) = -\Psi(t) > \varepsilon > 0$ for $t \in [a,b]$. From (3.6), according to [15: Theorem 5.8 on p. 84] , it follows that $\rho(K) < 1$. This completes the proof of the implication 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3). Consider the problem consisting of equation (1.1) and the zero boundary conditions (1.2). Let us now use the substitution

$$x(t) = \int_a^b G_k(t,s) z(s) \mathrm{d}s, \tag{3.8}$$

where $G_k(t,s)$ is Green's function of the problem consisting of the equation

$$({}^C D^\alpha_{a+} x)(t) = z(t) \tag{3.9}$$

with boundary conditions (1.2). It is clear from (3.4) that

$$\begin{cases} x'(t) = \displaystyle\int_a^b \frac{\partial}{\partial t} G_k(t,s) z(s) \mathrm{d}s, \\ \vdots \\ x^{(m)}(t) = \displaystyle\int_a^b \frac{\partial^m}{\partial t^m} G_k(t,s) z(s) \mathrm{d}s. \end{cases} \tag{3.10}$$

Substituting representation (3.8) and (3.10) into (1.1), we get (3.6), with $\Psi(t)$ defined by (3.7), where $\psi(t) = f(t)$. It is clear that $u^{(i)}(t) \equiv 0$ for $t \in [a,b]$ and $i = 0, \ldots, m$. If $\rho(K) < 1$, then (3.6) is uniquely solvable and its solution is

$$z(t) = (I-K)^{-1} \psi(t) = (I + K + K^2 + K^3 + \ldots) \psi(t). \tag{3.11}$$

We obtain that the solution x defined by (3.8) exists and is unique, and this proves that problem (1.1),(1.2) is uniquely solvable. We see also that $(I-K)^{-1}$ is a positive operator if K is positive. The assumption about positivity of the operator $T_i \colon C \to L_\infty$ and Lemma 2.4 imply that $K \colon L_\infty \to L_\infty$ is positive. Then from $\psi(t) \le 0$, it follows that $z(t) \le 0$, for $t \in [a,b]$. Thus, if $f(t) \le 0$, then $z(t) \le 0$ for $t \in [a,b]$. If $z(t) \le 0$, then from the fact of non-positivity of Green's function $G(t,s)$ and its derivatives $\frac{\partial^j G(t,s)}{\partial t^j}$ in the formulas of (3.8) and (3.10), we get $x(t) \ge 0$, $x'(t) \ge 0, \ldots, x^{(k)}(t) \ge 0$.

It is possible only in the case when Green's function $G(t,s)$ of problem (1.1),(1.2) and its derivatives satisfy the inequalities: $G(t,s) \le 0, \dots, \frac{\partial^m G(t,s)}{\partial t^m} \le 0$ for $t,s \in (a,b)$. The strict inequalities on Green's function $G_k(t,s)$ and its derivatives $\frac{\partial^j G_k(t,s)}{\partial t^j}$ imply positivity of the solution x and its derivatives $x^{(j)}(t)$ for $t \in (a,b)$ for every nonpositive right-hand side $f(t)$ in (1.1) such that $f(t) < 0$ on a set of positive measure. This allows to conclude negativity of Green's function $G(t,s)$ and its derivatives $\frac{\partial^j G(t,s)}{\partial t^j}$ for $t,s \in (a,b)$, $j = 1,\dots,k-1$ in the case of $j < k$. For $j = k$, we have negativity of $G(t,s)$, its derivatives $\frac{\partial^j G(t,s)}{\partial t^j}$ for $j < k$ are negative for $t,s \in (a,b)$ and the derivatives $\frac{\partial^k G(t,s)}{\partial t^k}$ are negative in the triangle $a < t < s < b$ and nonpositive in the triangle $a < s < t < b$. This completes the proof of the implication 2) $\Rightarrow$ 3).

3) $\Rightarrow$ 1). To prove this implication, we set

$$v(t) = -\int_a^b G(t,s)\mathrm{d}s. \tag{3.12}$$

We have $v'(t) = -\int_a^b \frac{\partial G(t,s)}{\partial t}\mathrm{d}s, \dots, v^{(i)}(t) = -\int_0^1 \frac{\partial^i G(t,s)}{\partial t^i}$, $i = 1,\dots,k$. Since $G(t,s) < 0$ and $\frac{\partial^i G(t,s)}{\partial t^i} < 0$, $i = 0,\dots,k-1$ for $(t,s) \in (a,b)\times(a,b)$ and $\frac{\partial^k G(t,s)}{\partial t^k} < 0$ for $a < t < s < b$, $\frac{\partial^k G(t,s)}{\partial t^k} \le 0$ for $a < s < t < b$, then $v(t) > 0$, $v'(t) > 0, \dots, v^{(k)}(t) > 0$, for $t \in (a,b)$. This completes the proof of implication 3) $\Rightarrow$ 1).

The proof is now complete. □

Corollary 3.1. *If $T_j\colon C \to L_\infty$ are positive operators for $j = 1,\dots,m$, $m \le k$, $n-1 < \alpha \le n$, $n \ge k+2$ and the inequality*

$$\begin{aligned} &T_0\left[\frac{(b-a)^{\alpha-k}(t-a)^k\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} - (t-a)^\alpha\right] \\ &\quad + \sum_{i=1}^m T_i\Bigg[\frac{(b-a)^{\alpha-k}(t-a)^{k-i}k(k-1)\dots(k-i+1)\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \\ &\qquad - \alpha(\alpha-1)\cdots(\alpha-i+1)(t-a)^{\alpha-i}\Bigg] < \Gamma(\alpha+1), \end{aligned} \tag{3.13}$$

then problem (1.1),(1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t,s)$ and their derivatives $\frac{\partial^j G(t,s)}{\partial t^j}$ satisfy the inequalities $G(t,s) < 0$ for $(t,s) \in (a,b)\times(a,b)$ and $\frac{\partial^j G(t,s)}{\partial t^j} < 0$ for $(t,s) \in (a,b)\times(a,b)$, $j < k$ and $\frac{\partial^k G(t,s)}{\partial t^k} < 0$ for $a < t < s < b$, $\frac{\partial^k G(t,s)}{\partial t^k} \le 0$ for $a \le s < t < b$.

P r o o f. Consider the auxiliary equation

$$({}^C D^\alpha_{a+} v)(t) = -\delta \tag{3.14}$$

with the boundary condition

$$v^{(i)}(a) = v^{(k)}(b) = 0 \qquad \text{for } \ 0 \le i \le n-1,\ i \ne k, \tag{3.15}$$

where δ is positive constant. Lemma 2.1 allows us to write (3.14)–(3.15) as

$$v(t) = b_k(t-a)^k - \frac{\delta}{\Gamma(\alpha+1)}(t-a)^\alpha. \tag{3.16}$$

Actually, from the representation of solutions of $({}^C D^\alpha_{a+} x)(t) = f(t)$ by formula (2.8) and using the boundary condition $v^{(i)}(a) = 0$, we obtain $b_i = 0$ for $0 \le i \le n-1$, $i \ne k$. The parameter b_k stays in representation (3.16) since $v^{(k)}(a) \ne 0$. The integral term in the representation (3.16) is

$(I^{\alpha}_{a+}f)(t) = (I^{\alpha}_{a+}(-\delta))(t)$ and can be easily calculated as

$$(I^{\alpha}_{a+}(-\delta))(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(-\delta)\mathrm{d}s = -\frac{\delta}{\Gamma(\alpha+1)}(t-a)^{\alpha}. \tag{3.17}$$

We have to "connect" b_k with δ to guarantee the inequalities

$$v^{(i)}(t) > 0 \qquad \text{for} \quad 0 \le i \le n-1, \quad i \ne k. \tag{3.18}$$

In order to continue the proof, let us describe in more detail an idea of this "connection". It is clear that condition (3.18) is fulfilled for sufficiently large b_k, but in (3.16) to achieve inequality (3.2), we need sufficiently small b_k. Thus, we have to choose a minimal possible b_k such that all inequalities in (3.18) are fulfilled.

Remark 1. Since $v(t) = b_k(t-a)^k - \frac{\delta}{\Gamma(\alpha+1)}(t-a)^{\alpha}$, we get $v'(t) = kb_k(t-a)^{k-1} - \frac{\delta\alpha}{\Gamma(\alpha+1)}(t-a)^{\alpha-1}$, $v''(t) = k(k-1)b_k(t-a)^{k-2} - \frac{\delta\alpha(\alpha-2)}{\Gamma(\alpha+1)}(t-a)^{\alpha-2}$, and

$$v^{(i)}(t) = k(k-1)\cdots(k-i+1)b_k(t-a)^{k-i} - \frac{\delta\alpha(\alpha-1)\cdots(\alpha-i+1)}{\Gamma(\alpha+1)}(t-a)^{\alpha-i}.$$

For every $k \le n-1$, we obtain

$$v^{(k)}(t) = k!b_k - \frac{\delta\alpha(\alpha-1)\ldots(\alpha-k+1)}{\Gamma(\alpha+1)}(t-a)^{\alpha-k},$$

so

$$b_k = \frac{\delta\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!\Gamma(\alpha+1)}(b-a)^{\alpha-k}.$$

Let us continue the proof of Corollary 3.1. Let us substitute the function $v^{(i)}$ in assertion 1) of Theorem 3.1, putting the value of b_k using Remark 1. Taking into account that $({}^{C}D^{\alpha}_{a+}v)(t) = -\delta$, we come to inequality (3.13) as a sufficient condition of existence, uniqueness and negativity of Green's function $G(t,s)$ and its derivatives. This completes the proof. □

4. Explicit sufficient conditions of negativity of Green's functions and their derivatives

Corollary 4.1. *Let $T_0\colon C \to L_\infty$ be a positive operator, $n-1 < \alpha \le n$, $n \ge 3$, $m = 0$ in equation (1.1), $k = 1$ in condition (1.2) and the inequality*

$$T_0[(b-a)^{\alpha-1}(t-a)\alpha - (t-a)^{\alpha}] < \Gamma(\alpha+1), \qquad t \in [a,b] \tag{4.1}$$

be fulfilled. Then problem (1.1),(1.2) is uniquely solvable for any $f \in L_\infty$, and its Green's function $G(t,s)$ is negative for $(t,s) \in (a,b)\times(a,b)$ and the derivatives $\frac{\partial G(t,s)}{\partial t}$ are negative for $a < t < s < b$ and nonpositive for $a < s < t < b$.

Consider the particular case of (1.1)

$$({}^{C}D^{\alpha}_{0+}x)(t) + q_0(t)x(h_0(t)) = f(t), \qquad t \in [a,b], \tag{4.2}$$

where $1 < \alpha \le 2$, $m = 0$, $q_0(t) \ge 0$ for $t \in [a,b]$.

Corollary 4.2. *If $2 < \alpha \le 3$ and the inequality*

$$q_0(t)\left[(b-a)^{\alpha-1}(h_0(t)-a)\alpha - (h_0(t)-a)^{\alpha}\right] < \Gamma(\alpha+1), \qquad t \in (a,b), \tag{4.3}$$

is fulfilled, then (4.2) *with the boundary conditions*

$$x(a) = 0, \qquad x'(b) = 0, \tag{4.4}$$

is uniquely solvable for any $f \in L_\infty$*, and its Green's function is negative for* $(t,s) \in (a,b) \times (a,b)$ *and the derivatives* $\frac{\partial G(t,s)}{\partial t}$ *are negative for* $a < t < s < b$ *and nonpositive for* $a < s < t < b$*.*

Corollary 4.3. *If* $2 < \alpha \le 3$*,* $k = 1$*,* $h_0(t) - a < \varepsilon$ *for* $a \le t \le b$*, then the inequality*

$$q_0(t) < \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha-1}\varepsilon\alpha - \varepsilon^\alpha} \tag{4.5}$$

implies the unique solvability for any $f \in L_\infty$ *of the problem* (4.2),(4.4)*, and its Green's function is negative for* $(t,s) \in (a,b) \times (a,b)$ *and the derivatives* $\frac{\partial G(t,s)}{\partial t}$ *are negative for* $a < t < s < b$ *and nonpositive for* $a < s < t < b$*.*

***Example* 1.** Assume that in the condition (4.5) of Corollary 4.3, $a = 0$, $b = 1$, $\alpha = 2.5$, $\varepsilon = 0.1,\ 0.01,\ 0.001$. Then we have the values given in Table 1.

Table 1

ε	$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha-1}\varepsilon\alpha-\varepsilon^\alpha}$
0.1	13.46370781
0.01	132.9872337
0.001	1329.357203

Corollary 4.4. *Let* $T_0, T_1 \colon C \to L_\infty$ *be positive operators,* $n-1 < \alpha \le n$*,* $n \ge 4$*,* $m = 1$ *in equation* (1.1)*,* $k = 2$ *in condition* (1.2)*, and the inequality*

$$T_0\left[\frac{(b-a)^{\alpha-2}(t-a)^2\alpha(\alpha-1)}{2} - (t-a)^\alpha\right] + T_1\Big[\alpha(\alpha-1)(b-a)^{\alpha-2}(t-a) - \alpha(t-a)^{\alpha-1}\Big] < \Gamma(\alpha+1), \qquad t \in [a,b], \tag{4.6}$$

be fulfilled. Then problem (1.1),(1.2) *is uniquely solvable, and its Green's function* $G(t,s)$ *and the derivative* $\frac{\partial G(t,s)}{\partial t}$ *are negative for* $(t,s) \in (a,b) \times (a,b)$ *and the second derivative* $\frac{\partial^2 G(t,s)}{\partial t^2} < 0$ *for* $a < t < s < b$ *and* $\frac{\partial^2 G(t,s)}{\partial t^2} \le 0$ *for* $a < s < t < b$*.*

Consider the particular case of (1.1)

$$({}^C D_{0+}^\alpha x)(t) + q_0(t)x(h_0(t)) + q_1(t)x'(h_1(t)) = f(t), \qquad t \in [a,b], \tag{4.7}$$

where $3 < \alpha \le 4$, $m = 1$, $q_0(t), q_1(t) \ge 0$ for $t \in [a,b]$.

Corollary 4.5. *If* $3 < \alpha \le 4$ *and the inequality*

$$q_0(t)\left[\frac{(b-a)^{\alpha-2}(h_0(t)-a)^2\alpha(\alpha-1)}{2} - (h_0(t)-a)^\alpha\right] + q_1(t)\Big[\alpha(\alpha-1)(b-a)^{\alpha-2}(h_0(t)-a) - \alpha(h_0(t)-a)^{\alpha-1}\Big] < \Gamma(\alpha+1), \qquad t \in (a,b), \tag{4.8}$$

is fulfilled, then (4.7) *with the boundary conditions*

$$x(a) = x'(a) = 0, \quad x''(b) = 0, \tag{4.9}$$

is uniquely solvable for any $f \in L_\infty$*, and its Green's function* $G(t,s)$*, the derivative* $\frac{\partial G(t,s)}{\partial t}$ *are negative for* $(t,s) \in (a,b) \times (a,b)$ *and the second derivative* $\frac{\partial^2 G(t,s)}{\partial t^2} < 0$ *for* $a < t < s < b$*, and* $\frac{\partial^2 G(t,s)}{\partial t^2} \le 0$ *for* $a < s < t < b$*.*

Corollary 4.6. *If $3 < \alpha \leq 4$ and the points $(q_0(t), q_1(t))$ are situated in the triangle $q_0 = 0$, $q_1 = 0$, and*

$$\frac{(b-a)^{\alpha-2}(h_0(t)-a)^2\alpha(\alpha-1) - 2(h_0(t)-a)^\alpha}{2\Gamma(\alpha+1)} q_0 + \frac{\alpha(\alpha-1)(b-a)^{\alpha-2}(h_0(t)-a) - \alpha(h_0(t)-a)^{\alpha-1}}{\Gamma(\alpha+1)} q_1 = 1$$

for $t \in [a,b]$, then Green's function $G(t,s)$ and its derivative $\frac{\partial G(t,s)}{\partial t}$ of problem (4.7),(4.9) are negative for $(t,s) \in (a,b) \times (a,b)$, and the derivative $\frac{\partial^2 G(t,s)}{\partial^2 t}$ is negative for $a < t < s < b$ and nonpositive for $a < s < t < b$.

Corollary 4.7. *If $3 < \alpha \leq 4$, $k = 2$, $h_0(t) - a < \varepsilon$, $h_1(t) - a < \varepsilon$ for $a \leq t \leq b$, then the inequality*

$$q_0(t)\left[\frac{(b-a)^{\alpha-2}\varepsilon^2\alpha(\alpha-1)}{2} - \varepsilon^\alpha\right] + q_1(t)\Big[\alpha(\alpha-1)(b-a)^{\alpha-2}\varepsilon - \alpha\varepsilon^{\alpha-1}\Big] < \Gamma(\alpha+1) \tag{4.10}$$

implies the unique solvability of problem (4.7),(4.9), and its Green's function $G(t,s)$ and the derivative $\frac{\partial G(t,s)}{\partial t}$ are negative for $(t,s) \in (a,b) \times (a,b)$, and the second derivative $\frac{\partial^2 G(t,s)}{\partial t^2} < 0$ for $a < t < s < b$ and $\frac{\partial^2 G(t,s)}{\partial t^2} \leq 0$ for $a < s < t < b$.

***Example* 2.** Assume that in the condition (4.10) of Corollary 4.6, $a = 0$, $b = 1$, $\alpha = 3.5$ and $\varepsilon = 0.1,\ 0.01,\ 0.001$. First, for $\varepsilon = 0.1$, (4.10) becomes

$$0.04343377223\, q_0(t) + 0.8639320282 q_1(t) < 11.63172839 \quad \text{(see Figure 1)}.$$

For $\varepsilon = 0.01$, (4.10) becomes

$$4.374 \times 10^{-4} q_0(t) + 0.087465 q_1(t) < 11.63172839 \quad \text{(see Figure 2)}.$$

For $\varepsilon = 0.001$, (4.10) becomes

$$4.374968377 \times 10^{-6} q_0(t) + 8.74988932 \times 10^{-3} q_1(t) < 11.63172839 \quad \text{(see Figure 3)}.$$

Here, we can see that for $\varepsilon = 0.1$, the area of the triangle is 1802.815922, for $\varepsilon = 0.01$, the area of the triangle is 1768257.179 and for $\varepsilon = 0.001$, the area of the triangle is 1767180992. It appears that the area of triangles tends to infinity when $\varepsilon \to 0$.

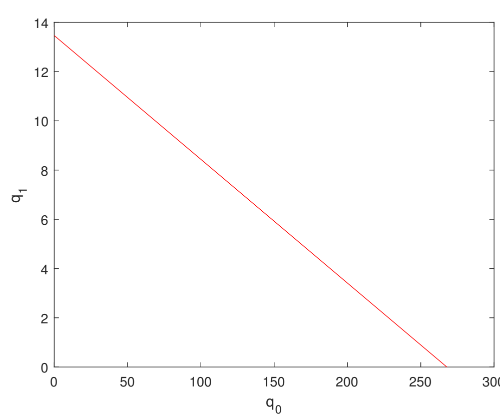

Figure 1. $\varepsilon = 0.1$ in Example 2

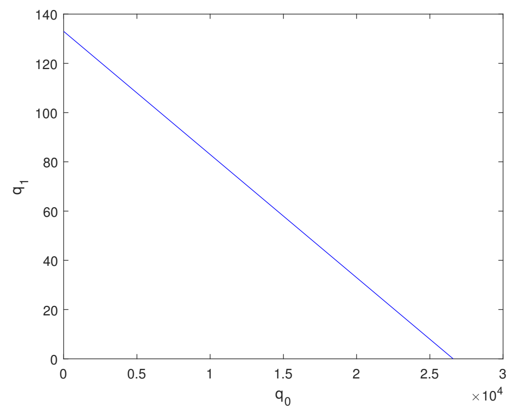

FIGURE 2. $\varepsilon = 0.01$ in Example 2

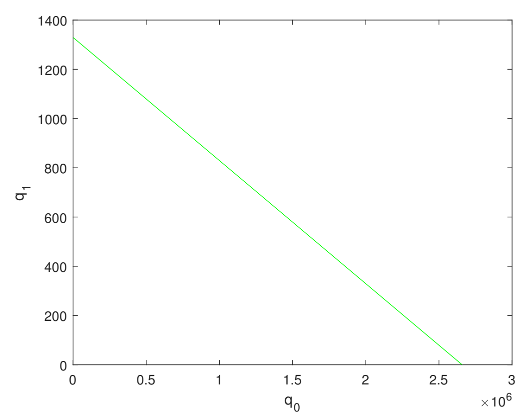

FIGURE 3. $\varepsilon = 0.001$ in Example 2

COROLLARY 4.8. *Let $T_0, T_1, T_2 \colon C \to L_\infty$ be positive operators, $n-1 < \alpha \le n$, $n \ge 5$, $m = 2$ in equation (1.1), $k = 3$ in condition (1.2) and the inequality*

$$\begin{aligned} & T_0\left[\frac{(b-a)^{\alpha-3}(t-a)^3\alpha(\alpha-1)(\alpha-2)}{6} - (t-a)^\alpha\right] \\ & + T_1\left[\frac{(b-a)^{\alpha-3}(t-a)^2\alpha(\alpha-1)(\alpha-2)}{2} - \alpha(t-a)^{\alpha-1}\right] \\ & + T_2\left[\frac{(b-a)^{\alpha-3}(t-a)\alpha(\alpha-1)(\alpha-2)}{3} - \alpha(\alpha-1)(t-a)^{\alpha-2}\right] < \Gamma(\alpha+1), \qquad t \in [0,1], \end{aligned} \tag{4.11}$$

be fulfilled. Then problem (1.1),(1.2) is uniquely solvable for any $f \in L_\infty$, and its Green's function $G(t,s)$, the derivatives $\frac{\partial G(t,s)}{\partial t}$, $\frac{\partial^2 G(t,s)}{\partial t^2}$ are negative for $(t,s) \in (a,b)\times(a,b)$ and the third derivative $\frac{\partial^3 G(t,s)}{\partial t^3} < 0$ for $a < t < s < b$ and $\frac{\partial^3 G(t,s)}{\partial t^3} \le 0$ for $a < s < t < b$.

Consider the particular case of (1.1)

$$({}^{C}D^{\alpha}_{0+}x)(t) + q_0(t)x(h_0(t)) + q_1(t)x'(h_1(t)) + q_2(t)x''(h_2(t)) = f(t), \qquad t \in [0,1], \tag{4.12}$$

where $4 < \alpha \le 5$, $m = 2$, $q_0(t), q_1(t), q_2(t) \ge 0$ for $t \in [a,b]$.

Corollary 4.9. *If $4 < \alpha \le 5$ and the inequality*

$$\begin{aligned} &q_0(t)\left[\frac{(b-a)^{\alpha-3}(h_0(t)-a)^3\alpha(\alpha-1)(\alpha-2)}{6} - (h_0(t)-a)^{\alpha}\right] \\ &+ q_1\left[\frac{(b-a)^{\alpha-3}(h_1(t)-a)^2\alpha(\alpha-1)(\alpha-2)}{2} - \alpha(h_1(t)-a)^{\alpha-1}\right] \\ &+ q_2\left[\frac{(b-a)^{\alpha-3}(h_2(t)-a)\alpha(\alpha-1)(\alpha-2)}{3} - \alpha(\alpha-1)(h_2(t)-a)^{\alpha-2}\right] < \Gamma(\alpha+1), \qquad t \in (a,b), \end{aligned} \tag{4.13}$$

is fulfilled, then problem (4.12) with the boundary conditions

$$x(a) = x'(a) = x''(a) = 0, \qquad x'''(b) = 0 \tag{4.14}$$

is uniquely solvable for any $f \in L_\infty$, and its Green's function $G(t,s)$ and the derivatives $\frac{\partial G(t,s)}{\partial t}$, $\frac{\partial^2 G(t,s)}{\partial t^2}$ are negative for $(t,s) \in (a,b)\times(a,b)$, and the third derivative $\frac{\partial^3 G(t,s)}{\partial t^3} < 0$ for $a < t < s < b$ and $\frac{\partial^3 G(t,s)}{\partial t^3} \le 0$ for $a < s < t < b$.

Corollary 4.10. *If $4 < \alpha \le 5$ and the coefficients $q_0(t)$, $q_1(t)$, $q_2(t)$ as coordinates are interior points of the pyramid $q_0 = 0$, $q_1 = 0$, $q_2 = 0$ and*

$$\begin{aligned} &\frac{(b-a)^{\alpha-3}(h_0(t)-a)^3\alpha(\alpha-1)(\alpha-2) - 6(h_0(t)-a)^{\alpha}}{6\Gamma(\alpha+1)}q_0 \\ &+ \frac{(b-a)^{\alpha-3}(h_1(t)-a)^2\alpha(\alpha-1)(\alpha-2) - 2\alpha(h_1(t)-a)^{\alpha-1}}{2\Gamma(\alpha+1)}q_1 \\ &+ \frac{(b-a)^{\alpha-3}(h_2(t)-a)\alpha(\alpha-1)(\alpha-2) - 3\alpha(\alpha-1)(h_2(t)-a)^{\alpha-2}}{3\Gamma(\alpha+1)}q_2 = 1 \end{aligned}$$

for $t \in [a,b]$, then problem (4.12),(4.14) is uniquely solvable for any $f \in L_\infty$, and its Green's function $G(t,s)$, the derivatives $\frac{\partial G(t,s)}{\partial t}$, $\frac{\partial^2 G(t,s)}{\partial t^2}$ are negative for $(t,s) \in (a,b)\times(a,b)$, and the third derivative $\frac{\partial^3 G(t,s)}{\partial t^3} < 0$ for $a < t < s < b$ and $\frac{\partial^3 G(t,s)}{\partial t^3} \le 0$ for $a < s < t < b$.

Corollary 4.11. *If $4 < \alpha \le 5$, $k = 3$, $h_0(t) - a$, $h_1(t) - a$, $h_2(t) - a < \varepsilon$ for $a \le t \le b$, then the inequality*

$$\begin{aligned} &\left[\frac{(b-a)^{\alpha-3}\epsilon^3\alpha(\alpha-1)(\alpha-2)}{6} - \epsilon^{\alpha}\right]q_0(t) \\ &+ \left[\frac{(b-a)^{\alpha-3}\epsilon^2\alpha(\alpha-1)(\alpha-2)}{2} - \alpha\epsilon^{\alpha-1}\right]q_1(t) \\ &+ \left[\frac{(b-a)^{\alpha-3}\epsilon\alpha(\alpha-1)(\alpha-2)}{3} - \alpha(\alpha-1)\epsilon^{\alpha-2}\right]q_2(t) < \Gamma(\alpha+1), \qquad t \in (a,b), \end{aligned} \tag{4.15}$$

implies the unique solvability of problem (4.12),(4.14) and negativity of its Green's function and its derivative $\frac{\partial G(t,s)}{\partial t}$, $\frac{\partial^2 G(t,s)}{\partial t^2}$ in $(t,s) \in (a,b)\times(a,b)$ and its derivative $\frac{\partial^3 G(t,s)}{\partial t^3}$ is negative for $a < t < s < b$ and nonpositive for $a < s < t < b$.

***Example* 3.** Assume that in the condition (4.15) of Corollary 4.11, $a = 0$, $b = 1$, $\alpha = 4.5$, $\varepsilon = 0.1,\ 0.01,\ 0.001$. For $\varepsilon = 0.1$, (4.15) becomes

$$6.530877223 \times 10^{-3} q_0(t) + 0.1954519751 q_1(t) + 1.262694127 q_2(t) < 52.34277778.$$

For $\varepsilon = 0.01$, (4.15) becomes

$$6.5615 \times 10^{-6} q_0(t) + 1.9683 \times 10^{-3} q_1(t) + 0.1310925 q_2(t) < 52.34277778.$$

For $\varepsilon = 0.001$, (4.15) becomes

$$6.562468377 \times 10^{-9} q_0(t) + 1.96873577 \times 10^{-5} q_1(t) + 0.01312450194 q_2(t) < 52.3427778.$$

Acknowledgement. The authors would like to thank the anonymous referee for many valuable comments and suggestions, leading to a better presentation of our results.

REFERENCES

[1] AGARWAL, R.P.—BOHNER, M.—ÖZBEKLER, A.: *Lyapunov Inequalities and Applications*, Springer, Cham, 2021.

[2] AGARWAL, R. P.—HRISTOVA, S.—O'REGAN, D.: *Iterative techniques for the initial value problem for Caputo fractional differential equations with non-instantaneous impulses*, Appl. Math. Comput. **334** (2018), 407–421.

[3] AGARWAL, R. P.—LIU, Y.—O'REGAN, D.—TIAN, C.: *Positive solutions of two-point boundary value problems for fractional singular differential equations*, Differ. Equ. **48** (2012), 619–629.

[4] AVERY, R.—HENDERSON, J.—O'REGAN, D.: *Functional compression-expansion fixed point theorem*, Electron. J. Differential Equations **2008**(22) (2008), 1–12.

[5] AZBELEV, N. V.—DOMOSHNITSKII, A. I.: *On de la Vallée Poussin's differential inequality*, Differentsial'nye Uravneniya **22**(12) (1986), 2041–2045.

[6] AZBELEV, N. V.—DOMOSHNITSKII, A. I.: *On the question of linear differential inequalities I*, Differentsial'nye Uravneniya **27**(3) (1991), 257–263.

[7] AZBELEV, N. V.—DOMOSHNITSKII, A. I.: *On the question of linear differential inequalities II*, Differentsial'nye Uravneniya **27**(6) (1991), 641–647.

[8] AZBELEV, N. V.—MAKSIMOV, V. P.—RAKHMATULLINA, L. F.: *Introduction to the Theory of Functional-differential Equations*, Nauka, Moscow, 1991.

[9] BAO, C.—PU, Y.—ZHANG, Y.: *Fractional-order deep backpropagation neural network*, Comput. Intell. Neurosci. (2018), Art. ID 7361628, 10.

[10] BEREZANSKY, L.—DOMOSHNITSKY, A.—KOPLATADZE, R.: *Oscillation, Nonoscillation, Stability and Asymptotic Properties for Second and Higher Order Functional Differential Equations*, CRC Press, 2020.

[11] CABRERA, I.—LOPEZ, B.—SADARANGANI, K.: *Lyapunov type inequalities for a fractional two-point boundary value problem*, Math. Methods Appl. Sci. **40**(10) (2017), 3409–3414.

[12] DE LA VALLÉE POUSSIN, C. J.: *Sur l'equation différentielle linéaire du second ordre. détermination d'une intégrale par deux valeurs assignées. extension aux équations d'ordre n*. J. Math. Pures Appl. **8** (1929), 125–144.

[13] GOODRICH, C. S.: *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. **23**(9) (2010), 1050–1055.

[14] KILBAS, A. A.—SRIVASTAVA, H. M.—TRUJILLO, J. J.: *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[15] KRASNOSEL'SKII, M. A.—VAINIKKO, G .M.—ZABREIKO, P. P.—RUTITSKII, Y. B.—STETSENKO, V. Y.: *Approximate Solution of Operator Equations*, Springer Science & Business Media, 2012.

[16] LAN, K.—LIN, W. *Positive solutions of systems of Caputo fractional differential equations*, Commun. Appl. Anal. **17**(1) (2013), 61–85.

[17] LAZOPOULOS, K. A.—LAZOPOULOS, A. K.: *Fractional vector calculus and fluid mechanics*, J. Mech. Behav. Mater. **26**(1–2) (2017), 43–54.

[18] MENDES, E. M. A. M.—SALGADO, G .H. O.—AGUIRRE, L. A.: *Numerical solution of Caputo fractional differential equations with infinity memory effect at initial condition*, Commun. Nonlinear Sci. Numer. Simul. **69** (2019), 237–247.

[19] PODLUBNY, I.: *Fractional Differential Equations*, Academic Press, Inc., San Diego, CA, 1999.

[20] REHMAN, H. U.—DARUS, M.—SALAH, J.: *A note on Caputo's derivative operator interpretation in economy*, J. Appl. Math. **2018** (2018), Art. ID 1260240.

[21] RIEWE, F.: *Mechanics with fractional derivatives*, Phys. Rev. E **55**(3) (1997), 3581–3592.
[22] SUN, Y.—ZHANG, X.: *Existence and nonexistence of positive solutions for fractional-order two-point boundary value problems.* Adv. Difference Equ. **2014**(53) (2014), 1–11.
[23] TARASOV, V. E.: *Fractional Dynamics.* Nonlinear Phys. Sci., Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[24] TARASOV, V. E.: *Mathematical economics: application of fractional calculus*, Mathematics **8**(5) (2020), Art. ID 660.
[25] TIAN, C.—LIU, Y.: *Multiple positive solutions for a class of fractional singular boundary value problems*, Mem. Differ. Equ. Math. Phys. **56** (2012), 115–131.
[26] WANG, J.—LIU, S.—FEČKAN, M.: *Iterative Learning Control for Equations with Fractional Derivatives and Impulses*, Springer Nature, 2021.
[27] WANG, N.—ZHANG, J.: *Existence and nonexistence of positive solutions for a class of Caputo fractional differential equation*, Sci. Asia **47**(1) (2021), 117–125.
[28] WARDAK, A.—GONG, P. *Fractional diffusion theory of balanced heterogeneous neural networks*, Phys. Rev. Research **3**(1) (2021), Art. ID 013083.

Received 13. 5. 2022
Accepted 5. 6. 2022

* *Department of Mathematics and Statistics*
Missouri S&T
Rolla, MO 65409
USA
E-mail: bohner@mst.edu

** *Department of Mathematics*
Ariel University
Ariel-40700
ISRAEL
E-mail: adom@ariel.ac.il
satyamsrivastava983@gmail.com

*** *Department of Mathematics*
Birla Institute of Technolology, Mesra
Ranchi-835215
INDIA
E-mail: spadhi@bitmesra.ac.in