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Article

Fractal Newton Methods

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Abstract: We introduce fractal Newton methods for solving $f(x) = 0$ that generalize and improve the classical Newton method. We compare the theoretical efficacy of the classical and fractal Newton methods and illustrate the theory with examples.

Keywords: fractal derivative; fractal Newton methods

MSC: 49M15

1. Introduction

Newton's method is one of the most universally known numerical algorithms. It was first introduced in around 1669 by Isaac Newton and then modified in 1690 by Joseph Raphson to obtain the real roots of polynomials. In 1740, Thomas Simpson extended the method to solve nonlinear equations, and in 1879 Arthur Cayley used the method for obtaining complex roots of polynomials [1]. Depending on the starting point, Newton's method may be convergent or divergent, has local quadratic convergence, and is undefined at critical points. Many modifications and enhancements of Newton's method exist in the literature [2–4]. In particular, Newton's method has recently been improved by using fractional derivatives instead of the classical derivative [5], and Wang and Tao [6] introduced a self-accelerating variable with memory into Newton's method. In this work, we replace the classical derivative in Newton's method with an α-fractal derivative where $0 < \alpha \leq 1$ and then modify the algorithm appropriately.

Fractal geometry and fractal and fractional calculus have become increasingly important in mathematics when one is confronted with certain types of nonlinear problems and the modeling of non-smooth phenomena. For example, fractal theory is the theoretical basis for fractal space-time [7], and in this setting fractal calculus can deal effectively with kinetics when fractal time replaces continuous time [8,9]. Candelario et al. [10] presented an optimal and low-computational-cost fractional Newton-type method for solving nonlinear equations. Golmankhaneh [11] discussed fractal calculus and its applications in detail. Blaszczyk et al. [12] investigated the approximation and application of the Riesz–Caputo fractional derivative of variable order with fixed memory.

Our manuscript is organized as follows: A deficiency of the classical Newton method is illustrated in Section 2, and fractal Newton methods are introduced to remedy it. The main definitions and theorems of fractal calculus are reviewed in Section 3. We prove our main convergence results for the fractal Newton methods in Section 4. We compare the convergence properties of the fractal and classical Newton method sequences in Section 5, and in Section 6 we illustrate their similarities and differences with examples. We state our conclusions in the final Section 7.

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2. Instance of Superiority of Fractal Newton Methods

Let $[a,b]$ be a closed, bounded interval in $[0,\infty)$. Let f be a continuous, real function on $[a,b]$ such that $f(a)f(b) \le 0$, so there exists at least one solution $t = \xi$ to $f(t) = 0$ in $[a,b]$. The classical Newton method sequence $\{t_n\}$, which is intended to successively approximate such a solution, is given by selecting $t_1 \in (a,b)$ and then defining

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} \quad (n = 1,2,3,\dots). \tag{1}$$

In order for this method to be robust, it is natural to assume that f is differentiable on (a,b) and $f'(t) \neq 0$ for all $t \in (a,b)$. The intermediate value property for differentiable, real functions then implies that either $f' > 0$ on (a,b) or $f' < 0$ on (a,b). In either case, f is strictly monotone on $[a,b]$, and there is precisely one solution $t = \xi$ to $f(t) = 0$ in $[a,b]$. Therefore, without a loss of generality, we may assume $f' > 0$ on (a,b), so f is monotonically increasing on $[a,b]$.

This is not sufficient, however, to guarantee the convergence of the Newton method sequence $\{t_n\}$ given by (1), no matter how $t_1 \neq \xi$ is chosen from (a,b). To understand this, consider $f(t) = t^{1/3}$ on $[0,1]$. If $t_1 \in (0,1)$, then

$$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = t_1 - \frac{t_1^{1/3}}{(1/3)t_1^{-2/3}} = t_1 - 3t_1 = -2t_1,$$

so $t_2 \in (-2,0)$. To remedy this, we reconsider $f(t) = t^{1/3}$ on $[-2,1]$, but then $t_3 = -2t_2 \in (0,4)$. Continuing in this manner, we are ultimately forced to consider $f(t) = t^{1/3}$ on $(-\infty,\infty)$. However, if $t_1 \neq 0$, then the Newton method sequence $\{t_n\} = \{(-2)^{n-1}t_1\}$ is divergent.

To remedy this deficiency of the classical Newton method, we begin by recalling the definition of the α-fractal derivative of a function.

Definition 1. *Let $\alpha > 0$ and $f : [0,b) \to \mathbb{R}$. The α-fractal derivative of f at $x_0 \in [0,b)$, denoted $D_\alpha f(x_0)$, is given by*

$$D_\alpha f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x^\alpha - x_0^\alpha} \tag{2}$$

provided the limit exists. In this case, we say that f is α-differentiable at x_0.

Fix $\alpha > 0$, and let f be a real, α-differentiable function on (a,b) that is continuous on $[a,b]$ and satisfies $f(a)f(b) \le 0$. If $D_\alpha f(x) \neq 0$ for all $x \in (a,b)$, then let $x_1 \in (a,b)$ and define the α-fractal Newton method sequence $\{x_n\}$ by

$$x_{n+1}^\alpha = x_n^\alpha - \frac{f(x_n)}{D_\alpha f(x_n)} \quad (n = 1,2,3,\dots). \tag{3}$$

It is clear that if $\alpha = 1$, then (3) reduces to the classical Newton method sequence (1).

Equipped with fractal Newton methods, let us return to the problem of finding the zero of $f(x) = x^{1/3}$ on $(-\infty,\infty)$. If α is a rational number in $(0,2/3)$ of the form $\alpha = p/q$ with q odd, then

$$D_\alpha f(x) = \frac{x^{1/3-\alpha}}{3\alpha}.$$

If $x_1 \neq 0$, then the α-fractal sequence (3) is given for $n \ge 1$ by

$$x_{n+1}^\alpha = x_n^\alpha - 3\alpha x_n^{1/3}/x_n^{1/3-\alpha} = x_n^\alpha - 3\alpha x_n^\alpha = (1-3\alpha)x_n^\alpha.$$

Since $\alpha \in (0,\frac{2}{3})$ implies $\lambda = 1 - 3\alpha \in (-1,1)$, it follows that $x_n = \lambda^{\frac{n-1}{\alpha}} x_1$ for $n \ge 1$, and thus $x_n \to 0$.

To summarize, when $f(x) = x^{1/3}$ and $x_1 \neq 0$, the classical Newton method sequence diverges, but if α is a rational number in $(0, 2/3)$ of the form $\alpha = p/q$ with q odd, then the α-fractal Newton method sequence converges to the solution $x = 0$ of $f(x) = 0$.

3. Preliminary Definitions and Theorems of Fractal Calculus

In this section, we formulate and prove the fractal analogues of many standard results of classical analysis. However, these fractal results are not deducible from the corresponding classical results, because it is possible for the fractal derivative $D_\alpha f(\xi)$ to exist at a point ξ, and the classical derivative of f fails to exist at ξ. Furthermore, the proofs in the literature of the fractal or conformable derivative results that we require for our fractal Newton method theorems are based on the assumption that $f(x)$ is expressible by a convergent power series in a neighborhood of the point ξ. That is, f is assumed to be an analytic function at ξ. To develop a robust theory, hypothesizing only the existence of the first and second fractal derivatives of f in a neighborhood of ξ led us to include the "standard" fractal results and their proofs. We know of no reference that states and proves these results in that generality.

As in this work, we usually insist $x_0 \geq 0$ in the definition of $D_\alpha f(x_0)$ to avoid technical difficulties. To illustrate these difficulties, suppose α is a positive rational number of the form $\alpha = p/q$, where q is odd and the greatest common divisor (gcd) of p and q is 1; i.e $\gcd(p, q) = 1$. Then, $x^\alpha = (\sqrt[q]{x})^p$ is clearly well-defined and real for all real x, and consequently $D_\alpha f(x_0)$ as given by (2) is meaningful. However, the examples $x^{\frac{1}{2}} = \sqrt{x}$ and $x^\pi = \exp(\pi \ln(x))$ illustrate why this need not be the case when $x_0 < 0$ and α is not a rational number of the aforementioned form.

It is clear that if f is α-differentiable at x_0, then f is continuous at x_0. Furthermore, it is easily seen that the fractal sum rule $D_\alpha(g + h)(x_0) = D_\alpha g(x_0) + D_\alpha h(x_0)$ and the fractal product rule $D_\alpha(gh)(x_0) = g(x_0)D_\alpha h(x_0) + h(x_0)D_\alpha g(x_0)$ hold when f and g are α-differentiable at x_0.

As an illustration of these facts, suppose $\alpha \in (0, 1]$, γ is a non-negative real number, and n is a positive integer. It is easy to show using the fractal product rule and mathematical induction that

$$D_\alpha(t^\alpha - \gamma^\alpha)^n = n(t^\alpha - \gamma^\alpha)^{n-1} \tag{4}$$

for all $t \geq 0$.

The proof of the next result is routine.

Theorem 1. *Let $\alpha > 0$, $f : [0, b) \to \mathbb{R}$, and $x_0 > 0$. Then, $f'(x_0)$ exists if and only if $D_\alpha f(x_0)$ exists. In this case,*

$$D_\alpha f(x_0) = \frac{1}{\alpha x_0^{\alpha-1}} f'(x_0).$$

The hypothesis $x_0 > 0$ is necessary in this theorem. To understand this, fix $\alpha \in (0, 1)$, let $0 < \gamma < \alpha < \beta \leq 1$, and define f on $[0, 1)$ by $f(x) = x^\alpha$. Then, an appeal to the definition of the (right) fractal derivative gives $D_\gamma f(0) = 0$, $D_\alpha f(0) = 1$, and $D_\beta f(0)$ does not exist. Specifically, $D_\alpha f(0)$ exists, but f is not differentiable at 0.

Using the previous theorem, it is easy to establish the fractal chain rule and the fractal Rolle's Theorem.

Lemma 1 (Fractal chain rule)**.** *Fix $\alpha > 0$. Let f be continuous on [a,b], $D_\alpha f(x)$ exist for some point $x \in [a, b]$, g be defined on an interval I that contains the range of f, and $D_\alpha g$ exist at the point $f(x)$. If $x > 0$ and $f(x) > 0$, then the composite function*

$$h(t) = g(f(t)) \quad (a \leq t \leq b)$$

is α-fractal differentiable at x and

$$D_\alpha h(x) = \alpha(f(x))^{\alpha-1} D_\alpha g(f(x)) D_\alpha f(x). \tag{5}$$

Lemma 2 (Fractal Rolle's theorem)**.** *Fix $\alpha > 0$ and $[a,b] \subseteq [0,\infty)$. Let $f : [a,b] \to \mathbb{R}$ be continuous, $f(a) = 0 = f(b)$, and let $D_\alpha f(t)$ exist for all $t \in (a,b)$. Then, there exists a point $\xi \in (a,b)$ such that $D_\alpha f(\xi) = 0$.*

The fractal mean value theorem is a consequence of the fractal Rolle's theorem. For completeness, we present a proof.

Theorem 2 (Fractal mean value theorem)**.** *Fix $\alpha > 0$ and $[a,b] \subseteq [0,\infty)$. Let $f : [a,b] \to \mathbb{R}$ be continuous, and let $D_\alpha f(t)$ exist for all $t \in (a,b)$. Then, there exists $\xi \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{b^\alpha - a^\alpha} = D_\alpha f(\xi). \tag{6}$$

Proof. We consider the function

$$h(t) = [f(b) - f(a)]t^\alpha - [b^\alpha - a^\alpha]f(t) + f(a)b^\alpha - f(b)a^\alpha \quad (a \le t \le b).$$

Then, h is continuous on $[a,b]$, $D_\alpha h(t)$ exists for all $t \in (a,b)$, and $h(a) = 0 = h(b)$. By the fractal Rolle's theorem, there exists a point $\xi \in (a,b)$ such that $D_\alpha h(\xi) = 0$. However,

$$D_\alpha h(\xi) = f(b) - f(a) - (b^\alpha - a^\alpha)D_\alpha f(\xi)$$

and the desired result (6) follows. □

The proof of our main result, the fractal Newton method convergence Theorem 4, relies on a fractal version of Taylor's theorem. For this, we rely on the notion of higher-order fractal derivatives.

Definition 2. *If f has an α-fractal derivative $D_\alpha f$ on an interval, and if $D_\alpha f$ itself has an α-fractal derivative, then we denote this derivative as $D_\alpha^{(2)} f$ and call it the second-order α-fractal derivative of f. Continuing in this manner when possible, we obtain functions*

$$f, D_\alpha f, \ldots, D_\alpha^{(n)} f,$$

each of which is the α-fractal derivative of the preceding one. $D_\alpha^{(n)} f$ is called the nth-order α-fractal derivative of f.

Theorem 3 (Fractal Taylor's theorem)**.** *Let $\alpha > 0$, $[a,b] \subset [0,\infty)$ and $f : [a,b] \to \mathbb{R}$. Let $n \ge 1$ be an integer, $D_\alpha^{(n-1)} f$ be continuous on $[a,b]$, and $D_\alpha^{(n)} f(t)$ exist for all $t \in (a,b)$. Let γ and β be distinct points in $[a,b]$, and for $a \le t \le b$ define*

$$p(t) = \sum_{k=0}^{n-1} \frac{D_\alpha^{(k)} f(\gamma)}{k!}(t^\alpha - \gamma^\alpha)^k.$$

Then, there exists a point x between γ and β such that

$$f(\beta) = p(\beta) + \frac{D_\alpha^{(n)} f(x)}{n!}(\beta^\alpha - \gamma^\alpha)^n.$$

Proof. Let M be the number defined by

$$f(\beta) = p(\beta) + M(\beta^\alpha - \gamma^\alpha)^n$$

and set

$$g(t) = f(t) - p(t) - M(t^\alpha - \gamma^\alpha)^n$$

for $a \le t \le b$. Using (4), it then follows that $D_\alpha^{(m)}(t^\alpha - \gamma^\alpha)^m = m!$, and if $0 \le k < m$, then $D_\alpha^{(m)}(t^\alpha - \gamma^\alpha)^k = 0$. Consequently, $D_\alpha^{(n)} p(t) = 0$, $D_\alpha^{(n)} M(\beta^\alpha - \gamma^\alpha)^n = Mn!$, and thus

$$D_\alpha^{(n)} g(t) = D_\alpha^{(n)} f(t) - Mn!$$

if $a < t < b$.

Next, observe that if k is an integer satisfying $0 \le k \le n-1$, then

$$\begin{aligned} D_\alpha^{(k)} p(\gamma) &= D_\alpha^{(k)} \left(\sum_{j=0}^{n-1} \frac{D_\alpha^{(j)} f(\gamma)}{j!} (t^\alpha - \gamma^\alpha)^j \right) |_{t=\gamma} \\ &= \sum_{j=0}^{n-1} \frac{D_\alpha^{(j)} f(\gamma)}{j!} D_\alpha^{(k)} (t^\alpha - \gamma^\alpha)^j \,|_{t=\gamma} \\ &= \frac{D_\alpha^{(k)} f(\gamma)}{k!} D_\alpha^{(k)} (t^\alpha - \gamma^\alpha)^k \,|_{t=\gamma} + \sum_{j>k} \frac{D_\alpha^{(j)} f(\gamma)}{j!} D_\alpha^{(k)} (t^\alpha - \gamma^\alpha)^j \,|_{t=\gamma} \\ &= D_\alpha^{(k)} f(\gamma). \end{aligned}$$

It follows that $0 = g(\gamma) = D_\alpha g(\gamma) = \ldots = D_\alpha^{(n-1)} g(\gamma)$.

Our choice of M shows that $g(\beta) = 0$. By the fractal Rolle's theorem, there exists x_1 between γ and β such that $D_\alpha g(x_1) = 0$. Since $D_\alpha g(\gamma) = 0$, the fractal Rolle's theorem implies that there exists x_2 between γ and x_1 such that $D_\alpha^{(2)} g(x_2) = 0$. After n steps, we arrive at the conclusion $D_\alpha^{(n)} g(x_n) = 0$ for some x_n between γ and x_{n-1}. Hence, $0 = D_\alpha^{(n)} g(x_n) = D_\alpha^{(n)} f(x_n) - n!M$; i.e.,

$$M = \frac{D_\alpha^{(n)} f(x_n)}{n!}.$$

□

4. Fractal Newton Method Convergence Theorem

Definition 3. *Let $\alpha > 0$ and let f be α-differentiable at $\xi \in (0, \infty)$. The α-tangent to f at $(\xi, f(\xi))$ is the function given by*

$$T_{f,\alpha,\xi}(t) = f(\xi) + D_\alpha f(\xi)(t^\alpha - \xi^\alpha).$$

Remark 1. *If $\alpha = 1$, then $T_{f,1,\xi}(t) = f(\xi) + f'(\xi)(t - \xi)$ is the classical tangent line function to the graph of $y = f(t)$ at the point $(\xi, f(\xi))$. For the general case $\alpha > 0$, if x_{n+1} is given by (3), then it is easy to see using Definition 3 that $T_{f,\alpha,x_n}(x_{n+1}) = 0$. Geometrically, this means that the graph of T_{f,α,x_n} intersects the t-axis at the point $(x_{n+1}, 0)$.*

Lemma 3. *Let $\alpha > 0$, $\eta \in (a, b) \subset [0, \infty)$, and f be twice α-differentiable with $D_\alpha^{(2)} f \ge 0$ on (a, b). Then, the graph of f lies on or above the graph of $T_{f,\alpha,\eta}$ on (a, b).*

Proof. Let η and t be distinct points in (a, b). Apply the fractal Taylor's theorem to f with $n = 2$, $\beta = t$, and $\gamma = \eta$. Then,

$$f(t) = p(t) + \frac{D_\alpha^{(2)} f(x)}{2!} (t^\alpha - \eta^\alpha)^2 \tag{7}$$

for some x between t and η. Here,

$$p(t) = f(\eta) + D_\alpha f(\eta)(t^\alpha - \eta^\alpha) = T_{f,\alpha,\eta}(t)$$

for all $t \in (a,b)$. It follows from (7) that

$$f(t) - T_{f,\alpha,\eta}(t) = \frac{D_\alpha^{(2)} f(x)}{2!}(t^\alpha - \eta^\alpha)^2 \geq 0$$

on (a,b). □

Theorem 4 (Fractal Newton method convergence)**.** *Fix $\alpha > 0$ and let $[a,b] \subset [0,\infty)$. Let f be twice α-differentiable with $D_\alpha^{(2)} f \geq 0$ on (a,b), and let f be α-differentiable with $D_\alpha f$ continuous on $[a,b]$ and positive on (a,b). If $f(a)f(b) < 0$ and $x_1 \in (a,b)$, then the α-fractal Newton method sequence given by (3) converges to the unique point ξ in (a,b) satisfying $f(\xi) = 0$.*

Proof. Note that $f(a)f(b) < 0$ and $D_\alpha f(x) > 0$ for $x \in (a,b)$ imply $f(a) < 0$ and $f(b) > 0$. By the intermediate value theorem and the fractal Rolle's theorem, there is a unique number ξ in (a,b) such that $f(\xi) = 0$. Let x_1 be any number in (a,b). If $f(x_1) = 0$, then $x_1 = \xi$, and it then follows from (3) that $x_n = \xi$ for all $n \geq 1$. Thus, $\xi = \lim_{n\to\infty} x_n$ is trivially true.

Assume $f(x_1) > 0$. It clearly follows that $\xi < x_1 < b$. We will show that the sequence $\{x_n\}$ given by (3) satisfies $\xi \leq x_{n+1} \leq x_n$ for all $n \geq 1$. To this end, note that when $n = 1$, we have

$$\frac{f(x_1)}{D_\alpha f(x_1)} > 0,$$

so

$$x_2^\alpha - x_1^\alpha = -\frac{f(x_1)}{D_\alpha f(x_1)} < 0.$$

Hence, $x_2^\alpha < x_1^\alpha$ and $x_2 < x_1$. Because $D_\alpha^{(2)} f(t) \geq 0$ for $t \in (a,b)$, it follows from Lemma 3 that the graph of f lies on or above the graph of T_{f,α,x_1} on (a,b). Hence, the intersection point $(x_2, 0)$ of the graph of T_{f,α,x_1} with the t-axis lies on or below the graph of f at the point $(x_2, f(x_2))$. That is, $f(x_2) \geq 0$, from which it follows that $\xi \leq x_2$.

Inductively, suppose $\xi \leq x_{m+1} \leq x_m$ for some integer $m \geq 1$. If $x_{m+1} = x_m$, then

$$\frac{f(x_m)}{D_\alpha f(x_m)} = {x_m}^\alpha - {x_{m+1}}^\alpha = 0.$$

Thus, $f(x_m) = 0$, and it follows from (3) that $x_n = \xi$ for all $n \geq m$. Similarly, if $\xi = x_{m+1}$ then $x_n = \xi$ for all $n \geq m+1$. In both these cases of equality, the induction is finished. Therefore, suppose $\xi < x_{m+1} < x_m$, and hence $f(x_{m+1}) > 0$. Consequently, (3) with $n = m+1$ implies that $x_{m+2} < x_{m+1}$. Furthermore, it follows from Lemma 3 that the graph of f lies on or above $T_{f,\alpha,x_{m+1}}$ on (a,b). Consequently, the intersection point $(x_{m+2}, 0)$ of the graph of $T_{f,\alpha,x_{m+1}}$ with the t-axis lies on or below the graph of f at the point $(x_{m+2}, f(x_{m+2}))$. Thus, $f(x_{m+2}) \geq 0$ so $\xi \leq x_{m+2}$. By induction, $\xi \leq x_{n+1} \leq x_n$ for all integers $n \geq 1$ when $f(x_1) > 0$.

Suppose $f(x_1) < 0$. Then, clearly $a < x_1 < \xi$ and

$$\frac{f(x_1)}{D_\alpha f(x_1)} < 0,$$

so

$$x_2^\alpha - x_1^\alpha = -\frac{f(x_1)}{D_\alpha f(x_1)} > 0.$$

Hence $x_1^\alpha < x_2^\alpha$ and $x_1 < x_2$. As before, Lemma 3 implies that the graph of f lies on or above the graph of T_{f,α,x_1} on (a,b). However, $T_{f,\alpha,x_1}(x_2) = 0$, so $f(x_2) \geq 0$, and thus $\xi \leq x_2 < b$. The proof for the case when $f(x_1) > 0$ now shows that $\xi \leq x_{n+1} \leq x_n$ for all $n \geq 2$.

Therefore, if $f(x_1) \neq 0$ and $n \geq 2$, the α-fractal Newton method sequence $\{x_n\}$ is monotonically decreasing and bounded below by ξ. Hence, $\eta = \lim_{n\to\infty} x_n$ exists, and clearly $\xi \le \eta$. Since $D_\alpha f$ is continuous on $[\xi, b]$, it follows that $B = \sup\{D_\alpha f(x) : \xi \le x \le b\} < \infty$, so

$$0 \le f(x_n) = D_\alpha f(x_n)(x_n^\alpha - x_{n+1}^\alpha) \le B(x_n^\alpha - x_{n+1}^\alpha)$$

for $n \ge 2$ by (3). Consequently, $f(\eta) = \lim_{n\to\infty} f(x_n) = 0$, and thus $\xi = \eta = \lim_{n\to\infty} x_n$. □

Corollary 1. *Fix $\alpha > 0$ and let $[a, b] \subset [0, \infty)$. Let f be twice α-differentiable with $D_\alpha^{(2)} f \ge 0$ on (a, b), and let f be α-differentiable with $D_\alpha f$ continuous on $[a, b]$ and positive on (a, b). If $f(a) \le 0$, $f(b) > 0$, $x_1 \in (a, b)$, and x_{n+1} is given by (3) for $n = 1, 2, 3, \ldots$, then the sequence $\{x_n\}_{n=1}^\infty$ converges to the unique point ξ in $[a, b]$ satisfying $f(\xi) = 0$.*

Proof. If $f(a) < 0$, then $f(a)f(b) < 0$, so the desired result follows directly from Theorem 4. Suppose $f(a) = 0$. The proof of Theorem 4 shows that $\{x_n\}$ is a monotone decreasing sequence that converges to $\xi = a$. □

It is well-known that the classical Newton method has local quadratic convergence. The next result shows that fractal Newton methods share this property.

Theorem 5 (Fractal Newton method convergence rate)**.** *Fix $\alpha \in (0, 1)$ and let $[a, b] \subset [0, \infty)$. Let f be twice α-differentiable with $D_\alpha^{(2)} f$, non-negative, and bounded on (a, b), and let f be α-differentiable with $D_\alpha f$, continuous, and non-zero on $[a, b]$. If $f(a)f(b) \le 0$ and $x_1 \in (a, b)$, then the α-fractal Newton method sequence given by (3) converges with a rate of at least 2 to the unique point $\xi \in [a, b]$ satisfying $f(\xi) = 0$.*

Proof. Suppose first that $f(a)f(b) < 0$. By Theorem 4, the α-fractal Newton method sequence $\{x_n\}_{n=1}^\infty$ given by (3) is convergent to $\xi \in (a, b)$. If $x_{n_0} = \xi$ for some integer $n_0 \ge 1$, then clearly $x_m = \xi$ for all $m \ge n_0$, and the proof is finished. Therefore, suppose $x_n \ne \xi$ for all $n \ge 1$. Apply the fractal Taylor's theorem with $\beta = \xi$ and $\gamma = x_n$ to obtain a point t_n between ξ and x_n such that

$$f(\xi) = f(x_n) + D_\alpha f(x_n)(x^\alpha - x_n^\alpha) + \frac{D_\alpha^{(2)} f(t_n)}{2}(x^\alpha - x_n^\alpha)^2. \tag{8}$$

However, $f(\xi) = 0$, so rearranging (8) and using (3) yields

$$x_n^\alpha - \xi^\alpha = \frac{f(x_n)}{D_\alpha f(x_n)} + \frac{D_\alpha^{(2)} f(t_n)}{2D_\alpha f(x_n)}(x_n^\alpha - \xi^\alpha)^2$$

and

$$x_{n+1}^\alpha - \xi^\alpha = \frac{D_\alpha^{(2)} f(t_n)}{2D_\alpha f(x_n)}(x_n^\alpha - \xi^\alpha)^2.$$

It follows that

$$|x_{n+1}^\alpha - \xi^\alpha| \le \frac{M}{2\delta}|x_n^\alpha - \xi^\alpha|^2 \quad (n = 1, 2, 3, \ldots)$$

where $|D_\alpha f(x)| \ge \delta > 0$ on $[a, b]$ and $0 \le D_\alpha^{(2)} f(x) \le M < \infty$ on (a, b). Observe that

$$|x_{n+1}^\alpha - \xi|\left|\frac{x_{n+1}^\alpha - \xi^\alpha}{x_{n+1}^\alpha - \xi}\right| \le \frac{M}{2\delta}\left|\frac{x_n^\alpha - \xi^\alpha}{x_n^\alpha - \xi}\right||x_n^\alpha - \xi|^2 \qquad (n = 1, 2, 3, \ldots). \tag{9}$$

Since

$$\lim_{x\to\xi} \frac{x^\alpha - \xi^\alpha}{x - \xi} = \alpha\xi^{\alpha-1},$$

it follows from (9) that to each $\epsilon > 0$ there corresponds an integer $N_0 = N_0(\epsilon) \ge 1$ such that

$$|x_{n+1}^\alpha - \xi|\alpha\xi^{\alpha-1} \le \left(\frac{M}{2\delta} + \epsilon\right)\left(\alpha\xi^{\alpha-1}\right)^2|x_n^\alpha - \xi|^2$$

for all $n \geq N_0$. Hence, there exists a positive constant A such that

$$|x_{n+1}^{\alpha} - \xi| \leq A\xi^{\alpha-1}|x_n^{\alpha} - \xi|^2 \qquad (n = 1, 2, 3, \dots). \tag{10}$$

If $f(a)f(b) = 0$, then the hypotheses on f imply that $f(x) \neq 0$ in (a, b) and $\{x_n\}_{n=1}^{\infty}$ converges to $\xi \in \{a, b\}$ as in Corollary 1. If $\xi = b$ or $\xi = a > 0$, then the argument leading to (10) still holds. If $\xi = a = 0$, then it follows from (8) that

$$x_{n+1}^{\alpha} = \frac{D_{\alpha}^{(2)} f(t_n)}{2D_{\alpha} f(x_n)} x_n^{2\alpha}$$

so

$$|x_{n+1}| \leq \left(\frac{M}{2\delta}\right)^{1/\alpha} |x_n|^2 \qquad (n = 1, 2, 3, \dots). \tag{11}$$

From (10), if $\xi \neq 0$ and (11) if $\xi = 0$, we see that the α-fractal Newton method sequence converges with a rate of at least 2. □

We remark that our proof of Theorem 5 shows that for real functions f on $[a, b]$ satisfying the hypotheses, there exists a constant $K = K(f, [a, b])$, independent of $\alpha \in (0, 1)$ and $\xi \in [a, b]$, such that for all $n \geq 1$,

$$|x_{n+1}^{\alpha} - \xi| \leq K\alpha\xi^{\alpha-1}|x_n^{\alpha} - \xi|^2 \qquad \text{if } \xi \neq 0, \tag{12}$$

and

$$|x_{n+1}^{\alpha}| \leq K|x_n^{\alpha}|^2 \qquad \text{if } \xi = 0. \tag{13}$$

5. Comparison of Fractal and Classical Newton Methods

In Section 2, we encountered an instance when certain α-fractal Newton method sequences $\{x_n\}$ given by (3) converge to the solution $x = \xi$ of $f(x) = 0$, but the classical Newton method sequence $\{t_n\}$ given by (1) diverges for every choice of $t_1 \neq \xi$. In Section 6, we will encounter infinite families of such instances. In this section, we explore conditions when the sequences $\{x_n\}$ and $\{t_n\}$ are equiconvergent.

Let $\alpha > 0$ and $\{x_n\}$ be an α-fractal Newton method sequence given by (3). If we assume that all $x_n > 0$, then (3) can be rearranged to yield

$$x_{n+1} = x_n \left(1 - \frac{\alpha f(x_n)}{x_n f'(x_n)}\right)^{1/\alpha} \qquad (n = 1, 2, 3, \dots).$$

On the other hand, assume that the classical Newton method sequence $\{t_n\}$ given by (1) consists entirely of positive terms. Then, (1) can be written as

$$t_{n+1} = t_n \left(1 - \frac{f(t_n)}{t_n f'(t_n)}\right) \quad (n = 1, 2, 3, \dots).$$

Comparing these identities, one expects that if $x_1 = t_1$ and α is sufficiently close to 1, then $|t_n - x_n| \to 0$ as $n \to \infty$.

To formulate a precise version, let f be a real function on $[a, b] \subset [0, \infty)$ satisfying:

(H0) $f(a)f(b) < 0$;
(H2) f'' is continuous and non-negative on the open interval (a, b);

and either

(H1+) f' is continuous and positive on the closed interval $[a, b]$;

or

(H1−) f' is continuous and negative on the closed interval $[a, b]$.

If $t_1 \in (a,b)$, then it is well-known that the classical Newton method sequence $\{t_n\}$ given by (1) converges to the unique solution $t = \xi$ of $f(t) = 0$ in the interval (a,b).

In the case when f satisfies (H0), (H1+), and (H2), the next theorem shows that the α-fractal Newton method obeys a complete analogue of the classical Newton method convergence theorem.

Theorem 6. *Let f satisfy (H0), (H1+), and (H2). If $0 < \alpha < 1$ and $x_1 \in (a,b)$, then the α-fractal sequence $\{x_n\}$ given by (3) converges to the unique solution $x = \xi$ of $f(x) = 0$ in the interval (a,b).*

Proof. Fix $\alpha \in (0,1)$. For all $x \in [a,b]$,

$$D_\alpha f(x) = \frac{x^{1-\alpha}}{\alpha} f'(x).$$

It follows from (H1+) that $D_\alpha f$ is continuous on $[a,b]$ and positive on (a,b). Furthermore, one computes

$$D_\alpha^{(2)} f(x) = \frac{x^{1-2\alpha}}{\alpha^2}\left[(1-\alpha)f'(x) + xf''(x)\right]$$

for all $x \in (a,b)$. Hence, $D_\alpha^{(2)} f$ is continuous and positive on (a,b) by (H1+) and (H2). Therefore, the α-fractal Newton method sequence $\{x_n\}$ is convergent to ξ by Theorem 4. □

The next theorem shows that if f satisfies (H0), (H1−), and (H2+) (f'' is continuous and positive on the closed interval $[a,b]$), then there is a non-empty open interval I in $(0,1)$ such that the α-fractal Newton method sequence $\{x_n\}$ given by (3) is also convergent to ξ when $\alpha \in I$. To make this precise, suppose that f satisfies (H1−). Then, $0 < M = \max\{|f'(x)| : x \in [a,b]\} < \infty$. Furthermore, suppose that f satisfies (H2+). Then, $m = \min\{f''(x) : x \in [a,b]\} > 0$. Now let μ be any positive number in the open interval $\left(1 - \frac{am}{M}, 1\right)$.

Theorem 7. *Let $[a,b] \subset (0,\infty)$ and let f satisfy (H0), (H1-), and (H2+). If $\alpha \in (\mu,1)$, then the α-fractal sequence $\{x_n\}$ given by (3) converges to the unique solution $x = \xi$ of $f(x) = 0$ in the interval (a,b).*

Proof. If $x \in [a,b]$, we have

$$(1-\alpha)f'(x) + xf''(x) > (1-\mu)(-M) + am > 0,$$

and it follows that

$$D_\alpha^{(2)} f(x) = \frac{x^{1-2\alpha}}{\alpha^2}\left[(1-\alpha)f'(x) + xf''(x)\right] > 0.$$

The proof of Lemma 3 shows that for any distinct points η and t in (a,b), there exists x between η and t such that

$$f(t) - T_{f,\alpha,\eta}(t) = \frac{D_\alpha^{(2)} f(x)}{2!}(t^\alpha - \eta^\alpha)^2 > 0.$$

That is, the graph of f lies on or above the graph of $T_{f,\alpha,\eta}$ on the interval $[a,b]$ for every $\eta \in (a,b)$ and every $\alpha \in (\mu,1)$.

Let $x_1 \in (a,b)$ and let $\{x_n\}$ be the α-fractal Newton method sequence given by (3). If $x_1 = \xi$, then $x_n = \xi$ for all $n \geq 1$, and the proof is finished in this case. Suppose $x_1 \neq \xi$. Then, either $f(x_1) > 0$ and $a < x_1 < \xi$, or $f(x_1) < 0$ and $\xi < x_1 < b$. Consider the first case, the proof in the second case being analogous. Arguing using induction and the identity (3) as in the proof of Theorem 4, we see that the sequence $\{x_n\}$ is increasing and bounded above by ξ. Hence, $\{x_n\}$ converges to a number η in (a,b). Furthermore, it follows from (3) and

$$0 < a < x_n \le x_{n+1} \le \xi$$

that

$$0 \le f(x_n) = -D_\alpha f(x_n)[x_{n+1}^\alpha - x_n^\alpha] \le M(x_{n+1}^\alpha - x_n^\alpha),$$

and thus $f(\eta) = 0$. However, then $\eta = \xi$. □

6. Numerical Results

Example 1. *Fix a rational number $\beta \in (0, 1/2]$ of the form $\beta = r/s$ where $\gcd(r,s) = 1$ and s is odd, and let $f(x) = x^\beta$ for $x \in (-\infty, \infty)$. If $x_1 \ne 0$ and α is a rational number in $(0, 2\beta)$ of the form $\alpha = p/q$ where $\gcd(p,q) = 1$ and q is odd, then the α-fractal Newton method sequence $\{x_n\}$ converges to the unique point $\xi = 0$ in $(-\infty, \infty)$ satisfying $f(\xi) = 0$. If $t_1 \ne 0$, then the classical Newton method sequence $\{t_n\}$ diverges.*

Proof. Suppose $x_1 \ne 0$ and let

$$x_{n+1}^\alpha = x_n^\alpha - \frac{f(x_n)}{D_\alpha f(x_n)}$$

for $n \ge 1$. It follows from $D_\alpha f(x) = \frac{\beta}{\alpha} x^{\beta-\alpha}$ for $x \ne 0$ that

$$x_{n+1}^\alpha = \left(1 - \frac{\alpha}{\beta}\right) x_n^\alpha$$

for $n \ge 1$. Setting $\gamma = 1 - \frac{\alpha}{\beta} \in (-1, 1)$, we have

$$x_n^\alpha = \gamma^{n-1} x_1^\alpha \tag{14}$$

for $n \ge 1$, and the desired conclusion for $\{x_n\}$ follows.

Suppose $t_1 \ne 0$ and let

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} = t_n - \frac{t_n^\beta}{\beta t_n^{\beta-1}} = \left(1 - \frac{1}{\beta}\right) t_n$$

for $n \ge 1$. Setting $\lambda = 1 - \frac{1}{\beta} \in (-\infty, -1]$, we have

$$t_n = \lambda^{n-1} t_1$$

for $n \ge 1$, and hence $\{t_n\}$ is divergent. □

Remark 2. *The observant reader will note that the example in Section 2 introducing fractal Newton methods is the case $\beta = 1/3$ and $\alpha \in (0, 2\beta) = (0, 2/3)$ of Example 1. When $\alpha \in (0, \beta)$, the result of Example 1 is a consequence of Corollary 1, since $f(0)f(1) = 0$, $D_\alpha f$ is continuous on $[0, \infty)$,*

$$D_\alpha f(x) = \frac{\beta}{\alpha} x^{\beta-\alpha} > 0$$

on $(0, \infty)$, and

$$D_\alpha^{(2)} f(x) = \frac{\beta(\beta-\alpha)}{\alpha^2} x^{\beta-2\alpha} > 0$$

for $x > 0$. Furthermore, it is clear in Example 1 that for $0 < \alpha < 2\beta$, the rate of convergence of the α-fractal Newton method sequence is faster the closer α is to β. In fact, if $\alpha = \beta$, then the α-fractal Newton method converges in one step; i.e., $x_n = x_2 = \xi = 0$ for all $n \ge 2$.

Example 2. *Let $f(x) = (x-1)^{\frac{1}{3}}$ on $(-\infty, \infty)$ and $x_1 > 0$. Then:*

1. *The classical Newton method sequence converges if and only if $x_1 = 1$;*
2. *The $\frac{1}{3}$ fractal Newton method sequence converges if and only if $x_1 = 1$;*
3. *The $\frac{1}{5}$ fractal Newton method sequence converges if and only if there exists an integer $N \geq 1$ such that $x_N = 1$.*

Proof. Let α be a positive rational number whose expression in lowest terms is $\alpha = p/q$, where $\gcd(p, q) = 1$ and both p and q are odd. The α-fractal Newton method recursion relation (3) in this case can be expressed as

$$x_{n+1} = x_n \left(1 - \frac{\alpha f(x_n)}{x_n f'(x_n)}\right)^{1/\alpha} \tag{15}$$

for $n \geq 1$. Because

$$\frac{f(x)}{f'(x)} = 3(x-1),$$

(15) reduces to

$$x_{n+1} = x_n \left(1 - 3\alpha + \frac{3\alpha}{x_n}\right)^{1/\alpha} \tag{16}$$

for $n \geq 1$.

Suppose $\alpha = 1$. Then, (16) is equivalent to $x_{n+1} - 1 = -2(x_n - 1)$, and hence $x_{n+1} - 1 = (-2)^{n-1}(x_1 - 1)$ for $n \geq 1$. Therefore, the classical Newton method sequence converges if and only if $x_1 = 1$.

Suppose $\alpha = 1/3$. Then, (16) is equivalent to $x_{n+1} = 1/x_n^2$, and hence

$$x_n = x_1^{(-2)^{n-1}}$$

for $n \geq 1$. If $x_1 > 1$, then $x_{2m} \to 0$ and $x_{2m-1} \to \infty$. If $0 < x_1 < 1$, then $x_{2m} \to \infty$ and $x_{2m-1} \to 0$. If $x_1 = 1$, then $x_n = 1$ for all $n \geq 1$. Consequently, the $\frac{1}{3}$ fractal Newton method sequence converges if and only if $x_1 = 1$.

Consider $\alpha = 1/5$. In this case, (16) is equivalent to

$$x_{n+1} = x_n \left(\frac{2}{5} + \frac{3}{5x_n}\right)^5 \tag{17}$$

for $n \geq 1$. Let

$$h(x) = x\left(\frac{2}{5} + \frac{3}{5x}\right)^5$$

for $x > 0$. Then, (17) can be expressed succinctly as

$$x_{n+1} = h(x_n) \tag{18}$$

for $n \geq 1$. Observe that the function h is continuous on $(0, \infty)$ and possesses the following properties:

1. h is decreasing on $(0, 6)$;
2. h is increasing on $(6, \infty)$;
3. $h(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$;
4. $3/16 = h(6) \leq h(x)$ for all $x > 0$;
5. There are exactly two positive solutions to $h(x) = 1$, namely $x = 1$ and an irrational solution $x = \xi$ that is approximately 89.9017661005;
6. $x = 1$ is the only positive solution to $h(x) = x$;
7. $h'(x) = \frac{2}{5}\left(1 - \frac{6}{x}\right)\left(\frac{2}{5} + \frac{3}{5x}\right)^4$;
8. $h''(x) = \frac{36}{25x^3}\left(\frac{2}{5} + \frac{3}{5x}\right)^3$.

Suppose there exists an integer $N \geq 1$ such that the $\frac{1}{5}$ fractal Newton method sequence $\{x_n\}$ satisfies $x_N = 1$. Then, (18) and property (vi) of h imply that $x_n = 1$ for all $n \geq N$ and $\{x_n\}$ converges, to 1 in fact. Conversely, suppose $x_1 > 0$ and the $\frac{1}{5}$ fractal Newton method sequence $\{x_n\}$ converges, say to the real number L. Then, Equation (18), together with continuity and the properties of the function h, imply that $L = h(L)$, and hence $L = 1$. Suppose, by way of contradiction, that $x_n \neq 1$ for all $n \geq 1$. By the convergence of $\{x_n\}$, there exists and integer $N \geq 1$ such that $|x_n - 1| < 1/10$ for all $n \geq N$.

By the classical Taylor's theorem, to each positive real number $x \neq 1$ there corresponds a real number ξ between x and 1 such that

$$h(x) = 1 - 2(x-1) + \frac{h''(\xi)}{2}(x-1)^2. \tag{19}$$

Let $n \geq N$. If $9/10 < x_n < 1$, then $h(x_n) > 1$ and $x_n < \xi_n < 1$ so $h''(\xi_n) > 0$. It follows from (19) and (18) that

$$x_{n+1} - 1 = h(x_n) - 1 > 2|x_n - 1|. \tag{20}$$

If $1 < x_n < 11/10$, then $0 < h(x_n) < 1$ and $1 < \xi_n < x_n$ so $0 < h''(\xi_n) < h''(1) = 36/25$ and

$$\left|-2 + \frac{h''(\xi_n)}{2}(x_n - 1)\right| > \left|-2 + \frac{36}{50}\frac{1}{10}\right| = 1.928. \tag{21}$$

Consequently, (19), (18), and (21) imply that

$$|x_{n+1} - 1| = |h(x_n) - 1| = \left|-2 + \frac{h''(\xi_n)}{2}(x_n - 1)\right||x_n - 1| > 1.928|x_n - 1|. \tag{22}$$

It follows from (20) and (22) that for all integers $m \geq 1$,

$$|x_{N+m} - 1| > (1.928)^m |x_N - 1|,$$

and hence $\{x_n\}$ diverges, a contradiction. We conclude that if $x_1 > 0$ and the $\frac{1}{5}$ fractal Newton method sequence converges, then there exists a positive integer N such that $x_N = 1$. □

Remark 3. *The $\frac{1}{5}$ fractal Newton method applied to $f(x) = (x-1)^{\frac{1}{3}}$ on $(-\infty, \infty)$ in Example 2 exhibits a curious phenomenon: There is an infinite set of positive values for x_1 for which the corresponding $\frac{1}{5}$ fractal Newton method sequence $\{x_n\}$ is convergent, necessarily to 1. To understand this, first observe that if $x_1 = \xi$, then $x_2 = h(x_1) = h(\xi) = 1$, and consequently $x_n = 1$ for all $n \geq 2$. Next, let g denote the inverse function of h on the interval $[6, \infty)$ (cf. property (ii) of h). If $x_1 = g(\xi)$, then $x_2 = h(x_1) = h(g(\xi)) = \xi$, $x_3 = h(x_2) = h(\xi) = 1$, and $x_n = 1$ for all $n \geq 3$. Continuing in this manner, if $g^{(m)}$ denotes the m-fold composition of g and $x_1 = g^{(m)}(\xi)$, then $x_n = 1$ for all $n \geq m+2$. Thus, if*

$$x_1 \in \{\xi, g(\xi), g^{(2)}(\xi), g^{(3)}(\xi), g^{(4)}(\xi), \dots\}$$

then the $\frac{1}{5}$ fractal Newton method sequence $\{x_n\}$ is convergent.

In its current form, the α-fractal Newton method has deficiencies that diminish its value for some practical applications. In Example 2, which considered the problem of solving $(x-1)^{\frac{1}{3}} = 0$, when $\alpha = 1/5$, the values of x_1 for which the $\frac{1}{5}$ fractal Newton method sequence converges are isolated and differ widely from the limit 1. For instance, ξ is approximately 89.9017661, and $g(\xi)$ is approximately 8771.9667803. However, in the problem of solving $x^{\frac{1}{3}} = 0$ near the end of Section 2, we saw that if α is a rational number in $(0, 2/3)$ of the form $\alpha = p/q$ with q odd, then for any value of x_1, the α-fractal Newton method sequence converges to the solution $x = 0$. It is an important and interesting problem for future researchers to modify the α-fractal Newton method in such a way that it works equally well on such problems.

7. Conclusions

We introduced a parameterized family of α-fractal Newton methods including the classical Newton method when $\alpha = 1$. We investigated fractal Newton methods in detail, providing general sufficient conditions for their convergence. We compared fractal and classical Newton methods and identified general circumstances when the classical and fractal Newton method sequences are equiconvergent. Moreover, we showed that, like the classical Newton method, local convergence is quadratic for fractal Newton method sequences. As a consequence of our methods, we obtained an upper bound for the error constant of α-fractal Newton method sequences as a function of $\alpha \in (0, 1)$. We illustrated with examples that fractal Newton method sequences can converge when the corresponding classical Newton method sequences diverge. Further research is needed to modify and improve the α-fractal Newton method, making it more robust in such instances. The experimental rate of convergence of fractal Newton method sequences and their numerical performance—e.g., CPU times and numbers of iterations—relative to the classical Newton method are interesting problems with practical consequences that are worthy of further investigation. The relaxation of the non-negativity hypothesis concerning the second fractal derivative in Theorem 4 is an interesting problem. However, it is a delicate one, since a change in the concavity of f at the root of $f(x) = 0$ leads easily to divergence for certain classical Newton method sequences.

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