15 Jan 1992

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Recommended Citation

R. H. Hering, "Boundedness And Periodic Solutions In Infinite Delay Systems," *Journal of Mathematical Analysis and Applications*, vol. 163, no. 2, pp. 521 - 535, Elsevier, Jan 1992.
The definitive version is available at https://doi.org/10.1016/0022-247X(92)90264-E

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **163**, 521–535 (1992)

Boundedness and Periodic Solutions in Infinite Delay Systems

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Submitted by V. Lakshmikantham

Received August 28, 1989

Liapunov methods are used to give conditions ensuring that solutions of infinite delay equations are uniformly bounded and uniformly ultimately bounded with respect to unbounded (C_g) initial function spaces; and the connection to proving existence of periodic solutions is examined. Several examples illustrate the application of these results, especially to integrodifferential equations. © 1992 Academic Press, Inc.

1. INTRODUCTION

It is known that for a functional differential equation with infinite delay which is sufficiently well posed in a certain phase space, existence of a periodic solution follows directly if solutions are uniformly bounded and uniformly ultimately bounded with respect to this phase space. The purpose of this paper is to develop conditions ensuring such boundedness properties for solutions of general infinite delay equations and techniques for applying these in the specific case of integrodifferential equations. Liapunov methods are used throughout.

In order to construct a phase space, let $C=\mathscr{C}([-h,0],\mathbb{R}^n)$, where $0<h\leqslant\infty$, and let $G=G^0\cup\{g_0\}$, where $g_0(r)\equiv 1$ for $r\in(-\infty,0]$ and $G^0=\{g\in\mathscr{C}((-\infty,0],\ [1,\infty))$: $g(0)=1$, g decreasing, $g(r)\to\infty$ as $r\to-\infty\}$. For a given $g\in G$, define the phase space $C_g=(C,|\cdot|_g)$, where $|\phi|_g=\sup_{s\leqslant 0}(|\phi(s)|/g(s))<\infty$. $C_0=(C,\|\cdot\|)$ is the space of bounded continuous functions with the sup norm, $\|\phi\|=\sup_{-h\leqslant s\leqslant 0}|\phi(s)|$, and for $h=\infty$, $C_0=C_{g_0}$.

DEFINITION 1.1. For g, $g^\circ\in G$, $g<g^\circ$ if $g(s)\leqslant g^\circ(s)$ for $s\leqslant 0$ and $\lim_{N\to\infty}[\sup_{s\leqslant 0}(g(s)/g^\circ(s-N))]=0$.

0022-247X/92 $3.00

Remark 1.2. Note that $g_0 < g$ for all $g \in G^0$, and that for $g_i \in G^0$, there exists $g \in G^0$ with $g < g_i$, $i = 1, 2$, for instance, $g = (\min(g_1, g_2))^{1/2}$. Moreover, for exponentially growing g, e.g., $g(r) = e^{-r}$, we can have $g < g$.

We consider the functional differential equation

$$x'(t) = F(t, x_t), \tag{DE}$$

where $x_t(s) = x(t+s)$, $-h \leqslant s \leqslant 0$ and $F \in \mathscr{C}(\mathbb{R} \times C_g, \mathbb{R}^n)$ for a given $g \in G$. Certain properties of solutions, such as existence, are needed for the theorems to follow. For the infinite delay case, the question of what conditions on the phase space and F ensure such properties is a complicated one and the reader is referred to [8, 10] for a discussion. Therefore, for brevity, only continuity is asked of F and the necessary properties of solutions are hypothesized explicitly in the theorems and then verified for examples to follow.

DEFINITION 1.3. Solutions of (DE)

(i) *exist* if for each $(t_0, \phi) \in \mathbb{R} \times C_g$, there is an $\alpha > 0$ and a continuous function x: $[t_0 - h, t_0 + \alpha) \to \mathbb{R}^n$, denoted by $x(t, t_0, \phi)$ or $x(t_0, \phi)$, such that $x(t)$ satisfies (DE) on $[t_0, t_0 + \alpha)$ and $x_{t_0} = \phi$,

(ii) are *continuable if bounded* if for each $(t_0, \phi) \in \mathbb{R} \times C_g$, $x(t_0, \phi)$ is defined on $[t_0, \infty)$ unless there exists $\beta > t_0$ such that $\overline{\lim}_{t \to \beta^-} |x(t, t_0, \phi)| = \infty$,

(iii) are *unique* if for each $(t_0, \phi) \in \mathbb{R} \times C_g$ and $\alpha > 0$, $x(t)$ and $y(t)$ are solutions of (DE) on $[t_0, t_0 + \alpha)$ with $x_{t_0} = y_{t_0} = \phi$ implies $x(t) \equiv y(t)$,

(iv) are *continuous in* ϕ if for each $(t_0, \phi) \in \mathbb{R} \times C_g$ and $\varepsilon, \beta > 0$, there exists a $\delta > 0$ such that $[\psi \in C_g, |\phi - \psi|_g < \delta]$ implies $|x_{t_0+\beta}(t_0, \phi) - x_{t_0+\beta}(t_0, \psi)|_g < \varepsilon$ for any $x(t_0, \phi)$, $x(t_0, \psi)$ defined on $[t_0, t_0 + \beta]$.

(v) If solutions satisfy (i)–(iv) for some $g \in G$, then (DE) is *g-well posed.*

Since Liapunov methods are used throughout, we defined a Liapunov functional and its derivative along a solution of (DE).

DEFINITION 1.4. A *Liapunov functional* is a continuous scalar functional V: $\mathbb{R} \times C_g \to [0, \infty)$, which for each $(t, \phi) \in \mathbb{R} \times C_g$ has a derivative along a solution $x(t, \phi)$ defined by

$$V'(t, \phi) = \overline{\lim}_{h \to 0^+} \frac{1}{h} \{V(t+h, x_{t+h}(t, \phi)) - V(t, \phi)\}.$$

Such a derivative is an upper right Dini derivative and discussion of its properties and conditions necessary to ensure its existence are found in [6, 11], respectively. Here, differentiability will be hypothesized for theorems and verified for examples.

Let $\mathscr{W} = \{W: \mathbb{R}^+ \to \mathbb{R}^+: W$ piecewise continuous, non-decreasing$\}$ and $N = \{\eta \in \mathscr{C}(\mathbb{R}, \mathbb{R}^+)$: there exist α, $L > 0$ such that $\int_t^{t+L} \eta(s)\, ds \geq \alpha$ for $t \in \mathbb{R}\}$, where α and L are said to *belong to* η.

2. Boundedness and Existence of Periodic Solutions

The boundedness properties to be studied here are:

Definition 2.1. Solutions of (DE) are *uniformly bounded in* C_g(g-UB) if for each $B_1 > 0$, there is a $B_2 > 0$ such that

$$(t_0, \phi) \in \mathbb{R} \times C_g \qquad \text{with} \quad |\phi|_g \leq B_1$$

implies

$$|x(t, t_0, \phi)| < B_2 \qquad \text{for} \quad t \geq t_0.$$

Definition 2.2. Solutions of (DE) are *uniformly ultimately bounded in* C_g(g-UUB) if there is a $B > 0$ and for each $B_3 > 0$, there is a $T > 0$ such that

$$(t_0, \phi) \in \mathbb{R} \times C_g \qquad \text{with} \quad |\phi|_g \leq B_3$$

implies

$$|x(t, t_0, \phi) < B \qquad \text{for} \quad t \geq t_0 + T.$$

In case $g = g_0$ or $h < \infty$, we simply say UB and UUB.

The following theorem, which may be found in [1, 2], demonstrates the strong connection between such boundedness properties and existence of periodic solutions for (DE).

Theorem 2.3. *Suppose there is a* $g \in G^0$ *such that* (DE) *is* g-*well posed,* F *is completely continuous in* $\mathbb{R} \times C_g$ *and* $F(t + \omega, \phi) = F(t, \phi)$ *for some* $\omega > 0$. *If, in addition, solutions of* (DE) *are* g-UB, UUB, *then there is a periodic solution with period* ω.

Therefore, under reasonable conditions on infinite delay FDEs, establishing g-UB, UUB is tantamount to establishing existence of a periodic solution.

Useful criteria exist for UB, UUB in ODEs [2, 11], however, as Hale notes [7, p. 139], analogous results are scarce for FDEs, particularly the infinite delay case. Some typical FDE results can be found in [2, 3, 7] for the finite delay case and [2, 5] for the infinite delay. Here, we will develop an idea introduced by Yoshizawa [11, p. 202] for finite delay:

THEOREM 2.4. *Suppose that $h<\infty$ and there exist a Liapunov functional V, functions $W_i \in \mathscr{W}$ and a constant $U>0$ such that in $\mathbb{R}\times C_0$*

(i) $W_1(|\phi(0)|) \leqslant V(t, \phi) \leqslant W_2(|\phi(0)|) + W_3(\|\phi\|)$,

(ii) $V'(t, \phi) \leqslant 0$ *whenever* $|\phi(0)| > U$,

(iii) $W_1(r) - W_3(r) \to \infty$ *as* $r \to \infty$.

Then solutions of (DE) *are* UB.

Thus, boundedness properties of solutions can follow from 2.4(i), (ii) if W_1 dominates W_3 on a neighborhood of infinity, a result which is also suggested for the $h=\infty$ case in C_0 by the conditions for UB, UUB given in [5]. Using this idea, we will extend Theorem 2.4 to g-UB and g-UUB.

3. BOUNDEDNESS RESULTS

THEOREM 3.1. *Suppose that for some $g^\circ \in G$, solutions of* (DE) *satisfy* 1.3(i), (ii) *and there exist a Liapunov functional V, functions $W_i \in \mathscr{W}$, $\eta \in N$, and constants U, r_0, $\beta>0$, $M \geqslant 0$ such that in $\mathbb{R}\times C_{g^\circ}$*

(i) $W_1(|\phi(0)|) \leqslant V(t, \phi) \leqslant W_2(|\phi(0)|) + W_3(|\phi|_{g^\circ})$,

(ii) $V'(t, \phi) \leqslant -\eta(t)\, W_5(|\phi(0)|) + M$,

(iii) $W_1(r) \to \infty$ *as* $r \to \infty$ *and* $\delta \overset{\text{def}}{=} \alpha W_5(U) - ML \geqslant 0$, *where* α, L *belong to* η,

(iv) $W_1(r) > \beta + ML + W_2(U) + W_3(r)$ *for* $r \geqslant r_0$.

Then, whenever $\delta \geqslant 0$, solutions of (DE) *are g-UB for any $g \in G$ with $g \leqslant g^\circ$, and whenever $\delta>0$, solutions of* (DE) *are g-UUB for any $g \in G$ with $g < g^\circ$.*

Remark 3.2. $W_1(r) - W_3(r) \to \infty$ as $r \to \infty$ suffices for 3.1(iv), and if $W_5(r)$ is unbounded, we can always choose U so that $\delta > 0$.

Remark 3.3. When $g^\circ = g_0$, the UUB conclusion of the theorem is vacuous, but when $g^\circ \in G^0$, it asserts that solutions are g-UB, UUB for some $g \in G^0$, since by Remark 1.2, there is a $g \in G^0$ with $g < g^\circ$. Moreover, if g° is exponentially growing, for instance, $g^\circ(r)=e^{-r}$, solutions are g°-UB, UUB since $g^\circ < g^\circ$.

For the finite delay case, we obtain the following result.

COROLLARY 3.4. *Suppose that* $h<\infty$, *solutions of* (DE) *satisfy* 1.3(i), (ii) *in* C_0, *and there exist a Liapunov functional* V, *functions* $W_i \in \mathscr{W}$, $\eta \in N$ *and constants* U, r_0, $\beta>0$, $M \geqslant 0$ *such that in* $\mathbb{R}\times C_0$

(i) $W_1(|\phi(0)|) \leqslant V(t,\phi) \leqslant W_2(|\phi(0)|)+W_3(\|\phi\|)$,

(ii) $V'(t,\phi) \leqslant -\eta(t)\, W_5(|\phi(0)|)+M$,

(iii) $W_1(r)\to\infty$ *as* $r\to\infty$ *and* $\delta \overset{\text{def}}{=} \alpha W_5(U)-ML \geqslant 0$, *where* α, L *belong to* η,

(iv) $W_1(r)>\beta+ML+W_2(U)+W_3(r)$ *for* $r\geqslant r_0$.

Then, solutions of (DE) *are* UB *whenever* $\delta\geqslant 0$ *and* UUB *whenever* $\delta>0$.

Proof. The proof of Theorem 3.1 suffices if we replace (4) by $W_2(B_2)+W_3(B_2)-N_1\delta<0$, $N_1>h$ and note that (5) is always satisfied when $h<\infty$. ∎

In applications, we often encounter a V' which can be shown to be negative definite for sufficiently large $|\phi(0)|$, but cannot be shown to satisfy (ii) of Theorem 3.1. For such cases, the following corollaries are useful.

COROLLARY 3.5. *Suppose that for some* $g^\circ\in G$, *solutions of* (DE) *satisfy* 1.3(i), (ii) *and there exist a Liapunov functional* V, *functions* $W_i\in\mathscr{W}$ *and constants* U, r_0, $\beta>0$, M, $\delta\geqslant 0$ *such that in* $\mathbb{R}\times C_{g^\circ}$

(i) $W_1(|\phi(0)|)\leqslant V(t,\phi)\leqslant W_2(|\phi(0)|)+W_3(|\phi|_{g^\circ})$,

(ii) $V'(t,\phi)\leqslant M$, *and* $V'(t,\phi)\leqslant -\delta$ *whenever* $|\phi(0)|\geqslant U$,

(iii) $W_1(r)\to\infty$ *as* $r\to\infty$,

(iv) $W_1(r)>\beta+W_2(U)+W_3(r)$ *for* $r\geqslant r_0$.

Then, the conclusions of Theorem 3.1 *hold.*

Proof. With the choices $\eta(t)\equiv 1$, $\alpha=L=\beta/2M$, $W_5(r)=\{\delta+M$ for $r\geqslant U$, 0 for $0\leqslant r<U\}$, all conditions of Theorem 3.1 are satisfied. ∎

COROLLARY 3.6. *Suppose that* $h<\infty$, *solutions of* (DE) *satisfy* 1.3(i), (ii) *in* C_0, *and there exist a Liapunov functional* V, *functions* $W_i\in\mathscr{W}$ *and constants* U, r_0, $\beta>0$, M, $\delta\geqslant 0$ *such that in* $\mathbb{R}\times C_0$

(i) $W_1(|\phi(0)|)\leqslant V(t,\phi)\leqslant W_2(|\phi(0)|)+W_3(\|\phi\|)$,

(ii) $V'(t,\phi)\leqslant M$, *and* $V'(t,\phi)\leqslant -\delta$ *whenever* $|\phi(0)|\geqslant U$,

(iii) $W_1(r) \to \infty$ *as* $r \to \infty$,

(iv) $W_1(r) > \beta + W_2(U) + W_3(r)$ *for* $r \geqslant r_0$.

Then, the conclusions of Corollary 3.4 *hold.*

Proof of Theorem 3.1. For the uniform boundedness, let $\delta \geqslant 0$ and fix $g \in G$ with $g \leqslant g^\circ$. Since $C_g \subseteq C_{g^\circ}$, $|\phi|_{g^\circ} \leqslant |\phi|_g$, 3.1(i) yields

$$V(t, \phi) \leqslant W_2(|\phi(0)|) + W_3(|\phi|_g). \tag{1}$$

Let $B_1 > 0$ be given, $B_1 > U$, $B_1 > r_0$. Define $P_1 = W_2(B_1) + W_3(B_1)$ and fix $t_0 \in \mathbb{R}$ and $\phi \in C_g$ with $|\phi|_g \leqslant B_1$. Denote $x(t, t_0, \phi)$ by $x(t)$ and $V(t, x_t)$ by $V(t)$. Suppose there is a $t_1 > t_0$ with

$$V(t_1) = P_1 + ML + 1 \quad \text{and} \quad V(s) < V(t_1) \quad \text{for all} \quad s \in [t_0, t_1). \tag{2}$$

Since $V' \leqslant M$ by 3.1(ii) and $V(t_0) \leqslant P_1$ by (1), it must be that $t_1 > t_0 + L$. Suppose that $|x(s)| > U$ for all $s \in [t_1 - L, t_1]$.

By 3.1(ii), (iii), we have $\int_{t_1-L}^{t_1} V'(s)\, ds \leqslant -W_5(U) \int_{t_1-L}^{t_1} \eta(s)\, ds + ML \leqslant -\alpha W_5(U) + ML = -\delta \leqslant 0$, and $V(t_1) \leqslant V(t_1 - L)$, a contradiction of (2). Thus, there exists $s_0 \in [t_1 - L, t_1]$ with $|x(s_0)| \leqslant U$; moreover, $V(t_1) \leqslant V(s_0) + ML$. Suppose that $|x_{s_0}|_g > B_1$. Since $\sup_{s \leqslant t_0 - s_0}(|x(s_0 + s)|/g(s)) = \sup_{s \leqslant 0}(|x(t_0 + s)|/g(t_0 - s_0 + s)) \leqslant \sup_{s \leqslant 0}(|x(t_0 + s)|/g(s)) = |\phi|_g \leqslant B_1$, then $|x_{s_0}|_g = \sup_{t_0 - s_0 \leqslant s \leqslant 0}(|x(s_0 + s)|/g(s)) \leqslant \sup_{t_0 - s_0 \leqslant s \leqslant 0} |x(s_0 + s)| = \sup_{t_0 \leqslant s \leqslant s_0} |x(s)|$ and $|x_{s_0}|_g \leqslant |x(s_1)|$ for some $s_1 \in [t_0, s_0]$. Using (1) and (2), $W_1(|x(s_1)|) \leqslant V(s_1) \leqslant V(t_1) \leqslant V(s_0) + ML \leqslant ML + W_2(|x(s_0)|) + W_3(|x_{s_0}|_g) < ML + W_2(U) + W_3\,(|x(s_1)|)$, which by 3.1(iv) implies $|x_{s_0}|_g \leqslant |x(s_1)| < r_0 < B_1$, a contradiction. So, it must be that $|x_{s_0}|_g \leqslant B_1$ and $V(t_1) \leqslant V(s_0) + ML \leqslant ML + W_2(U) + W_3(B_1) \leqslant P_1 + ML$, a contradiction of (2). Then, there is no t as in (2). Moreover, $V(t_0) \leqslant P_1$, so combining these gives that $W_1(|x(t)|) \leqslant V(t) < P_1 + ML + 1$ for any $t \geqslant t_0$. Using 3.1(iii), let $B_2 > 0$ be such that $W_1(r) > P_1 + ML + 1$ for $r \geqslant B_2$. For this $B_2 > 0$, which depends only on B_1, we have $|x(t)| < B_2$ for $t \geqslant t_0$. Then solutions are g-UB.

Note from 1.3(ii) that for any $g \leqslant g^\circ$ and $(t_0, \phi) \in \mathbb{R} \times C_g$, the solution $x(t_0, \phi)$ exists on $[t_0, \infty)$ since it remains bounded.

For the uniform ultimate boundedness, let $\delta > 0$, $g^\circ \in G^0$ and fix $g \in G$ with $g < g^\circ$. Let $B_3 > 0$ be given. Solutions are g-UB, so there is a $B_2 > 0$ ($B_2 \geqslant B_3$) for which $[t_0 \in \mathbb{R}, \phi \in C_g, |\phi|_g \leqslant B_3]$ implies that $|x(s, t_0, \phi)| < B_2$ for all $s \geqslant t_0$, and thus that for any $t \geqslant t_0$, $|x_t|_g = \max\{\sup_{s \leqslant t_0 - t}(|x(t+s)|/g(s)), \sup_{t_0 - t \leqslant s \leqslant 0}(|x(t+s)|/g(s))\} = \max\{\sup_{s \leqslant 0}(|x(t_0 + s)|/g(s + t_0 - t)), \sup_{t_0 \leqslant s \leqslant t}(|x(s)|/g(s - t))\} \leqslant \max\{|x_{t_0}|_g, \sup_{t_0 \leqslant s \leqslant t}(B_2/g(s-t))\} \leqslant B_2$. Then from (1),

$$[t_0 \in \mathbb{R}, \phi \in C_g, |\phi|_g \leqslant B_3] \text{ implies } V(t) \leqslant W_2(B_2) + W_3(B_2) \text{ for } t \geqslant t_0. \tag{3}$$

Choose $N_i \in \mathbb{N}$ to satisfy

$$W_2(B_2)+W_3(B_2)-N_1\delta<0, \qquad \sup_{s\leqslant 0}\frac{g(s)}{g^\circ(s-N_1)}<\frac{r_0}{B_3},$$

$$\frac{B_2}{g^\circ(-N_1)}<r_0, \qquad \text{and} \qquad W_2(B_2)+W_3(B_2)-N_2\beta<0, \tag{4}$$

which is possible since $g^\circ \in G^0$ and $g<g^\circ$. Fix $t_0\in\mathbb{R}$ and $\phi\in C_g$ with $|\phi|_g \leqslant B_3$ and let $P_0=W_2(U)+W_3(r_0)$.

LEMMA. *If there is a $t \geqslant t_0+N_1L+N_1$ for which $V(t)>P_0+ML$, then there is a $t_1\in[t-N_1L-N_1, t]$ for which $V(t_1)>V(t)+\beta$.*

Proof of Lemma. Suppose that $|x(s)|>U$ for all $s\in[t-N_1L, t]$. Then $\int_{t-N_1L}^t V'(s)\,ds \leqslant -W_5(U)\int_{t-N_1L}^t \eta(s)\,ds + MN_1L \leqslant -W_5(U)N_1\alpha + MN_1L = -N_1\delta$, and by (3) and (4), $V(t)\leqslant V(t-N_1L)-N_1\delta \leqslant W_2(B_2)+W_3(B_2)-N_1\delta<0$, a contradiction. Thus, there is an $s\in[t-N_1L, t]$ with $|x(s)|\leqslant U$. Let s_0 be the largest such, so that for some $j\in\{0, ..., N_1-1\}$ we have $s_0\in[t-(j+1)L, t-jL]$ and $|x(s)|\geqslant U$ for all $s\in[t-jL, t]$. Then $\int_{s_0}^t V'(s)\,ds = \int_{s_0}^{t-jL} V'(s)\,ds + \int_{t-jL}^t V'(s)\,ds \leqslant M[t-jL-s_0] - W_5(U)\int_{t-jL}^t n(s)\,ds + jLM \leqslant ML - j\delta \leqslant ML$, and by the hypothesis on t, $W_2(U)+W_3(r_0)+ML < V(t) \leqslant V(s_0)+ML \leqslant W_2(|x(s_0)|)+W_3(|x_{s_0}|_{g^\circ})+ML$, which implies that $|x_{s_0}|_{g^\circ}>r_0$. Then since $t_0-s_0\leqslant -N_1$ and (4) imply that

$$\begin{aligned}\sup_{s\leqslant -N_1}\frac{|x(s_0+s)|}{g^\circ(s)} &= \max\left\{\sup_{s\leqslant t_0-s_0}\frac{|x(s_0+s)|}{g^\circ(s)}, \sup_{t_0-s_0\leqslant s\leqslant -N_1}\frac{|x(s_0+s)|}{g^\circ(s)}\right\}\\ &= \max\left\{\sup_{s\leqslant 0}\frac{|x(t_0+s)|}{g^\circ(t_0-s_0+s)}, \sup_{t_0\leqslant s\leqslant s_0-N_1}\frac{|x(s)|}{g^\circ(s-s_0)}\right\}\\ &\leqslant \max\left\{\sup_{s\leqslant 0}\frac{|\phi(s)|}{g^\circ(s-N_1)}, \frac{B_2}{g^\circ(-N_1)}\right\}\\ &\leqslant \max\left\{\sup_{s\leqslant 0}\frac{B_3\,g(s)}{g^\circ(s-N_1)}, \frac{B_2}{g^\circ(-N_1)}\right\}<r_0,\end{aligned}$$

we must have that

$$|x_{s_0}|_{g^\circ}=\sup_{-N_1\leqslant s\leqslant 0}\frac{|x(s_0+s)|}{g^\circ(s)}\leqslant \sup_{s_0-N_1\leqslant s\leqslant s_0}|x(s)|.$$

Thus, there exists t_1 such that

$$t_1\in[s_0-N_1, s_0] \qquad \text{and} \qquad |x_{s_0}|_{g^\circ}\leqslant |x(t_1)|. \tag{5}$$

Note that $V(t_1) \leqslant V(t) + \beta$ implies that $W_1(|x(t_1)|) \leqslant V(t_1) \leqslant V(t) + \beta \leqslant V(s_0) + ML + \beta \leqslant W_2(|x(s_0)|) + W_3(|x_{s_0}|_{g^\circ}) + ML + \beta \leqslant \beta + ML + W_2(U) + W_3(|x(t_1)|)$, which by 3.1(iv) and (5) implies that $|x_{s_0}|_{g^\circ} \leqslant |x(t_1)| < r_0$, a contradiction. Then, $V(t_1) > V(t) + \beta$ and moreover, $t_1 \in [t - N_1 L - N_1, t]$. ∎

To finish the proof of the theorem, suppose there is a t with

$$t \geqslant t_0 + N_2[N_1 L + N_1] \qquad \text{and} \qquad V(t) > P_0 + ML. \tag{6}$$

Invoking the lemma N_2 times gives a $t_{N_2} \geqslant t_0$ with $V(t_{N_2}) > P_0 + ML + N_2\beta$, which implies by (3) and (4) that $P_0 + ML < V(t_{N_2}) - N_2\beta \leqslant W_2(B_2) + W_3(B_2) - N_2\beta < 0$, a contradiction. Then there is no t as in (6), and with $T = N_2(N_1 L + N_1)$, we have that $W_1(|x(t)|) \leqslant V(t) \leqslant P_0 + ML$ for $t \geqslant t_0 + T$. Let $B > 0$ be such that $W_1(r) > P_0 + ML$ for $r \geqslant B$. Then, $|x(t)| < B$ for $t \geqslant t_0 + T$. By the construction, B is independent of B_3, t_0, ϕ, and T depends only on B_3. Thus, solutions are g-UUB. ∎

4. Applications

In order to use Theorem 3.1 together with Theorem 2.3 to prove existence of periodic solutions, we must find some $g \in G^0$ for which (DE) and its solutions satisfy all the conditions there. Since, in applications, no g would be specified by the functional form of the FDE given, the required C_g space must somehow be produced. Using the following lemma, which extends a result in [4], we will illustrate how this can be done for a class of integrodifferential equations.

Lemma 4.1. *Let $W \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+)$ be increasing and unbounded and for $\Omega = (t, s)$: $-\infty < s \leqslant t < \infty$, let $C \in \mathscr{C}(\Omega, \mathbb{R}^+)$ satisfy $\sup_t \int_{-\infty}^{t} C(t, s)\, ds = J < \infty$. Then*

(i) *for each $\varepsilon > 0$, there exists $g \in G^0$ such that*

$$\sup_t \int_{-\infty}^{t} C(t, s)\, W(g(s-t))\, ds < W(1)J + \varepsilon,$$

if and only if

(ii) $\lim_{T \to \infty} [\sup_t \int_{-\infty}^{t-T} C(t, s)\, ds] = 0.$

Proof. ($\Leftarrow$) Fix $\varepsilon > 0$. Find $\gamma > 0$ with $e^{-\gamma} < \varepsilon / W(1)J$ and using (ii), define a sequence $\{r_j\}_0^\infty$ such that $r_0 = 0$, $r_j > 0$ for $j > 0$, $r_j \to \infty$ as $j \to \infty$, and $\sup_t \int_{-\infty}^{t-r_j} C(t, s)\, ds < J/(e^{1+2\gamma})\, j!$, $j = 1, 2, \ldots$. Define g^* by $g^*(r) = W^{-1}[W(1)[1+\gamma]^j]$ for $r \in (-r_{j+1}, r_j]$, $j = 0, 1, \ldots$. We can construct a

$g \in G^0$ with $g(r) \leqslant g^*(r)$ for $r \leqslant 0$, and for this g, $\sup_t \int_{-\infty}^t C(t, s) W(g(s-t))\, ds \leqslant \sum_{j=0}^{\infty} \sup_t \int_{t-r_{j+1}}^{t-r_j} C(t, s) W(1)[1+\gamma]^j\, ds \leqslant W(1) J + \sum_1^{\infty} (W(1)[1+\gamma]^j/e^{1+2\gamma} j!) J < W(1) J + \varepsilon$.

($\Rightarrow$) see [9]. ∎

Let $\mathscr{K} = \{C \in \mathscr{C}(\Omega, \mathbb{R}^+): \sup_t \int_{-\infty}^t C(t, s)\, ds = J < \infty, \lim_{T \to \infty} [\sup_t \int_{-\infty}^{t-T} C(t, s)\, ds] = 0\}$, where J is said to *belong* to C, and let $\mathscr{K}^n$ be the $n \times n$ matrices $C(t, s)$ with $|C| \in \mathscr{K}$.

Remark 4.2. $\mathscr{K}$, for instance, contains any continuous, non-negative kernel which satisfies $C(t+\omega, s+\omega) = C(t, s)$ for some $\omega > 0$ and is $L^1(-\infty, 0]$ in s uniformly for $t \in [-\omega, 0]$; in particular, any continuous $L^1[0, \infty)$ convolution kernel has $|C| \in \mathscr{K}$.

Let $\mathscr{W}_1 = \{W \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+)$: increasing, unbounded, $W(ab) \leqslant W(a) W(b)\}$, $P = \{p \in \mathscr{C}(\mathbb{R}, \mathbb{R}^n): \exists M > 0 \ni |p(t)| \leqslant M$ for $t \in \mathbb{R}\}$, where in the examples to follow, M will denote the bound of an element of P, $Q = \{q \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$: there exists $W \in \mathscr{W}_1$ with $|q(x)| \leqslant W(|x|)\}$, and $H = \mathscr{C}_0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, where $\mathscr{C}_0$ denotes the continuous functions locally Lipschitz in x. P_ω and H_ω denote the elements which are periodic in t with period ω and $\mathscr{K}_\omega$ denotes elements with $C(t+\omega, s+\omega) = C(t, s)$.

Then consider the following integrodifferential system

$$x'(t) = h(t, x(t)) + \int_{-\infty}^{t} C(t, s)\, q(x(s))\, ds + p(t), \qquad \text{(DE)}'$$

where $h \in H$, $C \in \mathscr{K}^n$, $q \in Q$, and $p \in P$.

LEMMA 4.3. *There is a $g_1 \in G^0$ for which solutions of* (DE)$'$ *satisfy* 1.3(i)–(iii) *and $F(t, \phi)$ is completely continuous in $\mathbb{R} \times C_{g_1}$. Moreover, if $h \in H_\omega$, $p \in P_\omega$, $C \in \mathscr{K}_\omega^n$ and solutions are g_2-*UB, UUB *for some $g_2 \in G^0$, then there is an ω-periodic solution.*

Proof. The first statement is proved in [9]. For $g \in G^0$ with $g < g_i$ $i = 1, 2$, (DE) is g-well posed [9], and Theorem 2.3 applies. ∎

The following examples illustrate the use of the above results.

EXAMPLE 4.4. Consider the scalar equation

$$x'(t) = -x^n(t) + \int_{-\infty}^{t} C(t, s)\, x^m(s)\, ds + p(t), \qquad \text{(DE)}'$$

where $|C| \in \mathscr{K}$, $p \in P$, $(\operatorname{sgn} x)\, x^n > 0$ for $x \neq 0$, $m \geqslant 1$ and $n \geqslant 2$. Let $D(t, s) = \int_t^{\infty} |C(u, s)|\, du$ and suppose that $D \in \mathscr{K}$, $\sup_t D(t, t) = J_1 < \infty$, and J_2, J_3 belong to D, $|C|$, respectively. If, in addition, either $m < (n+1)/2$ or

$\{m=(n+1)/2,\ J_3<2,\ J_2<2/(n+1)\}$, then solutions of (DE)' are g-UB, UUB for some $g\in G^0$. Moreover, if $|C|\in\mathscr{K}_\omega$, $p\in P_\omega$, then (DE)' has an ω-periodic solution.

Proof. For ε: $0<\varepsilon<1$, define the Liapunov functional

$$V(t,\phi)=\frac{2}{n+\varepsilon}|\phi(0)|^{n+\varepsilon}+\int_{-\infty}^{0}\int_{t}^{\infty}|C(u,s+t)|\ |\phi(s)|^{2m}\,du\,ds.$$

Using Lemma 4.1, find $g^\circ\in G^0$ such that $\int_{-\infty}^{t}D(t,s)[g^\circ(s-t)]^{2m}\,ds<J_2+\varepsilon_1$, where ε_1 is to be specified later. Then, $\int_{-\infty}^{0}\int_{t}^{\infty}|C(u,s+t)|\ |\phi(s)|^{2m}\,du\,ds=\int_{-\infty}^{t}D(t,s)\,|\phi(s-t)|^{2m}\,ds\leqslant\sup_{s\leqslant t}[|\phi(s-t)|/g^\circ(s-t)]^{2m}$ $\int_{-\infty}^{t}D(t,s)[g^\circ(s-t)]^{2m}\,ds\leqslant(J_2+\varepsilon_1)|\phi|_{g^\circ}^{2m}$, and

$$\frac{2}{n+\varepsilon}|\phi(0)|^{n+\varepsilon}\leqslant V(t,\phi)\leqslant\frac{2}{n+\varepsilon}|\phi(0)|^{n+\varepsilon}+(J_2+\varepsilon_1)|\phi|_{g^\circ}^{2m}. \tag{7}$$

In the notation of Corollary 3.5, $W_1(r)=(2/(n+\varepsilon))\,r^{n+\varepsilon}$ and $W_3(r)=(J_2+\varepsilon_1)\,r^{2m}$. Differentiating along solutions yields

$$\begin{aligned}V'(t,x_t)\leqslant{}&-2|x(t)|^{2n+\varepsilon-1}+2\int_{-\infty}^{t}|C(t,s)|\\&\times\{\tfrac{1}{2}[|x(s)|^{2m}+|x(t)|^{2n+2\varepsilon-2}]\}\,ds\\&+2M|x(t)|^{n+\varepsilon-1}+D(t,t)\,|x(t)|^{2m}\\&-\int_{-\infty}^{t}|C(t,s)|\,|x(s)|^{2m}\,ds,\end{aligned}$$

and

$$\begin{aligned}V'(t)\leqslant{}&-2|x(t)|^{2n+\varepsilon-1}+J_3|x(t)|^{2n+2\varepsilon-2}\\&+2M|x(t)|^{n+\varepsilon-1}+J_1|x(t)|^{2m}.\end{aligned} \tag{8}$$

For the $m<(n+1)/2$ case, take $\varepsilon_1=1$ and $\varepsilon>0$ with $2m-n<\varepsilon<1$. Then $n+\varepsilon>2m$ and

$$W_1(r)-W_3(r)\to\infty. \tag{9}$$

Since the negative term in (8) is of highest power, there exist δ, U, $M'>0$ such that

$$V'(t,\phi)\leqslant M' \quad\text{and}\quad V'(t,\phi)\leqslant-\delta \quad\text{whenever}\quad |\phi(0)|>U. \tag{10}$$

For the $m=(n+1)/2$ case, take $\varepsilon=1$ and $\varepsilon_1: 0<\varepsilon_1<2/(n+1)-J_2$. A calculation shows that (9) and (10) still hold. Then by (7), (9), (10), and Lemma 4.3, the conditions of Corollary 3.5 are satisfied and solutions are g-UB, UUB for some $g \in G^0$. The existence of an ω-periodic solution when $p \in P_\omega$, $|C| \in \mathscr{K}_\omega$ now follows from Lemma 4.3. ∎

EXAMPLE 4.5. The conclusions of Example 4.4 can be shown to hold also for the linear $n = m = 1$ case using

$$V(t, \phi) = |\phi(0)| + \int_{-\infty}^{0} \int_{t}^{\infty} |C(u, s+t)|\, |\phi(s)|\, ds\, du, \qquad \text{where} \qquad J_1, J_2 < 1.$$

EXAMPLE 4.6. Consider the pair of scalar equations

$$\begin{aligned} x'(t) &= x(t)[a - bx(t) - cy(t)] + p_1(t) \\ y'(t) &= y(t)\left[-d + \int_{-\infty}^{t} C(t, s)\, f(x(s))\, ds\right] + p_2(t). \end{aligned} \tag{DE)$'$}$$

This is the Lotka–Volterra predator–prey model and is of current interest in mathematical biology. It is known that with the conditions below, solutions starting in the positive quadrant remain there. Therefore, to confine our attention to the region of physical interest, we consider only those initial functions in C_g which map into the positive quadrant in $\mathbb{R}^2$. Suppose that $C \in \mathscr{K}_\omega$ and the $p_i \in P_\omega$ are positive valued. Define $D(t, s)$ as in Example 4.4 and suppose $D \in \mathscr{K}$, J_2 belongs to D, $\sup_t D(t, t) = J_1 < \infty$, $a, b, c, d > 0$,

$$-\frac{1}{2}a + \left[1 + \frac{a}{d}\right] M \stackrel{\text{def}}{=} -\delta < 0, \; \beta \stackrel{\text{def}}{=} \frac{a}{d} J_1, \text{ and } -b + \beta \stackrel{\text{def}}{=} -\gamma < 0.$$

If, in addition, either

$$f \in \mathscr{W}_1, f(r) \leqslant r, \quad f \text{ Lipschitz} \quad \text{and} \quad \frac{\ln(1+r)}{f(r)} \to \infty \quad \text{as} \quad r \to \infty, \tag{11}$$

or

$$f(r) = \ln(1+r) \qquad \text{and} \qquad J_2 < \frac{d}{a} \min\left(\frac{a}{d}, 1\right), \tag{12}$$

then solutions of (DE)$'$ are g-UB, UUB for some $g \in G^0$, and there is an ω-periodic solution which is not identically constant if the p_i are not.

Proof. For the system $z' = F(t, z_t)$, where $z = (x, y)$, differentiating the Liapunov functional

$$V(t, x, y) = \ln(x+1) + \frac{a}{d}\ln(y+1) + \frac{a}{d}\int_{-\infty}^{t} \int_{t}^{\infty} |C(u, s)|\, f(x(s))\, du\, ds$$

along solutions yields $V'(t, x, y) \leqslant (x/(x+1))[a - \gamma x - cy + \beta] - (ay/(y+1)) + M(1 + a/d)$, and with $U = \max\{2, 2(2a+\beta)/\min(\gamma, c)\}$ $M' = a + \beta + M[1 + a/d]$,

$$V'(t, \phi) \leqslant M' \qquad \text{and} \qquad V'(t, \phi) \leqslant -\delta \qquad \text{for} \quad |\phi(0)| > U.$$

In case (11) holds, find $g^\circ \in G^0$ such that $\sup_t \int_{-\infty}^t D(t, s) f(g^\circ(s-t))\, ds < f(1) J_2 + 1$. Then, proceeding as in the last example yields

$$\min\left(\frac{a}{d}, 1\right) \ln(1 + |\phi(0)|)$$
$$\leqslant V(t, \phi) \leqslant \left(1 + \frac{a}{d}\right) \ln(1 + |\phi(0)|) + \frac{a}{d}(f(1) J_2 + 1) f(|\phi|_{g^\circ}),$$

and for $W_1(r) = \min((a/d), 1) \ln(1 + r)$, $W_3(r) = (a/d)[f(1) J_2 + 1] f(r)$, we have $W_1(r) - W_3(r) \to \infty$ as $r \to \infty$. In case (12) holds, let $g^\circ \in G^0$ be such that

$$\sup_t \int_{-\infty}^t D(t, s) \ln(1 + g^\circ(s-t))\, ds < (\ln 2) J_2 + 1.$$

Then

$$(a/d) \int_{-\infty}^t D(t, s) f(x(s))\, ds$$
$$\leqslant \frac{a}{d} \int_{-\infty}^t D(t, s) \ln\left[\frac{1 + |z(s)|}{1 + g^\circ(s-t)}\right] ds$$
$$+ \frac{a}{d} \int_{-\infty}^t D(t, s) \ln(1 + g^\circ(s-t))\, ds$$
$$\leqslant \frac{a}{d} \sup_{s \leqslant t} \ln\left[\frac{1 + |z(t)|}{1 + g^\circ(s-t)}\right] \int_{-\infty}^t D(t, s)\, ds + \frac{a}{d}[J_2 \ln 2 + 1]$$
$$\leqslant \frac{a}{d}[J_2 \ln 2 + 1] + \frac{a}{d} J_2 \ln(1 + |z_t|_{g^\circ}),$$

and

$$\min\left(\frac{a}{d}, 1\right) \ln(1 + |\phi(0)|)$$
$$\leqslant V(t, \phi) \leqslant \left(1 + \frac{a}{d}\right) \ln(1 + |\phi(0)|) + \frac{a}{d}[J_2 \ln 2 + 1]$$
$$+ \frac{a}{d} J_2 \ln(1 + |\phi|_{g^\circ}),$$

so that $W_2(r) = (1 + a/d)\ln(1 + r) + (a/d)[J_2 \ln 2 + 1]$ and $W_1(r) - W_3(r) = [\min(a/d, 1) - (a/d) J_2] r \to \infty$ as $r \to \infty$. Then in both cases, the conditions of Corollary 3.5 are satisfied and Lemma 4.3 gives an ω-periodic solution which, by inspection is not constant if the p_i are not. ∎

EXAMPLE 4.7. Consider the scalar equation

$$x'(t) = -a(t)\, x(t) + b(t)\, x(t - r(t)) + p(t), \qquad \text{(DE)}'$$

where $p \in P$, $a \in \mathscr{C}(\mathbb{R}, \mathbb{R}^+)$, $b \in \mathscr{C}(\mathbb{R}, \mathbb{R})$, $r \in \mathscr{C}^1(\mathbb{R}, \mathbb{R}^+)$, $0 \leqslant r(t) \leqslant t$ and $r(t) \to \infty$ as $t \to \infty$. Suppose that there is a $d \in \mathscr{C}^1(\mathbb{R}, \mathbb{R}^+)$ such that $\int_0^\infty d(s)\, ds = L_1 < \infty$ and for $t \in \mathbb{R}$, $|b(t)| \leqslant d(t)$, $d'(t) \leqslant 0$,

$$|b(t)| - \frac{1}{2L_1}(1 - r'(t))\, d(r(t)) \leqslant 0 \qquad \text{and} \qquad a(t) - \frac{d(0)}{2L_1} \stackrel{\text{def}}{=} \eta(t) \in N.$$

Then solutions of (DE)′ are g-UB, UUB for some $g \in G^0$.

Proof. Since $d \in L^1[0, \infty)$, then $d(t - s) \in \mathscr{K}$ and there is a $g^\circ \in G^0$ with $\sup_t \int_{-\infty}^t d(t - s)\, g^\circ(s - t)\, ds < 3L_1/2$. For the Liapunov functional $V(t, \phi) = |\phi(0)| + (1/2L_1) \int_{-r(t)}^0 d(-s)\, |\phi(s)|\, ds$, we have $|\phi(0)| \leqslant V(t, \phi) \leqslant |\phi(0)| + \frac{3}{4} |\phi|_{g^\circ}$ and $W_1(r) - W_3(r) = \frac{1}{4} r \to \infty$ as $r \to \infty$. Along solutions

$$V'(t, x_t) \leqslant -\left[a(t) - \frac{d(0)}{2L_1}\right] |x(t)| + \left[|b(t)| - \frac{1}{2L_1}(1 - r'(t))\, d(r(t))\right].$$

$|x(t - r(t))| + (1/2L_1) \int_{t-r(t)}^t d'(t - s)\, |x(s)|\, ds + M$, so that $V'(t, \phi) \leqslant -\eta(t)\, |\phi(0)| + M$. Then Theorem 3.1 applies. ∎

EXAMPLE 4.8. Consider the scalar finite delay equation

$$x'(t) = -a(t)\, x(t) + b_1(t) \int_{t-h}^t b_2(s)\, x(s)\, ds$$
$$+ b_3(t)\, x(t - h) + p(t), \qquad \text{(DE)}'$$

where $h < \infty$, $a \in \mathscr{C}(\mathbb{R}, \mathbb{R}^+)$, $b \in \mathscr{C}(\mathbb{R}, \mathbb{R})$, $p \in P$. Suppose that

$$a(t) - |b_2(t)| \int_t^{t+h} |b_1(s)|\, ds - |b_3(t)| \stackrel{\text{def}}{=} \eta(t) \in N,$$

$$\sup_t \{|b_3(t)| - |b_3(t - h)|\} \leqslant 0$$

and

$$\sup_t \left\{ \left[\int_{t-h}^{t} |b_2(u)|\, du \right] \left[\int_{t-h}^{t+h} |b_1(s)|\, ds \right] + \int_{t-h}^{t} |b_3(s)|\, ds \right\} = \gamma < 1.$$

Then solutions of (DE)′ are UB, UUB. If in addition a, b_i, p are ω-periodic, then there is an ω-periodic solution.

Proof. For the Liapunov functional

$$V(t, \phi) = |\phi(0)| + \int_{-h}^{0} \int_{s}^{0} |b_2(u+t)|\, |b_1(u+t-s)|\, |\phi(u)|\, du\, ds$$

$$+ \int_{-h}^{0} |b_3(s+t)|\, |\phi(s)|\, ds,$$

we have $|\phi(0)| \leqslant V(t, \phi) \leqslant |\phi(0)| + \gamma \|\phi\|$, so that $W_1(r) - W_3(r) = (1-\gamma) r \to \infty$ as $r \to \infty$. Moreover,

$$V'(t, x_t) \leqslant -|x(t)| \left[a(t) - |b_2(t)| \int_{t}^{t+h} |b_1(s)|\, ds - |b_3(t)| \right]$$

$$+ [|b_3(t)| - |b_3(t-h)|]\, |x(t-h)| + M,$$

so that $V'(t, \phi) \leqslant -\eta(t)|\phi(0)| + M$. Then, Corollary 3.4 applies. The existence of the periodic solution follows from observing that for finite delay, Theorem 2.3 also applies in case the initial function space is C_0 [2]. ∎

Remark 4.9. The following are concrete instances of Examples 4.4, 4.7, 4.8, respectively:

$$x'(t) = -x^3(t) + \frac{1}{2} \int_{-\infty}^{t} \frac{x^2(s)}{(1+t-s)^3}\, ds + \cos t \tag{13}$$

$$x'(t) = -\frac{3}{2} x(t) + \frac{\sin t}{2(1+t)^3} x(t/2) + \sin t, \tag{14}$$

$$x'(t) = -\frac{1}{2} x(t) + \frac{1}{4} \int_{t-1}^{t} |\sin s|\, x(s)\, ds + \frac{1}{4} x(t-1) + \sin t. \tag{15}$$

Then, (13), (14) have solutions g-UB, UUB, for some $g \in G^0$, (15) has solutions UB, UUB, and (13) and (15) have 2π-periodic solutions.

References

1. O. A. Arino, T. A. Burton, and J. R. Haddock, Periodic solutions to functional differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **101A** (1985), 253–271.

2. T. A. BURTON, "Stability and Periodic Solutions of Ordinary and Functional Differential Equations," pp. 228, 272–277, 297–308, 317–322, Academic Press, Orlando, FL, 1985.
3. T. A. BURTON, Uniform boundedness for delay equations, *Acta. Math. Hungar.* **56** (1990), 259–268.
4. T. A. BURTON AND R. C. GRIMMER, Oscillation, continuation and uniqueness of solutions of retarded differential equations, *Trans. Amer. Math. Soc.* **179** (1973), 193–209.
5. T. A. BURTON AND R. H. HERING, Boundedness in infinite delay systems, *J. Math. Anal. Appl.* **144** (1989), 486–502.
6. P. HABETS, M. LALOY, AND N. ROUCH, "Stability Theory by Liapunov's Direct Method," p. 345, Springer-Verlag, New York, 1977.
7. J. K. HALE, "Theory of Functional Differential Equations," pp. 133–139, Springer-Verlag, New York, 1977.
8. J. K. HALE AND J. KATO, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11–41.
9. R. H. HERING, "Boundedness and Stability in Functional Differential Equations, dissertation," Southern Illinois University, Carbondale, IL, 1988.
10. F. KAPPEL AND W. SCHAPPACHER, Some considerations to the fundamental theory of infinite delay equations, *J. Differential Equations* **37** (1980), 141–183.
11. T. YOSHIZAWA, "Stability Theory by Liapunov's Second Method," Sects. 10, 32, 35, Math. Soc. Japan, Tokyo, 1966.