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A CLASS OF L^1 -CONVERGENCE

BY

R. BOJANIC AND Č. V. STANOJEVIĆ

ABSTRACT. It is proved that if the Fourier coefficients $\{a_n\}$ of $f \in L^1(0, \pi)$ satisfy $(*)$ $n^{-1} \sum_{k=n}^{2n} k^p |\Delta a_k| = o(1)$, for some $1 < p < 2$, then $\|s_n - f\| = o(1)$, if and only if $a_n \lg n = o(1)$. For cosine trigonometric series with coefficients of bounded variation and satisfying $(*)$ it is proved that a necessary and sufficient condition for the series to be a Fourier series is $\{a_n\} \in \mathcal{C}$, where \mathcal{C} is the Garrett-Stanojević [4] class.

1. Introduction. Let f be a 2π -periodic and even function in $L^1(0, \pi)$, and let $\{a_k\}$ be the sequence of its Fourier coefficients. Denote by \mathcal{F} the class of sequences of Fourier coefficients of all such functions. It is well known that, in general, it does not follow from $\{a_k\} \in \mathcal{F}$ that

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx$$

converges to f in the $L^1(0, \pi)$ -norm, i.e. it does not follow that $\|s_n - f\| = o(1)$, $n \rightarrow \infty$, where $\|\cdot\|$ is the $L^1(0, \pi)$ -norm.

However, there are examples of subclasses of \mathcal{F} for which $a_n \lg n = o(1)$, $n \rightarrow \infty$, is a necessary and sufficient condition for $\|s_n - f\| = o(1)$, $n \rightarrow \infty$.

DEFINITION 1.1. A subclass \mathcal{K} of \mathcal{F} is called a class of L^1 -convergence if

(1.1) $\|s_n - f\| = o(1)$, $n \rightarrow \infty$, is equivalent with $a_n \lg n = o(1)$, $n \rightarrow \infty$.

There are three classical examples of classes of L^1 -convergence. The first one is due to Young [1] and \mathcal{K} is defined to be the class of all convex sequences $\{a_k\}$ ($\Delta^2 a_k > 0$). The second one is the class of all sequences $\{a_k\}$ such that $\sum_{k=1}^{\infty} k |\Delta^2 a_k| < \infty$, introduced by Kolmogorov [2]. The latter one is an obvious extension of the first one, and both are subclasses of $\mathcal{B}\nabla$, the class of all null-sequences of bounded variation.

The third example is the Telyakovskii [3] class \mathcal{S} . A sequence $\{a_k\}$ belongs to \mathcal{S} if there exists a monotone sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \leq A_k$, for all k . Clearly $\mathcal{S} \subset \mathcal{B}\nabla$.

Garrett and Stanojević [4] introduced the following class \mathcal{C} . A null-sequence $\{a_k\}$ belongs to the class \mathcal{C} if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n , and such that

$$C_n(\delta) = \frac{1}{\pi} \int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon,$$

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for all n , where D_k is the Dirichlet kernel. As a corollary to their main result in [4], they proved that $\mathcal{C} \cap \mathcal{B}\mathcal{V}$ is a class of L^1 -convergence. Later, Garrett, Rees and Stanojević [5] proved that $\mathcal{S} \subset \mathcal{C} \cap \mathcal{B}\mathcal{V}$.

Recently Fomin [6] extended the Telyakovskii class \mathcal{S} in the following way. A sequence $\{a_k\}$ belongs to the class \mathcal{F}_p , if for some $1 < p \leq 2$,

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} < \infty.$$

Fomin [6] proved that \mathcal{F}_p is a class of L^1 -convergence and that $\mathcal{S} \subset \mathcal{F}_p$.

On the other hand Stanojević [7] proved that $\mathcal{F}_p \subset \mathcal{C} \cap \mathcal{B}\mathcal{V}$, and obtained several new classes of L^1 -convergence. A sequence $\{a_k\}$ of Fourier coefficients belongs to the class \mathcal{C}_p if, for some $1 < p \leq 2$, $\sum_{k=1}^{\infty} |\Delta a_k|^p < \infty$, and

$$(1.2) \quad n \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} = o(1), \quad n \rightarrow \infty.$$

Clearly $\{a_k\} \in \mathcal{F}_p$ implies that $\{a_k\} \in \mathcal{C}_p$. Hence the result of Stanojević [7] is stronger than the one of Fomin [6].

In [7], the class $\mathcal{B}\mathcal{V}$ is extended in the following manner.

DEFINITION 1.2. A null-sequence $\{a_k\}$ belongs to the class \mathcal{P} if

$$\frac{1}{n} \sum_{k=1}^n k |\Delta a_k| = o(1), \quad n \rightarrow \infty.$$

It is proved in [7] that if $\{a_k\} \in \mathcal{P} \cap \mathcal{F}$ and if $n\Delta a_n = O(1)$, $n \rightarrow \infty$, then (1.1) holds. The proof is based on the estimate

$$(1.3) \quad \|g_n - \sigma_n\| = O\left(\frac{1}{n} \sum_{k=1}^n k |\Delta a_k|\right), \quad n \rightarrow \infty,$$

where $g_n(x) = s_n(x) - a_{n+1}D_n(x)$, and σ_n is the Fejér's sum of s_n .

We have a threefold objective in this paper. First, we shall, within a subclass of \mathcal{P} , remove the condition $n\Delta a_n = O(1)$, $n \rightarrow \infty$. The subclass \mathcal{V}_p of \mathcal{P} is defined as follows.

DEFINITION 1.3. A sequence $\{a_k\}$ belongs to the class \mathcal{V}_p if, for some $1 < p \leq 2$,

$$(1.4) \quad \frac{1}{n} \sum_{k=1}^n k^p |\Delta a_k|^p = o(1), \quad n \rightarrow \infty.$$

Secondly we shall estimate $\|s_n - f\| - |a_n| \|D_n\|$ directly without using the estimate (1.3).

Finally we shall obtain a necessary and sufficient condition for $a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$, $\{a_n\} \in \mathcal{V}_p \cap \mathcal{B}\mathcal{V}$, to be a Fourier series.

2. Lemmas. In the proof of our main theorem, instead of the condition (1.4) we shall use an equivalent one

$$(2.1) \quad \frac{1}{n} \sum_{k=n}^{2n-1} k^p |\Delta a_k|^p = o(1), \quad n \rightarrow \infty.$$

The following lemma establishes that equivalence.

LEMMA 2.1. Let $\{c_k\}$ be a sequence of nonnegative numbers. Then

$$\alpha_n = \frac{1}{n} \sum_{k=1}^n c_k = o(1), \quad n \rightarrow \infty,$$

if and only if

$$\beta_n = \frac{1}{n} \sum_{k=n}^{2n} c_k = o(1), \quad n \rightarrow \infty.$$

PROOF. The “only if” part is obvious. For the “if” part notice that

$$2^n \beta_{2^n} = 2^{n+1} \alpha_{2^{n+1}} - 2^n \alpha_{2^n}$$

or

$$(2.2) \quad \alpha_{2^m} = \frac{\alpha_1}{2^m} + \frac{1}{2^m} \sum_{k=0}^{m-1} 2^k \beta_{2^k}.$$

For $2^m \leq n < 2^{m+1}$, we have

$$(2.3) \quad \alpha_n \leq \frac{2^{m+1}}{n} \alpha_{2^{m+1}} \leq 2 \alpha_{2^{m+1}}.$$

Hence from (2.2) and (2.3) it follows that $\beta_n \rightarrow 0$, $n \rightarrow \infty$, implies that $\alpha_n \rightarrow 0$, $n \rightarrow \infty$.

The next lemma provides an identity that we need in order to avoid using (1.3) explicitly in our proof.

LEMMA 2.2. Let

$$s_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^n a_k \cos kx$$

and let $\{\sigma_n(x)\}$ be the sequence of Fejér sums of the sequence $\{s_n(x)\}$. Then

$$(2.4) \quad \begin{aligned} s_n(x) - f(x) &= 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x)) \\ &\quad - \frac{1}{n} \sum_{k=n+1}^{2n} (2n-k) a_k \cos kx. \end{aligned}$$

The core of the proof is supplied by the following lemma.

LEMMA 2.3. Let $\{c_k\}$ be a sequence of real numbers. Then for any $1 < p \leq 2$ and $n \geq 1$

$$\frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx \leq A_p \left(\frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p},$$

where A_p is an absolute constant.

PROOF. We write

$$\begin{aligned}
 (2.5) \quad \frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx &= \frac{1}{n} \int_0^{\pi/n} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx \\
 &\quad + \frac{1}{n} \int_{\pi/n}^\pi \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx \\
 &= I_1 + I_2.
 \end{aligned}$$

Since $|D_k| \leq k + 1$, for the first integral in (2.5) we have $I_1 \leq n^{-1} \sum_{k=n}^{2n-1} |c_k|$, and by Hölder's inequality

$$I_1 \leq \left(\frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p}.$$

Let $1/p + 1/q = 1$, $p > 1$. Then by applying the Hölder inequality to the second integral in (2.5) we get

$$\begin{aligned}
 I_2 &= \frac{1}{n} \int_{\pi/n}^\pi \frac{1}{2 \sin(x/2)} \left| \sum_{k=n}^{2n-1} c_k \sin\left(k + \frac{1}{2}\right)x \right| dx \\
 &\leq \frac{1}{2n} \left(\int_{\pi/n}^\pi \frac{dx}{(2 \sin(x/2))^p} \right)^{1/p} \left(\int_{\pi/n}^\pi \left| \sum_{k=n}^{2n-1} c_k \sin\left(k + \frac{1}{2}\right)x \right|^q dx \right)^{1/q}.
 \end{aligned}$$

Since

$$\int_{\pi/n}^\pi \frac{dx}{(\sin(x/2))^p} \leq \pi^p \int_{\pi/n}^\pi \frac{dx}{x^p} \leq \frac{\pi}{p-1} n^{p-1},$$

it follows that

$$I_2 \leq \frac{1}{2} \left(\frac{\pi}{p-1} \right)^{1/p} n^{-1/p} \left(\int_0^\pi \left| \sum_{k=n}^{2n-1} c_k \sin\left(k + \frac{1}{2}\right)x \right|^q dx \right)^{1/q}.$$

Let $1 < p \leq 2$. Then using the Hausdorff-Young inequality we get

$$\begin{aligned}
 \left(\frac{1}{\pi} \int_0^\pi \left| \sum_{k=n}^{2n-1} c_k \sin\left(k + \frac{1}{2}\right)x \right|^q dx \right)^{1/q} &\leq \left(\frac{1}{2\pi} \int_{-\pi}^\pi \left| \sum_{k=n}^{2n-1} c_k e^{ikx} \right|^q dx \right)^{1/q} \\
 &\leq \left(\sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p}.
 \end{aligned}$$

Thus $I_2 \leq (\pi/2)(p-1)^{-1/p} (n^{-1} \sum_{k=n}^{2n-1} |c_k|^p)^{1/p}$. Combining all estimates we get

$$I_1 + I_2 \leq \pi \left(2 + \frac{1}{2}(p-1)^{-1/p} \right) \left(\frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p},$$

and the proof of Lemma 2.3 is completed.

3. Main result. In our main theorem we shall prove that for some $1 < p \leq 2$, the class $\mathcal{V}_p \cap \mathcal{F}$ is an L^1 -convergence class. To make use of our lemmas we shall reformulate that result in the following manner.

THEOREM 3.1. Let $f \in L^1(0, \pi)$ be an even and 2π -periodic function and for some $1 < p \leq 2$ let the sequence $\{a_k\}$ of its Fourier coefficients satisfy

$$(2.1) \quad \frac{1}{n} \sum_{k=n}^{2n} k^p |\Delta a_k|^p = o(1), \quad n \rightarrow \infty.$$

Then (1.1) holds.

PROOF. Since

$$\begin{aligned} \sum_{k=n+1}^{2n-1} (2n-k)a_k \cos kx &= \sum_{k=n+1}^{2n-1} (2n-k)a_k (D_k(x) - D_{k-1}(x)) \\ &= -na_n D_n(x) + \sum_{k=n}^{2n-1} ((2n-k)a_k - (2n-k-1)a_{k+1}) D_k(x), \end{aligned}$$

from (2.4) we get

$$\begin{aligned} (3.1) \quad s_n(x) - f(x) &= a_n D_n(x) + 2(\sigma_{2n-1}(x) - f(x)) \\ &\quad - (\sigma_{n-1}(x) - f(x)) - \frac{1}{n} \sum_{k=n}^{2n-1} a_{k+1} D_k(x) \\ &\quad + \frac{1}{n} \sum_{k=n}^{2n-1} (2n-k-1) \Delta a_k D_k(x); \end{aligned}$$

hence

$$\begin{aligned} | \|s_n - f\| - |a_n| \|D_n\| | &\leq 2\|\sigma_{2n-1} - f\| + \|\sigma_{n-1} - f\| \\ &\quad + \frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} a_{k+1} D_k(x) \right| dx \\ &\quad + \frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} (2n-k-1) \Delta a_k D_k(x) \right| dx. \end{aligned}$$

Applying Lemma 2.3 to the last two integrals we get

$$\frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} a_k D_k(x) \right| dx \leq A_p \left(\frac{1}{n} \sum_{k=n}^{2n-1} |a_k|^p \right)^{1/p}$$

and

$$\frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} (2n-k-1) \Delta a_k D_k(x) \right| dx \leq A_p \left(\frac{1}{n} \sum_{k=n}^{2n-1} k^p |\Delta a_k|^p \right)^{1/p}.$$

Since $\|D_n\| = (4/\pi^2) \lg n + O(1)$, $n \rightarrow \infty$, and since $f \in L^1(0, \pi)$ implies that $\|\sigma_n - f\| = o(1)$, $n \rightarrow \infty$, we finally get

$$| \|s_n - f\| - |a_n| \lg n | = O \left(\left(\frac{1}{n} \sum_{k=n}^{2n} k^p |\Delta a_k|^p \right)^{1/p} \right), \quad n \rightarrow \infty.$$

In view of (2.1) this completes the proof of Theorem 3.1. We have now a corollary to this theorem.

COROLLARY 3.1. *The class $\mathcal{C}_p \cap \mathfrak{F}$ is a L^1 -convergence class.*

PROOF. Since $n^{p-1} \sum_{k=n}^{2n-1} |\Delta a_k|^p = o(1)$, $n \rightarrow \infty$, implies

$$\frac{1}{n} \sum_{k=n}^{2n-1} k^p |\Delta a_k|^p = o(1), \quad n \rightarrow \infty,$$

it follows that $\mathcal{C}_p \subset \mathcal{V}_p$.

4. Necessary and sufficient integrability conditions. Let $\{a_k\} \in \mathfrak{B} \mathcal{V}$. Then

$$(4.1) \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

converges pointwise to some f in $(0, \pi]$. In order to prove that (4.1) is a Fourier series it suffices to show that $f \in L^1(0, \pi)$.

THEOREM 4.1. *Let $\{a_k\} \in \mathcal{V}_p \cap \mathfrak{B} \mathcal{V}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.*

PROOF. *The “if” part.* Garrett and Stanojević [4] proved that if $\{a_k\} \in \mathfrak{B} \mathcal{V}$, then $\|g_n - f\| = o(1)$, $n \rightarrow \infty$, if and only if $\{a_k\} \in \mathcal{C}$. Hence $\{a_k\} \in \mathcal{C}$ is always a sufficient condition for $f \in L^1(0, \pi)$ whenever $\{a_k\} \in \mathfrak{B} \mathcal{V}$.

The “only if” part. The identity (3.1) can be rewritten as

$$\begin{aligned} s_n(x) - a_{n+1} D_n(x) - f(x) &= g_n(x) - f(x) \\ &= 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x)) \\ &\quad - \frac{1}{n} \sum_{k=n}^{2n-1} a_{k+1} D_k(x) - \frac{1}{n} \sum_{k=n-1}^{2n-1} \Delta a_k D_k(x) + \frac{\Delta a_n D_n(x)}{n}. \end{aligned}$$

Assuming that $f \in L^1(0, \pi)$ and applying Lemma 2.3 we get

$$\|g_n - f\| = O\left(\left(\frac{1}{n} \sum_{k=n}^{2n} k^p |\Delta a_k|^p\right)^{1/p}\right), \quad n \rightarrow \infty.$$

Since $\{a_k\} \in \mathcal{V}_p \cap \mathfrak{B} \mathcal{V}$ we have that $\|g_n - f\| = o(1)$, $n \rightarrow \infty$, and finally that $\{a_k\} \in \mathcal{C}$. This completes the proof of Theorem 4.1.

COROLLARY 4.1. *Let $\{a_k\} \in \mathcal{C}_p \cap \mathfrak{B} \mathcal{V}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.*

PROOF. $\mathcal{C}_p \subset \mathcal{V}_p$.

Let \mathfrak{M} be a class of all monotone null-sequences $\{a_k\}$. As a consequence to Corollary 4.1 we obtain a partial answer to the classical outstanding question: Let $\{a_k\} \in \mathfrak{M}$. What are necessary and sufficient conditions for (4.1) to be a Fourier series?

COROLLARY 4.2. *Let $\{a_k\} \in \mathcal{V}_p \cap \mathfrak{M}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.*

Theorem 4.1, Corollary 4.1 and Corollary 4.2 extend the integrability classes found by Stanojević [7].

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