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Convergent Factorial Series Solutions of Linear Difference Equations*

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1. INTRODUCTION

Although the analytic theory of linear difference equations dates back to an 1885 memoir of Poincaré, continuing work essentially began around 1910 with publication of several papers of G. D. Birkhoff, R. D. Carmichael, J. Horn, and N. E. Nörlund; see, for instance, [1, 2, 4, 5]. Since that time, C. R. Adams, W. J. Trjitzinsky, and others have carried on the development of the theory; numerous contributions have been made in recent years by W. A. Harris, Jr., Y. Sibuya, and H. L. Turrittin; see, for instance, [7–9, 16–17]. The monographs of Nörlund [14] and L. M. Milne-Thomson [13] give a comprehensive account of earlier work, and references to more recent literature are given in the survey paper of Harris [8].

The similarity between difference equations and differential equations has been noted by many investigators; for instance, Birkhoff [3] pointed out that the analytic theory of linear difference equations provides a "methodological pattern" for the "essentially simpler but analogous" theory of linear differential equations. In this paper, we apply a projection method which has been useful in the treatment of analytic differential equations [6, 10] to obtain existence theorems for convergent factorial series solutions of analytic difference equations. Our setting is similar to that used by Harris [7] in the development of a Frobenius method for constructing solutions of difference systems of the form

$$(z-1)\,\Delta_{-1}y(z) = A(z)\,y(z), \tag{1.1}$$

where

$$\Delta_{-1}y(z) = y(z) - y(z-1),$$

and $A(z)$ is an $n \times n$ matrix whose elements admit convergent factorial series

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expansions. Our methods are completely different from those used in [7], and our results include existence theorems for solutions of a class of q-difference equations [2] as well as for equations of the form (1.1).

2. Preliminaries: Factorial Series

In this paper, factorial series play a role similar to that of power series in the theory of ordinary differential equations. Since computations with factorial series are neither so familiar nor so elementary as the corresponding computations with power series, we summarize results which we shall use from the theory of factorial series. Further details and additional results may be found in the monograph of Nörlund [15]. We also define here a Banach space of functions with convergent factorial series representations.

A series of the form

$$f(z) = f_0 + \sum_{k=0}^{\infty} \frac{f_{k+1}k!}{z(z+1)\cdots(z+k)} \tag{2.1}$$

is called a factorial series. The domain of convergence of the series (2.1) is a half-plane, $\{z\colon \operatorname{Re} z > \gamma\}$; γ is called the abscissa of convergence of f. The series converges absolutely in a half-plane $\{z\colon \operatorname{Re} z > \sigma\}$, where σ, the abscissa of absolute convergence, satisfies $\gamma \leqslant \sigma \leqslant \gamma + 1$. The series (2.1) represents a holomorphic function in the interior of its half-plane of convergence, with the possible exceptions of poles at the nonpositive integers.

The form of series (2.1) can be simplified by introducing the factorial function, $z^{(k)}$, defined by

$$z^{(k)} = z(z-1)(z-2)\cdots(z-k+2)(z-k+1)$$

if k is a positive integer, and

$$z^{(k)} = 1/(z-k)^{(-k)}$$

if k is a negative integer. Thus the series (2.1) can be written as

$$f(z) = f_0 + \sum_{k=0}^{\infty} f_{k+1}k!(z-1)^{(-k-1)}. \tag{2.2}$$

By a theorem of unique development [13], any function which has a factorial series expansion of the form (2.2) has a unique expansion of this form.

The transformation $(z, z+1)$. The function f defined by (2.2) also admits a representation of the form

$$f(z) = f_0 + \sum_{k=0}^{\infty} (f_1 + f_2 + \cdots + f_{k+1})\, k!z^{(-k-1)}. \tag{2.3}$$

If γ_1 denotes the abscissa of convergence of the transformed series, then $\gamma_1 \leqslant \gamma$, if $\gamma \geqslant 0$. Equation (2.3) is a special case of the transformation $(z, z + m)$; see [13].

The transformation $(z/q, z)$. Let $\sigma > 1$ be the abscissa of absolute convergence of (2.1) and let $q > 1$. Then

$$f(qz) = f_0 + \sum_{k=0}^{\infty} \frac{f_{k+1} k!}{q^{k+1} z(z + 1/q) \cdots (z + k/q)}$$

is well-defined for $\operatorname{Re} z > \gamma$, where γ represents the abscissa of convergence of (2.1). It is advantageous to express $f(qz)$ as a factorial series of the form (2.2). To this end, note that for each nonnegative integer k (see [15]),

$$(1 - t^{1/q})^k = \sum_{s=k}^{\infty} \psi_s^{(k)}(q)(1 - t)^s, \tag{2.4}$$

where

$$\psi_s^{(k)}(q) = \frac{1}{k!} \sum_{m=1}^{k} (-1)^{s+m} \binom{k}{m} \left(\frac{m}{q}\right)^{(s)}. \tag{2.5}$$

For each $\epsilon > 0$, there exists a positive constant M such that

$$0 \leqslant \psi_s^{(k)}(q) < \frac{M}{s^{1+1/q-\epsilon}}, \qquad s = 1, 2, \ldots \tag{2.6}$$

Using the binomial expansion, a straightforward calculation shows that

$$\frac{(1/q)^k k!}{z(z + 1/q) \cdots (z + k/q)} = \int_0^1 t^{z-1}(1 - t^{1/q})^k \, dt. \tag{2.7}$$

Because of the inequality (2.6), it is permissible to substitute (2.4) into the integral of (2.7) and to integrate term by term. This yields

$$\frac{(1/q)^k k!}{z(z + 1/q) \cdots (z + k/q)} = \sum_{s=k}^{\infty} \psi_s^{(k)}(q) \frac{s!}{z(z + 1) \cdots (z + s)}. \tag{2.8}$$

Thus

$$f(qz) = f_0 + \frac{1}{q} \sum_{k=0}^{\infty} f_{k+1} \sum_{s=k}^{\infty} \psi_s^{(k)}(q)\, s!(z - 1)^{(-s-1)}. \tag{2.9}$$

For $\operatorname{Re} z \geqslant \sigma$,

$$| f_{k+1} \psi_s^{(k)}(q)\, s!(z - 1)^{(-s-1)} | \leqslant | f_{k+1} | \, \psi_s^{(k)}(q)\, s!(\sigma - 1)^{(-s-1)},$$

and from (2.8) it follows that

$$\sum_{s=k}^{\infty} | f_{k+1}\psi_s^{(k)}(q)\, s!(z-1)^{(-s-1)} | \leqslant \frac{q\, | f_{k+1} |\, k!}{(q\sigma)(q\sigma+1)\cdots(q\sigma+k)}.$$

The series

$$\sum_{k=0}^{\infty} \frac{| f_{k+1} |\, k!}{q\sigma(q\sigma+1)\cdots(q\sigma+k)}$$

converges, since $q > 1$. Hence the double series

$$\sum_{k=0}^{\infty} \sum_{s=k}^{\infty} f_{k+1}\psi_s^{(k)}(q)\, s!(z-1)^{(-s-1)} \tag{2.10}$$

is absolutely convergent for $\operatorname{Re} z \geqslant \sigma$. Interchanging the order of summation in (2.9), we arrive at

$$f(qz) = f_0 + \frac{1}{q} \sum_{k=0}^{\infty} g_{k+1} k!(z-1)^{(-k-1)}, \tag{2.11}$$

where

$$g_{k+1} = f_1\psi_k^{(0)}(q) + f_2\psi_k^{(1)}(q) + \cdots + f_{k+1}\psi_k^{(k)}(q). \tag{2.12}$$

A Banach space for factorial series. Let $X(\delta)$ denote the set of all complex-valued functions which have factorial series expansions absolutely convergent for $\operatorname{Re} z \geqslant \delta > 1$. For $h \in X(\delta)$,

$$h(z) = h_0 + \sum_{k=0}^{\infty} h_{k+1} k!(z-1)^{(-k-1)}, \tag{2.13}$$

define

$$\| h \| = | h_0 | + \sum_{k=0}^{\infty} | h_{k+1} |\, k!(\delta-1)^{(-k-1)}.$$

With addition and scalar multiplication defined on $X(\delta)$ in the usual way, $(X(\delta), \| \cdot \|)$ is a Banach space isomorphic to l^1. Let $X_n(\delta)$ represent the set of all n-vector functions $f = f(z)$ whose components are elements of $X(\delta)$. For $f(z) = (f^1(z), f^2(z), \ldots, f^n(z))^T$, define

$$\| f \|_n = \sum_{j=1}^{n} \| f^j \|.$$

It follows that $(X_n(\delta), \| \cdot \|_n)$ is also a Banach space.

The product of factorial series. The product of two factorial series is also representable by a factorial series. In particular, suppose f and g are elements of $X(\delta)$. Then h, defined by $h(z) = f(z)\,g(z)$, also belongs to $X(\delta)$, and will have an expansion of the form (2.13) with coefficients satisfying the equations

$$h_0 = f_0 g_0 ,$$
$$h_k = h_k(f_0, \ldots, f_k, g_0, \ldots, g_k), \qquad k \geqslant 1.$$

Here h_k represents a function linear in each of its arguments. For our purposes, only the explicit form of h_0 is needed. The explicit form of the remaining coefficients is given by Nörlund [15]. For convenience, we shall write

$$h_k = (fg)_k .$$

Remark. A different formula for the coefficients of the product function appears in [13]. However, the formula stated there on page 295 is in error. For example, it yields the expansion

$$\left(\frac{1}{z}\right)^2 = \frac{1}{z(z+1)} .$$

3. Convergent Factorial Series Solutions

In this section, we obtain existence theorems for convergent factorial series solutions of linear difference systems. As corollaries, we obtain results for a class of q-difference equations analogous to functional-differential systems treated by Grimm and Hall [6]. The principal result is the following theorem, the proof of which uses a method developed by Harris, Sibuya and Weinberg [10] for singular differential systems.

Theorem 1. *Let $A(z)$ and $B(z)$ be $n \times n$ matrices whose elements belong to $X(\delta)$, $\delta > 1$. Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, with each d_i equal to 1 or 2, and let $q > 1$. Denote by $(z-1)^{(D)}$ the matrix $\operatorname{diag}((z-1)^{(d_1)}, \ldots, (z-1)^{(d_n)})$. For every N sufficiently large and every vector-valued function $\phi(z)$ with $(z-1)^{(D)}\,\Delta_{-1}\phi(z)$ a factorial polynomial of degree N, there exists a factorial polynomial $f(z;\phi)$ (also depending upon A, B, q, and N) of degree $N-1$ such that the linear difference system*

$$(z-1)^{(D)}\,\Delta_{-1}y(z) = A(z)\,y(z) + B(z)\,y(qz) + f(z;\phi) \tag{3.1}$$

has a factorial series solution belonging to $X_n(\delta)$. Further, f and y are linear and homogeneous in ϕ, and $(z-1)^{(D)}(y-\phi) = O((z-1)^{(-N)})$ as $z \to \infty$ in $\operatorname{Re} z \geqslant \delta$.

Proof. For N a sufficiently large positive integer, define L_N: $X_n(\delta) \to X_n(\delta)$ by

$$L_N y = g, \qquad y = (y^1, ..., y^n)^T, \qquad g = (g^1, ..., g^n)^T,$$

$$y^j(z) = y_0^j + \sum_{k=0}^{\infty} y_{k+1}^j k!(z-1)^{(-k-1)},$$

$$g^j(z) = -\sum_{k=N}^{\infty} \frac{y_{k+1}^j k!}{k+d_j}(z-d_j)^{(-k-d_j)}. \tag{3.2}$$

If $d_j = 1$, then

$$\| g^j \| = \sum_{k=N}^{\infty} \left| \frac{y_{k+1}^j}{k+1} \right| k!(\delta-1)^{(-k-1)}$$

$$\leqslant \frac{1}{N+1} \sum_{k=N}^{\infty} | y_{k+1}^j | k!(\delta-1)^{(-k-1)}$$

$$\leqslant \frac{1}{N+1} \| y^j \|. \tag{3.3}$$

If $d_j = 2$, use the transformation (2.3) on the right side of (3.2) to get

$$g^j(z) = -\sum_{k=N+1}^{\infty} \left(\frac{y_{N+1}^j}{(N+1)(N+2)} + \cdots + \frac{y_k^j}{k(k+1)} \right) k!(z-1)^{(-k-1)}. \tag{3.4}$$

Thus

$$\| g^j \| \leqslant \sum_{k=N+1}^{\infty} \left(\left| \frac{y_{N+1}^j}{(N+1)(N+2)} \right| + \cdots + \left| \frac{y_k^j}{k(k+1)} \right| \right) k!(\delta-1)^{(-k-1)}$$

$$= \sum_{k=N}^{\infty} \frac{| y_{k+1}^j |}{k+2} k!(\delta-2)^{(-k-2)},$$

$$\| g^j \| \leqslant \frac{\| y^j \|}{(N+2)(\delta-1)}. \tag{3.5}$$

Define $\hat{y}(z) = (y^1(qz), y^2(qz), ..., y^n(qz))^T$, where

$$\hat{y}^j(z) = y_0^j + \sum_{k=0}^{\infty} \frac{y_{k+1}^j k!}{qz(qz+1)\cdots(qz+k)}.$$

Use the transformation (2.11) to obtain

$$\hat{y}^j(z) = y_0^j + \frac{1}{q} \sum_{k=0}^{\infty} b_{k+1}^j k!(z-1)^{(-k-1)},$$

where

$$b_{k+1}^j = y_1^j \psi_k^{(0)}(q) + y_2^j \psi_k^{(1)}(q) + \cdots + y_{k+1}^j \psi_k^{(k)}(q).$$

For each $h \in X(\delta)$, define

$$h^*(z) = | h_0 | + \sum_{k=0}^{\infty} | h_{k+1} | k!(z-1)^{(-k-1)}.$$

By the transformation (2.11), we obtain

$$\hat{y}^{j*}(z) = | y_0^j | + \frac{1}{q} \sum_{k=0}^{\infty} c_{k+1}^j k!(z-1)^{(-k-1)},$$

where

$$c_{k+1}^j = | y_1^j | \psi_k^{(0)}(q) + | y_2^j | \psi_k^{(1)}(q) + \cdots + | y_{k+1}^j | \psi_k^{(k)}(q).$$

Since all $\psi^{(s)}(q)$ are nonnegative, $c_{k+1} \geqslant | b_{k+1} |$. Thus

$$\| \hat{y}^j \| \leqslant \hat{y}^{j*}(\delta) = y^{j*}(q\delta) \leqslant y^{j*}(\delta) = \| y^j \|. \tag{3.6}$$

Note that if the elements of the $n \times n$ matrix $A(z) = (a^{ij}(z))$ are in $X(\delta)$, then for $f \in X_n(\delta)$, $Af \in X_n(\delta)$ and

$$\| Af \|_n \leqslant n^2 \max_{i,j} \| a^{ij} f^j \|.$$

From the form of the product coefficients [15], we have

$$\| a^{ij} f^j \| \leqslant a^{ij*}(\delta)\, f^{j*}(\delta) = \| a^{ij} \| \| f^j \|.$$

Hence

$$\| Af \|_n \leqslant n^2 \max_{i,j} \| a^{ij} \| \| f \|_n . \tag{3.7}$$

Let $\phi = (\phi^1, \phi^2,..., \phi^n)^T$ be the function with components

$$\phi^j(z) = \phi_0^j + \sum_{k=0}^{N-2+d_j} k!\, \phi_{k+1}^j (z - d_j)^{(-k-1)}. \tag{3.8}$$

Note that if $(z-1)^{(D)} \Delta_{-1} \phi(z)$ is a factorial polynomial of degree N, the components ϕ^j of ϕ must be of the form (3.8).

Consider the functional equation in $X_n(\delta)$,

$$y = \phi + T_N[y], \tag{3.9}$$

where $T_N[y] = L_N(Ay + B\hat{y})$. For N chosen sufficiently large, according to (3.3) and (3.5)–(3.7), we have $\| T_N \| < 1$. Hence (3.9) has a unique solution $y \in X_n(\delta)$, where

$$y(\cdot\,; \phi) = (I - T_N)^{-1} \phi. \tag{3.10}$$

It follows from the form of the functional equation (3.9) that its factorial series solution (3.10) satisfies the linear difference system

$$(z-1)^{(D)} \Delta_{-1} y(z) = A(z)\, y(z) + B(z)\, y(qz) + f(z; \phi),$$

where

$$f(z; \phi) = (z-1)^{(D)} \Delta_{-1} \phi(z) - (Ay)_0 - \sum_{k=0}^{N-1} (Ay)_{k+1} k! (z-1)^{(-k-1)}$$

$$- (B\hat{y})_0 - \sum_{k=0}^{N-1} (B\hat{y})_{k+1} k! (z-1)^{(-k-1)}. \tag{3.11}$$

Since the coefficients of $y(z; \phi)$ are linear in the coefficients of ϕ, the coefficients of f are linear in the coefficients of ϕ also. This completes the proof of the theorem.

Corollary 1. *Let d denote the number of d_j which equal 2. Then the system*

$$(z-1)^{(D)} \Delta_{-1} y(z) = A(z)\, y(z) + B(z)\, y(qz) \tag{3.12}$$

has at least d linearly independent factorial series solutions in $X_n(\delta)$.

Proof. Factorial series solutions of (3.12) may be inferred from solutions of the determining equation

$$f(z; \phi) \equiv 0. \tag{3.13}$$

This equation represents $n(N+1)$ linear homogeneous algebraic equations in $nN + \sum_{j=1}^{n} d_j$ unknowns. Hence the corollary follows.

Remark. (i) Corollary 1 is a partial analog of a theorem of F. Lettenmeyer for ordinary differential systems; see [6].

(ii) If $d_j = 2$, $j = 1, 2, \ldots, n$, then (3.12) has n linearly independent factorial series solutions. In this case, the solutions of (3.12) are analogous to solutions of a linear differential system which has $z = \infty$ as an ordinary point.

(iii) By the use of Waring's formula (see Section 4), the case where some $d_j > 2$ can be reduced to the case $d_j = 2$ before application of Theorem 1. Thus Theorem 1 can be used in all cases where each d_j is a positive integer.

(iv) A result of Harris and Turrittin [11] on factorial series representation of reciprocals of factorial series permits reduction of a system of the form

$$z^k F(z)\, \Delta_{-1} y(z) = A(z)\, y(z) + B(z)\, y(qz),$$

where F, A, and B have factorial series representations, and k is a positive integer, to a system of the type we are considering here.

Formal factorial series solutions. We now seek a solution of the equation

$$(z-1)\, \Delta_{-1} y(z) = A(z)\, y(z) + B(z)\, y(qz) \tag{3.14}$$

of the form

$$y(z) = y_0 + \sum_{k=0}^{\infty} y_{k+1} k! (z-1)^{(-k-1)},$$

where A, B, and q are as in Theorem 1.

After substituting the formal solution into (3.14) and equating coefficients, we obtain the relations

$$(A_0 + B_0)\, y_0 = 0,$$

and for $k \geqslant 0$,

$$-y_{k+1}(k+1)! = [A_0 y_{k+1} + (1/q)\, B_0 b_{k+1}]\, k! + U_k(y_0, ..., y_k),$$

where for each k, U_k is a linear function in each of its arguments. Further, recall that

$$b_{k+1} = y_{k+1}\psi_k^{(k)}(q) + \cdots + y_1 \psi_k^{(0)}(q).$$

From (2.5) it follows that $\psi_k^{(k)}(q) = 1/q^k$. Hence

$$\left[(k+1)I + A_0 + \frac{1}{q^{k+1}} B_0\right] y_{k+1} = l_k(A_0, ..., A_{k+1}, B_0, ..., B_{k+1}, y_0, ..., y_k),$$

where l_k is a linear function in each of its arguments; I is the $n \times n$ identity matrix.

These are the equations which are satisfied by each formal factorial series solution. If a formal solution exists, the following corollary holds.

COROLLARY 2. *Let $y(z) = y_0 + \sum_{k=0}^{\infty} y_{k+1} k!\, (z-1)^{(-k-1)}$ be a formal solution of (3.14). Then $y \in X_n(\delta)$.*

Proof. If $d_j = 1$, $j = 1, ..., n$, then ϕ takes the form

$$\phi(z) = \phi_0 + \sum_{k=0}^{N-1} \phi_{k+1} k! (z-1)^{(-k-1)}.$$

In this case, the determining equation (3.13) is the system

$$(A_0 + B_0)\,\phi_0 = 0,$$

$$\left((k+1)I + A_0 + \frac{\psi_k^{(k)}(q)}{q}\,B_0\right)\phi_{k+1}$$
$$= l_k(A_0, \ldots, A_{k+1}, B_0, \ldots, B_{k+1}, \phi_0, \ldots, \phi_k), \qquad k = 0, 1, 2, \ldots, N-1.$$

These are the first $N + 1$ equations for the existence of a formal solution. Since the matrices $A_0 + B_0/q^{k+1}$ are uniformly bounded in norm for all k, the matrices on the left are nonsingular for k sufficiently large. For all such k, the coefficients y_{k+1} are determined uniquely by the preceding coefficients; thus every formal solution is convergent.

4. More General Convergent Solutions

The results of the preceding section are applied and extended here to yield existence theorems for more general convergent solutions of (1.1). The gamma function and the reciprocal gamma function [12] will be useful in representing such solutions. Define the reciprocal gamma function by

$$(1/\Gamma)(z) = 1/\Gamma(z).$$

This is an entire function of z, which satisfies the functional relation

$$(1/\Gamma)(z) = z(1/\Gamma)(z+1). \tag{4.1}$$

The reciprocal gamma function is important here in light of the equation

$$(z-1)\,\Delta_{-1}y(z) = 1, \tag{4.2}$$

which has a solution

$$\Omega(z) = -\Gamma(z)(1/\Gamma)'(z).$$

The digamma function $\Omega(z)$ plays essentially the same role in the theory of linear difference equations as that played by the solution $\log z$ of $zy' = 1$ in the theory of linear differential equations.

Now consider the equation

$$(z-1)\,\Delta_{-1}y(z) = A(z)\,y(z), \tag{4.3}$$

where A is as in Section 3. Suppose λ is an eigenvalue of A_0. Make the change of variables $y(z) = \Gamma(z)(1/\Gamma)(z - \lambda)\, w(z)$. Then

$$\begin{aligned}(z-1)\,\Delta_{-1}y(z) = {} & (z-1)\,\Gamma(z)(1/\Gamma)(z-\lambda)\,w(z)\\ & - (z-1)\,\Gamma(z-1)(1/\Gamma)(z-\lambda-1)\,w(z)\\ & + (z-1)\,\Gamma(z-1)(1/\Gamma)(z-\lambda-1)\,\Delta_{-1}w(z).\end{aligned}$$

Using (4.1), together with well-known properties of the gamma function, we write

$$\begin{aligned}(z-1)\,\Delta_{-1}y(z) = {} & \lambda\Gamma(z)(1/\Gamma)(z-\lambda)\,w(z)\\ & + (z-\lambda-1)\,\Gamma(z)(1/\Gamma)(z-\lambda)\,\Delta_{-1}w(z).\end{aligned}$$

Hence, (4.3) becomes

$$(z-\lambda-1)\,\Delta_{-1}w(z) = (A(z) - \lambda I)\,w(z). \tag{4.4}$$

The function $(z-1)/(z-\lambda-1)$ has a factorial series expansion absolutely convergent for $\operatorname{Re} z > \operatorname{Re}\lambda + 1$; in fact,

$$\frac{z-1}{z-\lambda-1} = 1 + \frac{\lambda}{z} + \frac{\lambda(\lambda+1)}{z(z+1)} + \frac{\lambda(\lambda+1)(\lambda+2)}{z(z+1)(z+2)} + \cdots.$$

This expansion is known as Waring's formula [13].

Multiply (4.4) by $(z-1)/(z-\lambda-1)$. Equation (4.4) then has the form

$$(z-1)\,\Delta_{-1}w(z) = \left(A_0 - \lambda I + \sum_{k=0}^{\infty} \bar{A}_{k+1} k!(z-1)^{(-k-1)}\right) w(z). \tag{4.5}$$

The components of the factorial series in (4.5) are elements of $X(\beta)$, where $\beta > \max\{\delta, \operatorname{Re}\lambda + 1\}$. The form of (4.5) coupled with Theorem 1 yields the following corollary.

COROLLARY 3. *Let n_λ be the number of linearly independent vectors y satisfying $A_0 y = \lambda y$. The number N_λ of linearly independent solutions of system* (4.3) *of the form $y(z) = \Gamma(z)(1/\Gamma)(z-\lambda)\, w(z)$, where $w(z) = w_0 + \sum_{k=0}^{\infty} w_{k+1} k!(z-1)^{(-k-1)}$ is in $X_n(\beta)$, satisfies* $\max(n_\lambda, n_{\lambda-1}, \ldots) \leqslant N_\lambda \leqslant n_\lambda + n_{\lambda-1} + \cdots$.

The proof parallels the proof of Corollary 3 in [10].

Existence of the formal solution. For the time being, assume that A_0 has no eigenvalues which differ from λ by a negative integer. In this case, Corollary 3 yields $n_\lambda = N_\lambda$. If n_λ is strictly less than the algebraic multiplicity of the eigenvalue λ, then we can show the existence of solutions of a more general type than those mentioned in Corollary 3.

Without loss of generality assume that, for equation (4.3), zero is an eigenvalue of A_0. For the present, also assume that A_0 has no other integer eigenvalues. Suppose that the eigenvalue zero does not have a full complement of linearly independent eigenvectors. We will look for a solution of the form

$$y(z) = y^{[1]}(z) + \Omega(z)\, y^{[2]}(z), \tag{4.6}$$

where $y^{[1]}, y^{[2]} \in X_n(\delta)$. Write

$$y^{[i]}(z) = y_0^{[i]} + \sum_{k=0}^{\infty} y_{k+1}^{[i]} k!(z-1)^{(-k-1)}, \qquad i = 1, 2.$$

The substitution of (4.6) into (4.3) yields

$$\begin{aligned}(z-1)\,\Delta_{-1}y^{[1]}(z) &+ (z-1)(\Delta_{-1}\Omega(z))\, y^{[2]}(z) + (z-1)\,\Omega(z-1)\,\Delta_{-1}y^{[2]}(z)\\ &= A(z)\, y^{[1]}(z) + A(z)\,\Omega(z)\, y^{[2]}(z).\end{aligned}$$

If we recall (4.2), and use the identity $\Omega(z) = 1/(z-1) + \Omega(z-1)$, the preceding equation becomes

$$\begin{aligned}(z-1)\,\Delta_{-1}y^{[1]}(z) &+ y^{[2]}(z) + (z-1)\,\Omega(z-1)\,\Delta_{-1}y^{[2]}(z)\\ &= A(z)\, y^{[1]}(z) + A(z)\left(\frac{1}{z-1} + \Omega(z-1)\right) y^{[2]}(z).\end{aligned} \tag{4.7}$$

Equate the coefficients of $\Omega(z-1)$ to obtain

$$(z-1)\,\Delta_{-1}y^{[2]}(z) = A(z)\, y^{[2]}(z).$$

As shown before, this equation has a formal factorial series solution. In fact, since A_0 is singular, there is a formal solution of the form

$$y_0^{[2]} + \sum_{k=0}^{\infty} y_{k+1}^{[2]} k!(z-1)^{(-k-1)},$$

where

$$A_0 y_0^{[2]} = 0.$$

Equate the remaining terms in (4.7) to obtain

$$(z-1)\,\Delta_{-1}y^{[1]}(z) = A(z)\, y^{(1)}(z) + \left(\frac{1}{z-1} A(z) - I\right) y^{[2]}(z). \tag{4.8}$$

Set

$$\frac{1}{z-1} A(z) = C(z) = \sum_{k=0}^{\infty} C_{k+1} k!(z-1)^{(-k-1)}.$$

Then equation (4.8) takes the form

$$-\sum_{k=0}^{\infty} y_{k+1}^{[1]}(k+1)!(z-1)^{(-k-1)}$$

$$= \left[A_0 + \sum_{k=0}^{\infty} A_{k+1}k!(z-1)^{(-k-1)}\right]\left[y_0^{[1]} + \sum_{k=0}^{\infty} y_{k+1}^{[1]}k!(z-1)^{(-k-1)}\right]$$

$$+ \left[-I + \sum_{k=0}^{\infty} C_{k+1}k!(z-1)^{(-k-1)}\right]\left[y_0^{[2]} + \sum_{k=0}^{\infty} y_{k+1}^{[2]}k!(z-1)^{(-k-1)}\right].$$

Equate coefficients in the above expression. First we obtain $y_0^{[2]} = A_0 y_0^{[1]}$. By the hypothesis on A_0, there exists a generalized eigenvector $y_0^{[1]}$ satisfying this equation. For $k \geqslant 0$, we obtain

$$[A_0 + (k+1)I]\, y_{k+1}^{[1]} = h_k(A_1, \ldots, A_{k+1}, y_0^{[1]}, \ldots, y_k^{[1]}, y_0^{[2]}, \ldots, y_{k+1}^{[2]})$$

where h_k is a linear function in each of its arguments. By assumption, the matrices $A_0 + (k+1)\,I$, $k \geqslant 0$, are nonsingular. Hence the equation (4.3) has a formal solution of the form (4.6).

Convergence of the formal solution. We now introduce a new Banach space in order to prove the convergence of (4.6). Let $u^{[1]}, u^{[2]} \in X_n(\delta)$. Then

$$u^{[1]}(z) = u_0^{[1]} + \sum_{k=0}^{\infty} u_{k+1}^{[1]}k!(z-1)^{(-k-1)},$$

$$u^{[2]}(z) = u_0^{[2]} + \sum_{k=0}^{\infty} u_{k+1}^{[2]}k!(z-1)^{(-k-1)}.$$

With $u^{[1]}$ and $u^{[2]}$ we can associate the sequences $\{u_k^{[1]}\}_{k=0}^{\infty}$ and $\{u_k^{[2]}\}_{k=0}^{\infty}$. Define $H_n^2(\delta) = \{u\colon u = \{(u_k^{[1]}, u_k^{[2]})\}_{k=0}^{\infty},$ where $u^{[1]}, u^{[2]} \in X_n(\delta)\}$. $H_n^2(\delta)$ is a set of sequences of ordered pairs of complex vectors of dimension n, i.e., a sequence of elements from C_n^2. On $H_n^2(\delta)$ define a norm by

$$\| u \|_n = \| u^{[1]} \|_n + \| u^{[2]} \|_n .$$

Under this norm, $H_n^2(\delta)$ is a Banach space.

We write out (4.7) in detail.

$$y_0^{[2]} + \sum_{k=0}^{\infty} [y_{k+1}^{[2]} - (k+1)\, y_{k+1}^{[1]}]\, k!(z-1)^{(-k-1)}$$

$$- \Omega(z-1) \sum_{k=0}^{\infty} y_{k+1}^{[2]}(k+1)!(z-1)^{(-k-1)} \tag{4.9}$$

$$= (Ay^{[1]})_0 + \sum_{k=0}^{\infty} [(Ay^{[1]})_{k+1} + (Cy^{[2]})_{k+1}]\, k!(z-1)^{(-k-1)}$$

$$+ \Omega(z-1)\left[(Ay^{[2]})_0 + \sum_{k=0}^{\infty} (Ay^{[2]})_{k+1} k!(z-1)^{(-k-1)}\right].$$

We will express (4.9) as a functional equation in $H_n{}^2(\delta)$. Define the operator A': $H_n{}^2(\delta) \to H_n{}^2(\delta)$ by $A'u = v$, where

$$v_k^{[1]} = (Au^{[1]})_k + (Cu^{[2]})_k$$

and

$$v_k^{[2]} = (Au^{[2]})_k .$$

From (3.7), it follows that there exist constants K_1 and K_2 such that

$$\begin{aligned} \| A'u \|_n^2 &= \| v^{[1]} \|_n + \| v^{[2]} \|_n \\ &\leqslant K_1 \| u^{[1]} \|_n + K_2 \| u^{[2]} \|_n + K_1 \| u^{[2]} \|_n \\ &\leqslant (2K_1 + K_2) \| u \|_n^2 . \end{aligned}$$

Thus A' is a bounded operator.

Define J_m: $H_n{}^2(\delta) \to H_n{}^2(\delta)$ by $v = J_m u$,

$$\begin{aligned} v_k &= 0, && k < m, \\ &= D_k u_k/k, && k \geqslant m, \end{aligned}$$

where D_k: $C_n{}^2 \to C_n{}^2$ is defined by $D_k p = -(p^{[1]} + p^{[2]}/k, p^{[2]})$ and $p = (p^{[1]}, p^{[2]})$. If $r = (r^{[1]}, r^{[2]})$ is in $C_n{}^2$, and $r = (1/k)\, D_k p$, then

$$(r^{[2]}, 0) - k(r^{[1]}, r^{[2]}) = (r^{[2]} + p^{[1]} + p^{[2]}/k, p^{[2]}).$$

But $r^{[2]} = -p^{[2]}/k$, hence

$$(r^{[2]}, 0) - k(r^{[1]}, r^{[2]}) = (p^{[1]}, p^{[2]});$$

i.e., D_k has the property that $r = (1/k)\, D_k p \Rightarrow (r^{[2]}, 0) - kr = p$. From the form of D_k, it is clear that there exists a constant K_3 such that

$$\| J_m u \|_n^2 \leqslant (K_3/m) \| u \|_n^2 .$$

Let $\overline{v^{[1]}}(z) + \Omega(z)\, \overline{v^{[2]}}(z)$ be a formal solution of (4.3), and set

$$\bar{v} = \{(\overline{v_k^{[1]}}, \overline{v_k^{[2]}})\}_{k=0}^{\infty} = \{\bar{v}_k\}_{k=0}^{\infty} .$$

Consider the functional equation

$$v = J_m A' v + (\bar{v}_0, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{m-1}, 0, 0, 0, \ldots). \tag{4.10}$$

Now $\| J_m A' v \|_n^2 \leqslant ((2K_1 + K_2) K_3/m) \| v \|_n^2$. Thus for m sufficiently large, $J_m A'$ is a contraction mapping. Hence equation (4.10) has a unique solution in $H_n^2(\delta)$. Equating coefficients in (4.9), we have

$$\begin{aligned}(y_k^{[2]} - k y_k^{[1]}) - \Omega(z-1)\, k y_k^{[2]} \\ = (Ay^{[1]})_k + (Cy^{[2]})_k + \Omega(z-1)(Ay^{[2]})_k .\end{aligned} \tag{4.11}$$

The solution of (4.10) also satisfies (4.11) since $v_k = \bar{v}_k$ for $k < m$, and for $k \geqslant m$, $v_k = (1/k) D_k p$, with $p = (A'v)_k$. Hence $(v_k^{[2]}, 0) - k(v_k^{[1]}, v_k^{[2]}) = ((Av^{[1]})_k + (Cv^{[2]})_k, (Av^{[2]})_k)$. Thus $v(z) = v^{[1]}(z) + \Omega(z)\, v^{[2]}(z)$ is also a formal solution of (4.3). Since the coefficients v_k of a formal solution are uniquely determined by the preceding ones for k sufficiently large, $v_k = \bar{v}_k$ for all k. Since $v \in H_n^2(\delta)$, the formal factorial series $\overline{v^{[1]}}(z)$ and $\overline{v^{[2]}}(z)$ are absolutely convergent for $\operatorname{Re} z \geqslant \delta$.

If this procedure does not yield a full complement of solutions (corresponding to the multiplicity of the zero eigenvalue of A_0), we can obtain still other solutions of the form

$$y(z) = y^{[1]}(z) + y^{[2]}(z)\, \Omega(z) + y^{[3]}(z)\, \Omega_2(z) + \cdots + y^{[j+1]}(z)\, \Omega_j(z),$$

where

$$\Omega_j(z) = (-1)^j \Gamma(z) \frac{d^j}{dz^j} \left(\frac{1}{\Gamma}\right)(z).$$

The procedure is similar to that above, but the Banach space is now $H_n^{j+1}(\delta)$, i.e., the elements are ordered $(j+1)$-tuples.

We summarize the results of this section in the following theorem.

THEOREM 2. *Let $A(z)$ be an $n \times n$ matrix whose components are elements of $X(\delta)$. If the distinct eigenvalues of $A_0, \lambda_1, \ldots, \lambda_k$, $k \leqslant n$, do not differ by integers, then* (4.3) *has n linearly independent solutions of the form*

$$\Gamma(z)(1/\Gamma)(z - \lambda_i)[y_i^{[1]}(z) + y_i^{[2]}(z)\, \Omega(z) + \cdots + y_i^{[s_i]} \Omega_{s_i - 1}(z)].$$

Here $s_i \leqslant$ algebraic multiplicity (λ_i), and each $y^{[j]}(z)$ is in $X_n(\beta_i)$, where $\beta_i >$ $\max\{\delta, \operatorname{Re} \lambda_i + 1\}$.

The case in which A_0 has eigenvalues which differ by integers may be reduced to the above case by the following theorem, proved in [7].

THEOREM 3. *Let the distinct eigenvalues of A_0 in* (4.3) *be $\lambda_1, \ldots, \lambda_k$, $k \leqslant n$. There exists a matrix function V of z, nonsingular for $1/z \neq 0$, and linear in $1/z$, such that the transformation $y = Vw$ transforms* (4.3) *into a system with the same properties as* (4.3),

$$(z-1)\,\Delta_{-1} w(z) = \left(\tilde{A}_0 + \sum_{k=0}^{\infty} \tilde{A}_{k+1} k!(z-1)^{(-k-1)}\right) w(z),$$

and where $\tilde{A}_0$ has eigenvalues $\lambda_1 + 1, \lambda_2, \ldots, \lambda_k$.

By using sufficiently many transformations V_i of the type mentioned in Theorem 3, we finally obtain a system satisfying the hypothesis of Theorem 2. Hence n linearly independent solutions may be inferred no matter what the form of A_0 is.

Remark. A result for ordinary differential equations similar to Theorem 2 was obtained by W. Walter [18] by extending the Harris-Sibuya-Weinberg procedure to logarithmic solutions.

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