01 Jan 1981

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Recommended Citation

Č. V. Stanojevic, "Classes Of L¹-convergence Of Fourier And Fourier-stieltjes Series," *Proceedings of the American Mathematical Society*, vol. 82, no. 2, pp. 209 - 215, American Mathematical Society, Jan 1981.
The definitive version is available at https://doi.org/10.1090/s0002-9939-1981-0609653-4

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, June 1981

CLASSES OF L^1-CONVERGENCE OF FOURIER AND FOURIER-STIELTJES SERIES

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ABSTRACT. It is shown that the Fomin class $\mathcal{F}_p$ $(1 < p < 2)$ is a subclass of $\mathcal{C} \cap \mathcal{BV}$, where $\mathcal{C}$ is the Garrett-Stanojević class and $\mathcal{BV}$ the class of sequences of bounded variation. Wider classes of Fourier and Fourier-Stieltjes series are found for which $a_n \lg n = o(1)$, $n \to \infty$, is a necessary and sufficient condition for L^1-convergence. For cosine series with coefficients in $\mathcal{BV}$ and $n\Delta a_n = O(1)$, $n \to \infty$, necessary and sufficient integrability conditions are obtained.

1. Introduction. Telyakovskiĭ [1] extended the classical result of Kolmogorov [2] concerning the L^1-convergence of the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

with $\{a_k\}$ in the class of quasi-convex null-sequences $(\sum_{k=1}^{\infty}(k+1)|\Delta^2 a_k| < \infty)$. Paraphrasing a sufficient integrability condition of Sidon [3], Telyakovskiĭ [1] obtained a new class $\mathcal{S}$ containing the class of quasi-convex null-sequences. A sequence $\{a_k\}$ belongs to $\mathcal{S}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| \leq A_k$, for all k. Let $\mathcal{BV}$ be the class of all null-sequences of bounded variation. It is plain that $\mathcal{S} \subset \mathcal{BV}$.

Telyakovskiĭ [1] proved that if $\{a_k\} \in \mathcal{S}$, then (1.1) is a Fourier series of some $f \in L^1(0, \pi)$ and that

$$\|s_n - f\| = o(1), \qquad n \to \infty, \tag{1.2}$$

if and only if

$$a_n \lg n = o(1), \qquad n \to \infty, \tag{1.3}$$

where s_n are the partial sums of (1.1) and $\|\cdot\|$ is the $L^1(0, \pi)$-norm.

Garrett and Stanojević [4] introduced the following class $\mathcal{C}$ of null-sequences.

DEFINITION 1.1. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{C}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, independent of n, and such that

$$C_n(\delta) = \frac{1}{\pi}\int_0^{\delta}\left|\sum_{k=n}^{\infty} \Delta a_k D_k(x)\right| dx < \varepsilon, \tag{1.4}$$

for all n, where D_k is the Dirichlet kernel.

Received by the editors July 9, 1980; presented to the 7th Congress of Yugoslav Mathematicians, October 1980, Bečići, Montenegro, Yugoslavia.

AMS (MOS) subject classifications (1970). Primary 42A20, 42A32.

Key words and phrases. L^1-convergence of Fourier series and Fourier-Stieltjes series, integrability of cosine series.

Let

$$g_n(x) = s_n(x) - a_{n+1}D_n(x), \tag{1.5}$$

where $\{a_k\} \in \mathfrak{B}\mathcal{V}$. Garrett and Stanojević [4] proved that $\|g_n - f\| = o(1)$, $n \to \infty$, if and only if $\{a_k\} \in \mathcal{C}$.

The next theorem is a corollary to that result.

THEOREM A. *Let* $\{a_k\} \in \mathcal{C} \cap \mathfrak{B}\mathcal{V}$. *Then* (1.1) *is a Fourier series of some* $f \in L^1(0, \pi)$ *and* (1.2) $\Leftrightarrow$ (1.3).

Redefining the class $\mathcal{S}$ and using the Sidon-Fomin lemma [1], Garrett, Rees and Stanojević [5] proved that

$$\mathcal{S} \subset \mathcal{C} \cap \mathfrak{B}\mathcal{V}.$$

Recently Fomin [6] extended the class $\mathcal{S}$ in the following manner.

DEFINITION 1.2. A null-sequence $\{a_k\}$ belongs to the class $\mathfrak{F}_p$, if for some $p > 1$,

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} < \infty. \tag{1.6}$$

Notice that the class $\mathfrak{F}_p$ is wider when p is closer to 1. Hence, without loss of generality we may assume that $1 < p \leq 2$, in all subsequent considerations. It can be also shown that

$$\mathcal{S} \subset \mathfrak{F}_p \subset \mathfrak{B}\mathcal{V}.$$

The following theorem of Fomin [6] generalizes the Telyakovskiĭ result [1].

THEOREM B. *For some* $1 < p \leq 2$, *let* $\{a_k\} \in \mathfrak{F}_p$. *Then* (1.1) *is a Fourier series of some* $f \in L^1(0, \pi)$ *and* (1.2) $\Leftrightarrow$ (1.3).

In this paper we shall show that

$$\mathfrak{F}_p \subset \mathcal{C} \cap \mathfrak{B}\mathcal{V},$$

and give necessary and sufficient conditions for L^1-convergence for certain larger classes of Fourier and Fourier-Stieltjes series. As a by-product we shall obtain necessary and sufficient conditions for (1.1) to be a Fourier series.

Most of our results can be adapted for the sine series either directly or after appropriate modifications.

2. Classes of L^1-convergence. A closer scrutiny of the proof of our first theorem will reveal some new classes of Fourier coefficients for which (1.2) $\Leftrightarrow$ (1.3). Such classes of Fourier (or Fourier-Stieltjes) coefficients we call classes of L^1-convergence.

THEOREM 2.1. *Let* $1 < p \leq 2$. *Then*

$$\mathfrak{F}_p \subset \mathcal{C} \cap \mathfrak{B}\mathcal{V}.$$

PROOF. From (1.4) we have

$$C_n(\delta) \leq C_n(\pi) = C_n.$$

Hence, it suffices to show that from $\{a_k\} \in \mathfrak{F}_p$, it follows that $C_n = o(1)$, $n \to \infty$.

Before we proceed, we notice two consequences of $\{a_k\} \in \mathcal{F}_p$, i.e.

$$f \in L^1(0, \pi), \tag{2.1}$$

$$\sum_{k=1}^{\infty} k^{p-1}|\Delta a_k|^p < \infty. \tag{2.2}$$

In the rest of the proof we shall show that (2.1) and (2.2) imply that $C_n = o(1)$, $n \to \infty$.

Consider

$$C_{n+1} = \frac{1}{\pi}\int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx$$
$$= \frac{1}{\pi}\int_0^{\pi}|g_n(x) - f(x)|\, dx.$$

Because of $\{a_k\} \in \mathcal{B}\mathcal{V}$, f is the pointwise limit of s_n in $(0, \pi]$, and because of (1.5) f is also the pointwise limit of g_n in $(0, \pi]$.

The integral C_{n+1} we split in the following way.

$$C_{n+1} = \frac{1}{\pi}\int_0^{1/n}|g_n(x) - f(x)|\, dx + \frac{1}{\pi}\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx. \tag{2.3}$$

The first integral in (2.3) we estimate as

$$\frac{1}{\pi}\int_0^{1/n}|g_n(x) - f(x)|\, dx < \frac{1}{\pi}\int_0^{1/n}|\sigma_n(x) - f(x)|\, dx$$
$$+ \frac{1}{\pi}\int_0^{1/n}|g_n(x) - \sigma_n(x)|\, dx,$$

where σ_n is the Fejér sum of s_n. It is plain that

$$\frac{1}{\pi}\int_0^{1/n}|\sigma_n(x) - f(x)|\, dx = O(\|\sigma_n - f\|), \qquad n \to \infty.$$

From (1.5) we get

$$g_n(x) - \sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n} k\Delta a_k D_k(x) - \frac{1}{n+1}\sum_{k=1}^{n} a_k D_k(x).$$

Hence

$$\frac{1}{\pi}\int_0^{1/n}|g_n(x) - \sigma_n(x)|\, dx < \frac{1}{\pi}\frac{1}{n+1}\sum_{k=1}^{n} k|\Delta a_k|\int_0^{1/n}|D_k(x)|\, dx$$
$$+ \frac{1}{\pi}\frac{1}{n+1}\sum_{k=1}^{n} |a_k|\int_0^{1/n}|D_k(x)|\, dx$$

or

$$\frac{1}{\pi}\int_0^{1/n}|g_n(x) - \sigma_n(x)|\, dx = O\left(\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right), \qquad n \to \infty,$$

where the trivial $o(1)$ term is omitted. (Trivial $o(1)$ terms will be omitted from here on.)

Altogether, for the first integral in (2.3) we have

$$\frac{1}{\pi}\int_0^{1/n}|g_n(x)-f(x)|\,dx = O(\|\sigma_n - f\|) + O\left(\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right), \qquad n\to\infty.$$

The second integral in (2.3) we write as

$$I_n = \frac{1}{\pi}\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx$$
$$= \frac{1}{\pi}\int_{1/n}^{\pi}\frac{1}{2\sin\frac{x}{2}}\left|\sum_{k=n+1}^{\infty}\Delta a_k \sin\left(k+\frac{1}{2}\right)x\right| dx.$$

Applying the Hölder inequality we get

$$I_n < \frac{1}{\pi}\left[\int_{1/n}^{\pi}\frac{dx}{2^p\sin^p\frac{x}{2}}\right]^{1/p}\left(\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k\sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q},$$

where $1/p + 1/q = 1$. Or

$$I_n < A_p((n+1)^{p-1})^{1/p}\left(\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k\sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q}, \tag{2.4}$$

(A_p, B_p and C_p are absolute constants depending on p).

For $\{a_k\}\in\mathfrak{B}\mathcal{V}$ and fixed n, the sequence $\sum_{k=n+1}^{N}\Delta a_k\sin(k+\frac{1}{2})x$ converges uniformly to $\sum_{k=n+1}^{\infty}\Delta a_k\sin(k+\frac{1}{2})x$, as $N\to\infty$. Thus

$$\left(\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k\sin\left(k+\tfrac{1}{2}\right)x\right|^q dx\right)^{1/q}$$
$$= \lim_{N\to\infty}\left(\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{N}\Delta a_k\sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q}.$$

Applying the Hausdorff-Young inequality to the last integral we get

$$\left(\int_{1/n}^{\pi}\left|\sum_{k=n+1}^{N}\Delta a_k\sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q} < B_p\left(\sum_{k=n+1}^{N}|\Delta a_k|^p\right)^{1/p}.$$

For the integral (2.4) we now have the estimate

$$I_n < C_p((n+1)^{p-1})^{1/p}\left(\sum_{k=n+1}^{\infty}|\Delta a_k|^p\right)^{1/p} < C_p\left(\sum_{k=n+1}^{\infty}k^{p-1}|\Delta a_k|^p\right)^{1/p}. \tag{2.5}$$

Combining all estimates we obtain

$$C_{n+1} = O(\|\sigma_n - f\|) + O\left(\frac{1}{n}\sum_{k=1}^{n}k|\Delta a_k|\right) + O\left(\sum_{k=n+1}^{\infty}k^{p-1}|\Delta a_k|^p\right), \qquad n\to\infty.$$

However the first term is $o(1)$ because of (2.1), the second because of $\{a_k\}\in\mathfrak{B}\mathcal{V}$, and the third because of (2.2). Finally

$$C_n = o(1), \qquad n\to\infty.$$

This completes the proof of Theorem 2.1.

The right-hand side of the first estimate for I_n in (2.5) can be rewritten as

$$(n+1)\left(\frac{\sum_{k=n+1}^{\infty}|\Delta a_k|^p}{n+1}\right)^{1/p}. \tag{2.6}$$

Since (1.6) implies (2.2), it follows that (1.6) implies (2.6). That motivates a new class of L^1-convergence.

DEFINITION 2.1. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{C}_p$ if for some $1 < p \leq 2$,

$$n\left(\frac{\sum_{k=n}^{\infty}|\Delta a_k|^p}{n}\right)^{1/p} = o(1), \qquad n \to \infty.$$

The following theorem corresponds to the class $\mathcal{C}_p$.

THEOREM 2.2. *Let* (1.1) *be a Fourier series of some* $f \in L^1(0, \pi)$ *and let* $\{a_k\} \in \mathcal{C}_p \cap \mathcal{BV}$. *Then* (1.2) $\Leftrightarrow$ (1.3).

The class $\mathcal{C}_p$ has an interesting subclass $\mathcal{C}_p^*$.

DEFINITION 2.2. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{C}_p^*$ if for some $1 < p \leq 2$,

$$\sum_{k=1}^{\infty} k^{p-1}|\Delta a_k|^p < \infty. \tag{2.7}$$

The next theorem is a corollary to Theorem 2.2.

THEOREM 2.3. *Let* (1.1) *be a Fourier series of some* $f \in L^1(0, \pi)$ *and let* $\{a_k\} \in \mathcal{C}_p^* \cap \mathcal{BV}$. *Then* (1.2) $\Leftrightarrow$ (1.3).

The proofs of both Theorem 2.2 and Theorem 2.3 are similar to the proof of Theorem 2.1.

Relations defining classes $\mathcal{F}_p$, $\mathcal{C}_p^*$ and $\mathcal{C}_p$ are not explicit enough, although each of them tells in an obscure way something about the nature of possible gaps of $\{a_k\}$. The following corollary to Theorem 2.3 makes that gap condition quite explicit.

COROLLARY 2.1. *Let* (1.1) *be a Fourier series with* $\{a_k\} \in \mathcal{BV}$, *and let*

$$n\Delta a_n = O(1), \qquad n \to \infty. \tag{2.8}$$

Then (1.2) $\Leftrightarrow$ (1.3).

PROOF. The condition (2.8) together with $\{a_k\} \in \mathcal{BV}$ implies that $\{a_k\} \in \mathcal{C}_p^*$.

A natural extension of $\mathcal{BV}$ is the following class.

DEFINITION 2.3. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{S}$ if

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k| = o(1), \qquad n \to \infty. \tag{2.9}$$

The class $\mathcal{S}$ extends not only $\mathcal{BV}$, but the class $\mathcal{QM}$ of quasi-monotone ($a_k/k^{\alpha}\downarrow 0$, for some $\alpha > 0$) sequences, as well. Combining the class $\mathcal{S}$ with the condition (2.8) we obtain a theorem for L^1-convergence of Fourier-Stieltjes series.

THEOREM 2.4. *Let* (1.1) *be a Fourier-Stieltjes series with* $\{a_k\} \in \mathcal{S}$, *and let* (2.8) *hold. Then it converges in* L^1 *if and only if*

$$a_n \lg n = o(1), \qquad n \to \infty.$$

PROOF. Observe that if $f \in L^1(0, \pi)$ and $\|g_n - \sigma_n\| = o(1)$, $n \to \infty$, then

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$\|s_n - \sigma_n\| = o(1), \qquad n \to \infty.$$

On the other hand, if $\|g_n - \sigma_n\| = o(1)$, $n \to \infty$, then $\|s_n - \sigma_n\| = o(1)$, $n \to \infty$, is equivalent with $a_n \lg n = o(1)$, $n \to \infty$. Thus it remains to show that from $\{a_k\} \in \mathcal{S}$ and (2.8) it follows that $\|g_n - \sigma_n\| = o(1)$, $n \to \infty$. Using the same technique as in the proof of Theorem 2.1 we get

$$\|g_n - \sigma_n\| = O\left(\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right), \qquad n \to \infty,$$

and the proof of Theorem 2.4 follows.

A corollary to Theorem 2.4 is a slightly weaker form of a theorem of Telyakovskiĭ and Fomin [7], (for a different proof see [8]), concerning the L^1-convergence of Fourier series with coefficients in $\mathcal{Q}\mathcal{M}$.

COROLLARY 2.2. *Let* (1.1) *be a Fourier series with* $\{a_k\} \in \mathcal{Q}\mathcal{M}$, *and let* (2.8) *hold. Then*

$$\|s_n - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$a_n \lg n = o(1), \qquad n \to \infty.$$

The interest of this corollary lies in the fact that both the necessity and sufficiency parts of the proof are obtained using the same technique, i.e. the sharp estimates of $\|s_n - \sigma_n\|$.

3. Necessary and sufficient integrability conditions. Throughout this section we shall consider (1.1) with $\{a_k\} \in \mathcal{B}\mathcal{V}$. Since in this case the pointwise limit f of s_n exists in $(0, \pi]$, to prove that (1.1) is a Fourier series, it suffices, by a standard argument, to show that $f \in L^1(0, \pi)$.

THEOREM 3.1. *Let* $\{a_k\} \in \mathcal{B}\mathcal{V}$ *and let* $n\Delta a_n = O(1)$, $n \to \infty$. *Then* (1.1) *is a Fourier series if and only if* $\{a_k\} \in \mathcal{C}$.

PROOF. Let $\{a_k\} \in \mathcal{B}\mathcal{V}$. Then $\{a_k\} \in \mathcal{C}$ is always a sufficient condition for $f \in L^1(0, \pi)$. Thus it remains to show that under the condition of Theorem 3.1 from $f \in L^1(0, \pi)$, it follows that $\{a_k\} \in \mathcal{C}$.

Notice that for $\|g_n - \sigma_n\| = o(1)$, $n \to \infty$, from $\|\sigma_n - f\| = o(1)$, $n \to \infty$, it follows $\|g_n - f\| = o(1)$, $n \to \infty$. But $f \in L^1(0, \pi)$ implies that $\|\sigma_n - f\| = o(1)$,

$n \to \infty$, and $\| g_n - f \| = o(1)$, $n \to \infty$ is equivalent to $\{a_k\} \in \mathcal{C}$. However $\{a_k\} \in \mathcal{B}\mathcal{V}$ implies that $\{a_k\} \in \mathcal{P}$. Hence from

$$\| g_n - \sigma_n \| = O\left(\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| \right), \qquad n \to \infty,$$

it follows that $\| g_n - \sigma_n \| = o(1)$, $n \to \infty$. This completes the proof of Theorem 3.1.

As a corollary of Theorem 3.1, we have a partial answer to the classical outstanding question: Let $\{a_k\}$ be a monotone null-sequence. What are the necessary and sufficient conditions for (1.1) to be a Fourier series?

COROLLARY 3.1. *Let* $\{a_k\}$ *be a monotone null-sequence and let* $n\Delta a_n = O(1)$, $n \to \infty$. *Then* (1.1) *is a Fourier series if and only if* $\{a_k\} \in \mathcal{C}$.

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