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# A First Order Method for Differential Equations of Neutral Type

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**Abstract.** A first order method is presented for solution of the initial-value problem for a differential equation of neutral type with implicit delay in the critical case where the time-lag is zero and the method of stepwise integration does not apply. A convergence theorem is proved, and numerical examples are given.

**1. Introduction.** In this note, we present a first order method for the numerical solution of the initial-value problem (IVP) for a neutral-type functional-differential equation without previous history:

$$x'(t) = f(t, x(t), x(g(t, x(t))), x'(g(t, x(t)))), \tag{1}$$

$$x(a) = x_0, \qquad x'(a) = z_0, \tag{2}$$

where $z_0$ is a real root of the algebraic equation

$$z = f(a, x_0, x_0, z). \tag{3}$$

Here, $x(t)$ is a scalar function to be determined on some finite interval $[a, b]$. We shall make the following assumptions regarding $f$ and $g$:

(H1) $f$ and $g$ are continuous and satisfy uniform Lipschitz conditions of the form

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leqq L\{|x_1 - x_2| + |y_1 - y_2|\} + L_z\,|z_1 - z_2|,$$

$$|g(t, x_1) - g(t, x_2)| \leqq L_g|x_1 - x_2|$$

in their respective domains $E$ and $E'$, where

$$E = \{(t, x, y, z)\colon a \leqq t \leqq b,\ |x - x_0| \leqq c,\ |y - x_0| \leqq c,\ |z| \leqq M\}$$

and $E'$ is the projection of $E$ in the $(t, x)$ space; $c$, $M$, $L$, $L_g$, $L_z$ are constants, with $L_z < 1$, $M$ is such that $\sup_{(t,x,y,z)\in E} |f(t, x, y, z)| < M$, and $M(b - a) < c$.

(H2) $a \leqq g(t, x) \leqq t$ for $(t, x) \in E'$.

Our hypotheses, together with additional smoothness and growth conditions on $f$ and $g$, ensure the local existence of a solution of the IVP (1)–(2). Furthermore, $x(t)$ is the only solution having a bounded derivative on $[a, b]$; see [2], [4]. Our result extends a method developed by Feldstein [3] for the equation of retarded type

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$$x'(t) = f(t, x(t), x(g(t)))$$

to the neutral-type equation with implicit delay (1). Other methods for implicit-delay equations are given in [1].

**2. The Algorithm $\mathfrak{A}$.** Let $y(t) = x(g(t, x(t)))$; $z(t) = x'(g(t, x(t)))$. Let $N$ be a positive integer, and let $h = (b - a)/N$. For each nonnegative integer $n \leqq N$, let $t_n = a + nh$. Let $[s]$ denote the integer part of $s$. Define the algorithm $\mathfrak{A}$ as follows:

$$f_n = f(t_n, x_n, y_n, z_n), \qquad g_n = g(t_n, x_n), \tag{4}$$

$$q(n) = [(g_n - a)/h], \qquad r(n) = (g_n - a)/h - q(n), \tag{5}$$

$$y_0 = x_0, \qquad y_n = x_{q(n)} + hr(n)f_{q(n)}, \tag{6}$$

$$z_n = f_{q(n)}, \tag{7}$$

$$x_{n+1} = x_n + hf_n. \tag{8}$$

Note that condition (H2) implies $q(n) \leqq n$, thus, the algorithm is well defined. For $n = 0$, $g_0 = a$, $q(0) = 0$, and $r(0) = 0$. Thus, $y_0 = x_0$ and $z_0 = f(a, x_0, x_0, z_0)$. Let $u_0$, an approximation of the root $z_0$, be chosen independently of $h$. It is of interest to note that such an approximation does not destroy the order $h$ convergence of the algorithm. It is of further interest that (6) may be simplified to $y_n = x_{q(n)}$. The error bound established in the convergence theorem for this "simplified" algorithm is larger but still of order $h$, as noted following the proof of convergence of the algorithm $\mathfrak{A}$. The second numerical example of Section 4 demonstrates both the algorithm $\mathfrak{A}$ and the simplified algorithm.

If $g_n = t_n$ for any $n$, $1 \leqq n \leqq N$, then $q(n) = n$, $r(n) = 0$, and (7) becomes $z_n = f(t_n, x_n, y_n, z_n)$ which has exactly one root $z$ in the interval $[-M, M]$ under the conditions (H1)–(H2) together with the smoothness and growth conditions mentioned in Section 1. We must in general include a procedure for finding this root, and this in turn will affect the error estimate. As before, such an estimate does not destroy the order $h$ convergence of the algorithm. For simplicity, we do not take this into account, since our aim is to show the convergence of the algorithm $\mathfrak{A}$.

Thus, we shall assume in the convergence proof that (7) will not reduce to $z_n = f(t_n, x_n, y_n, z_n)$, $n \geqq 1$.

**3. Convergence.**

THEOREM. *Let $f$ and $g$ satisfy* (H1)–(H2) *and suppose, in addition, that there exists a unique solution $x(t)$ of* (1)–(2) *with* $\sup_{[a,b]} |x''(t)| \leqq B$. *Then, for each $t_n \in [a, b]$, $0 < n \leqq N$,*

$$|x_n - x(t_n)| \leqq h\left\{L_z\, |z_0 - u_0|\, e^{s(b-a)} + \frac{B}{2s}\left(\frac{1 + L_z}{1 - L_z}\right)(e^{s(b-a)} - 1)\right\} + O(h^2)$$

*where*

$$s = L(1 + c_0) + L_z c_1,$$

$$c_0 = 1 + ML_g,$$

$$c_1 = (L(2 + ML_g) + BL_g)/(1 - L_z),$$

*$u_0$ is the approximation to $z_0$ mentioned above, and $x_n$ is given by algorithm $\mathfrak{A}$.*

*Proof.* Let $e_n = |x_n - x(t_n)|$; $e_n^* = |y_n - y(t_n)|$; $e_n^{**} = |z_n - z(t_n)|$. From (8) and Taylor's formula, we obtain

$$e_{n+1} \leqq e_n + h(L(e_n + e_n^*) + L_z e_n^{**}) + h^2 B/2. \tag{9}$$

Equation (5) implies that $g_n = t_{q(n)} + hr(n)$, and hence, in a similar manner, we have (after replacing $n$ by $(n + 1)$)

$$\begin{aligned} e_{n+1}^* \leqq\ & ML_g e_{n+1} + e_{q(n+1)} \\ & + hr(n + 1)\{L(e_{q(n+1)} + e_{q(n+1)}^*) + L_z e_{q(n+1)}^{**}\} + h^2 r^2(n + 1)B/2, \end{aligned} \tag{10}$$

$$e_{n+1}^{**} \leqq BL_g e_{n+1} + L(e_{q(n+1)} + e_{q(n+1)}^*) + L_z e_{q(n+1)}^{**} + hr(n + 1)B. \tag{11}$$

We then have two cases to consider:

*Case* 1. $q(n + 1) = n + 1$ and $r(n + 1) = 0$. Under these conditions, (9) is unchanged:

$$e_{n+1} \leqq e_n(1 + hL) + e_n^* hL + e_n^{**} hL_z + h^2 B/2. \tag{9a}$$

(10) becomes

$$e_{n+1}^* \leqq e_{n+1}(1 + ML_g) = e_{n+1}c_0. \tag{10a}$$

And (11) becomes

$$e_{n+1}^{**} \leqq (L + BL_g)e_{n+1} + Le_{n+1}^* + L_z e_{n+1}^{**}$$

or

$$e_{n+1}^{**} \leqq \left(\frac{L + BL_g + L(1 + ML_g)}{1 - L_z}\right)e_{n+1} = e_{n+1}c_1. \tag{11a}$$

Define the partial ordering for vectors: $v_1 = (v_1^1, \cdots, v_1^k) \leqq v_2 = (v_2^1, \cdots, v_2^k)$ if $v_1^i \leqq v_2^i$, $i = 1, \cdots, k$. Then, in vector form, (9a), (10a), and (11a) become

$$\begin{bmatrix} e_{n+1} \\ e_{n+1}^* \\ e_{n+1}^{**} \end{bmatrix} \leqq \begin{bmatrix} 1 + hL & hL & hL_z \\ (1 + hL)c_0 & hLc_0 & hL_z c_0 \\ (1 + hL)c_1 & hLc_1 & hL_z c_1 \end{bmatrix} \begin{bmatrix} e_n \\ e_n^* \\ e_n^{**} \end{bmatrix} + hB \begin{bmatrix} h/2 \\ hc_0/2 \\ hc_1/2 \end{bmatrix}$$

which is of the form $d_{n+1} \leqq A_1 d_n + b_1$.

*Case* 2. $q(n + 1) \leqq n$ and $0 \leqq r(n + 1) < 1$.

Let

$$\delta_n = \max_{1 \leqq i \leqq n} e_i, \qquad \delta_n^* = \max_{1 \leqq i \leqq n} e_i^*, \qquad \delta_n^{**} = \max_{1 \leqq i \leqq n} e_i^{**}.$$

Then, (9) becomes

$$\delta_{n+1} \leqq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_z + h^2 B/2. \tag{9b}$$

And (10) becomes

$$\delta_{n+1}^* \leqq ML_g\delta_{n+1} + \delta_n(1 + hL) + hL\delta_n^* + hL_z\delta_n^{**} + h^2B/2.$$

Using (9b), we have

$$\delta_{n+1}^* \leqq (\delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_z + h^2B/2)(1 + ML_g)$$

or

$$\text{(10b)} \qquad \delta_{n+1}^* \leqq \delta_n(1 + hL)c_0 + \delta_n^* hLc_0 + \delta_n^{**} hL_z c_0 + h^2 c_0 B/2.$$

Finally, (11) becomes

$$\delta_{n+1}^{**} \leqq \delta_{n+1}BL_g + \delta_n L + \delta_n^* L + \delta_n^{**} L_z + hB.$$

Further, enlarging $\delta_n$ to $\delta_{n+1}$ and $\delta_n^*$ to $\delta_{n+1}^*$ on the right, and using $1 - L_z > 0$, we find

$$\delta_{n+1}^{**} \leqq \delta_{n+1}\left(\frac{L + BL_g}{1 - L_z}\right) + \delta_{n+1}^* \frac{L}{1 - L_z} + \frac{hB}{1 - L_z}.$$

Using (9b) and (10b), we have

$$\delta_{n+1}^{**} \leqq \left(\frac{L + BL_g + Lc_0}{1 - L_z}\right)\left(\delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_z + \frac{h^2B}{2}\right) + \frac{hB}{1 - L_z}$$

or

$$\text{(11b)} \qquad \delta_{n+1}^{**} \leqq \delta_n(1 + hL)c_1 + \delta_n^* hLc_1 + \delta_n^{**} hL_z c_1 + \frac{hB}{1 - L_z} + \frac{h^2 c_1 B}{2}.$$

Then, as a vector system, (9b), (10b), and (11b) become

$$\text{(12)} \qquad \begin{bmatrix} \delta_{n+1} \\ \delta_{n+1}^* \\ \delta_{n+1}^{**} \end{bmatrix} \leqq \begin{bmatrix} 1 + hL & hL & hL_z \\ (1 + hL)c_0 & hLc_0 & hL_z c_0 \\ (1 + hL)c_1 & hLc_1 & hL_z c_1 \end{bmatrix} \begin{bmatrix} \delta_n \\ \delta_n^* \\ \delta_n^{**} \end{bmatrix} + hB \begin{bmatrix} h/2 \\ hc_0/2 \\ hc_1/2 + 1/(1 - L_z) \end{bmatrix}$$

which is of the form $d_{n+1} \leqq A_2 d_n + b_2$. Comparing this with the result obtained in Case 1, we find that $A_1$ and $A_2$ are identical and that $b_1 \leqq b_2$. Thus, any bound obtained here in Case 2 for $d_{n+1}$ will also bound $d_{n+1}$ in Case 1.

To complete the proof, we shall use the following lemmas [3] which may be verified by induction:

LEMMA 1. *Suppose $A$ is a $k \times k$ real matrix and $b$ is a real $k$-vector. Let $\{d_n\}$ ($n = 0, 1, \cdots$) satisfy $d_{n+1} \leqq Ad_n + b$. Then*

$$d_{n+1} \leqq A^{n+1} d_0 + \left(\sum_{i=0}^{n} A^i\right) b.$$

LEMMA 2. *Let $p = (p_1, \cdots, p_k)$, $q = (q_1, \cdots, q_k)$. Suppose the $k \times k$ matrix $A$ has the form $A = p^T q$. Then*

$$A^n = \left(\sum_{i=1}^{k} p_i q_i\right)^{n-1} A.$$

By Lemma 1,

$$d_{n+1} \leqq A_2^{n+1} d_0 + \left(\sum_{i=0}^{n} A_2^i\right) b_2,$$

where

$$d_0 = \begin{bmatrix} e_0 \\ e_0^* \\ e_0^{**} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ |z_0 - u_0| \end{bmatrix}.$$

Then, because

$$A_2 = \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} (1 + hL,\ hL,\ hL_z),$$

we can make use of Lemma 2 to obtain

$$A_2^i = (1 + hL + hLc_0 + hL_z c_1)^{i-1} A_2 = (1 + hs)^{i-1} A_2.$$

Two results follow from this: $A_2^{n+1} = (1 + hs)^n A_2 \leqq e^{s(b-a)} A_2$, and

$$\sum_{i=1}^{n} A_2^i = A_2 \sum_{i=1}^{n} (1 + hs)^{i-1} = \frac{((1 + hs)^n - 1)}{hs} A_2 \leqq \frac{1}{hs} (\exp(s(b - a)) - 1) A_2.$$

Finally,

$$\begin{aligned} d_{n+1} &\leqq A_2^{n+1} d_0 + \left(\sum_{i=0}^{n} A_2^i\right) b_2 \\ &\leqq h \left\{ |z_0 - u_0|\ L_z e^{s(b-a)} \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \right. \\ &\qquad \left. + \frac{B}{2s}\left(hs + \frac{1 + L_z}{1 - L_z}\right)(e^{s(b-a)} - 1) \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} + B \begin{bmatrix} \dfrac{h}{2} \\ \dfrac{hc_0}{2} \\ \dfrac{hc_1}{2} + \dfrac{1}{1 - L_z} \end{bmatrix} \right\} \end{aligned}$$

which gives

$$e_{n+1} \leqq \delta_{n+1} \leqq h \left\{ |z_0 - u_0|\ L_z e^{s(b-a)} + \frac{B}{2s}\left(hs + \frac{1 + L_z}{1 - L_z}\right)(e^{s(b-a)} - 1) + \frac{hB}{2} \right\}$$

and the theorem follows.

For the simplified algorithm, where (6) is replaced by $y_n = x_{q(n)}$ the following bound is possible:

$$
\begin{aligned}
d_{n+1} \leqq h\Bigg\{ & |z_0 - u_0|\, L_z e^{s(b-a)} \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \\
& + \left(\frac{B}{2s}\left(hs + \frac{1+L_z}{1-L_z}\right) + \frac{1}{s}\left(\frac{ML}{1-L_z}\right)\right)(e^{s(b-a)} - 1) \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \\
& + B \begin{bmatrix} \dfrac{h}{2} \\ \dfrac{hc_0}{2} \\ \dfrac{hc_1}{2} + \dfrac{1}{1-L_z} \end{bmatrix} + \begin{bmatrix} 0 \\ M \\ \dfrac{ML}{1-L_z} \end{bmatrix} \Bigg\},
\end{aligned} \tag{13}
$$

and hence

$$
\begin{aligned}
e_{n+1} \leqq h\Bigg\{ & |z_0 - u_0|\, L_z e^{s(b-a)} \\
& + \left(\frac{B}{2s}\left(hs + \frac{1+L_z}{1-L_z}\right) + \frac{1}{s}\left(\frac{ML}{1-L_z}\right)\right)(e^{s(b-a)} - 1) + \frac{hB}{2}\Bigg\}.
\end{aligned}
$$

TABLE I. *$x_n(h)$ denotes the value of $x_n$ for step size $h$.*

| $t_n$ | $x(t_n)$ | $x_n(2^{-4})$ | $x_n(2^{-6})$ | $x_n(2^{-8})$ | $x_n(2^{-10})$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| .0625 | .0039 | 0 | .0029 | .0034 | .0039 |
| .1250 | .0158 | .0078 | .0138 | .0153 | .0157 |
| .1875 | .0360 | .0238 | .0329 | .0352 | .0358 |
| .2500 | .0653 | .0484 | .0610 | .0642 | .0650 |
| .3125 | .1048 | .0825 | .0990 | .1032 | .1044 |
| .3750 | .1562 | .1275 | .1485 | .1541 | .1556 |
| .4375 | .2224 | .1853 | .2119 | .2196 | .2217 |
| .5000 | .3078 | .2593 | .2942 | .3043 | .3069 |
| .5625 | .4206 | .3547 | .4026 | .4159 | .4194 |
| .6250 | .5771 | .4856 | .5518 | .5705 | .5754 |
| .6875 | .8185 | .6707 | .7778 | .8080 | .8159 |
| .7500 | 1.3244 | .9860 | 1.2205 | 1.2968 | 1.3174 |

TABLE II. *$x_n^{(1)}(h)$ denotes the value of $x_n$ for step size h by algorithm $\mathfrak{A}$; $x_n^{(2)}(h)$ denotes the value of $x_n$ for step size h by the simplified algorithm.*

| $t_n$ | $x(t_n)$ | $x_n^{(1)}(2^{-2})$ | $x_n^{(2)}(2^{-2})$ | $x_n^{(1)}(2^{-4})$ | $x_n^{(2)}(2^{-4})$ |
|---|---|---|---|---|---|
| .25 | .2474 | .2500 | .2500 | .2483 | .2478 |
| .50 | .4794 | .4930 | .4892 | .4838 | .4759 |
| .75 | .6816 | .7180 | .6866 | .6942 | .6739 |
| 1.00 | .8414 | .9228 | .8569 | .8697 | .8273 |
| $t_n$ | $x(t_n)$ | $x_n^{(1)}(2^{-8})$ | $x_n^{(2)}(2^{-8})$ | $x_n^{(1)}(2^{-12})$ | $x_n^{(2)}(2^{-12})$ |
| .25 | .2474 | .2475 | .2471 | .2474 | .2474 |
| .50 | .4794 | .4797 | .4787 | .4794 | .4794 |
| .75 | .6816 | .6825 | .6802 | .6817 | .6815 |
| 1.00 | .8414 | .8435 | .8390 | .8416 | .8413 |

**4. Examples.** (a) We solve the IVP

$$x'(t) = \frac{-4tx^2(t)}{4 + \log^2 \cos t} + \tan 2t + \tfrac{1}{2} \tan^{-1} z$$

($z_0 = 0$, $x_0 = 0$, $z = x'(g(t, x(t))) \equiv x'(tx^2(t)/(1 + x^2(t))))$ on the interval [0, .75]. The existence and uniqueness of the solution is guaranteed by the results of [2] mentioned earlier. The only solution is $x(t) = -\frac{1}{2} \log \cos 2t$.

The results of the computation by algorithm $\mathfrak{A}$ are given in Table I.

(b) Consider the IVP

$$x'(t) = \cos t(1 + y) + xz - \sin(t(1 + \sin^2 t)),$$

with $y = x(tx^2(t))$, $z = x'(tx^2(t))$, $z_0 = 1$, $x_0 = 0$, on the interval [0, 1]. As in example (a), existence and uniqueness of the solution are guaranteed by the results of [2]. Here, the solution is $x(t) = \sin t$.

The results of the computation by the algorithm $\mathfrak{A}$ and by the simplified algorithm are given in Table II.

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