A Coefficient Inequality For Convex Univalent Functions

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ABSTRACT. A short proof of $|a_2^2 - a_3| \leq (1 - |a_2|^2)/3$ is given for normalized convex univalent functions.

If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent in $|z| < 1$, it is well known that $|a_2^2 - a_3| \leq 1$. If f maps $|z| < 1$ onto a convex domain, the above inequality can be improved to

$$|a_2^2 - a_3| \leq (1 - |a_2|^2)/3. \tag{1}$$

This inequality follows from a result of J. A. Hummel [1, p. 1388, (11)] by using the fact that $g(z) = zf'(z)$ is starlike if f is convex. From (1), it follows that $|a_2^2 - a_3| \leq 1/3$. This inequality has appeared in [2], [4], and [5].

Hummel's proof uses variational techniques. The purpose of this note is to give an elementary proof of (1). Let $h(z) = 1 + zf''(z)/f'(z)$. Let $\Phi(z) = [1 - h(z)]/[1 + h(z)]$. Then $|\Phi(z)| < 1$ and $\Phi(z) = -a_2 z + 3(a_2^2 - a_3)z^2 + \cdots$. From [3, p. 172, Exercise 9], it follows that $|3(a_2^2 - a_3)| \leq 1 - |a_2|^2$. The proof is finished. Note that (1) is sharp for all permissible values of $|a_2|$: If $0 \leq \lambda \leq 1$ and $|z| < 1$, let

$$f_\lambda(z) = \int_0^z \left(\frac{1 + \tau}{1 - \tau}\right)^\lambda \frac{1}{1 - \tau^2}\, d\tau = z + \lambda z^2 + \frac{1}{3}(2\lambda^2 + 1)z^3 + \cdots .$$

If $\lambda < 1$, then $zf_\lambda'(z)$ is starlike with two slits. Since $zf_1'(z)$ is the Koebe function, all the f_λ are univalent and convex in $|z| < 1$. Their coefficients give equality in (1).

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