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A COEFFICIENT INEQUALITY FOR CONVEX UNIVALENT FUNCTIONS

S. Y. TRIMBLE

ABSTRACT. A short proof of $|a_2^2 - a_3| \leq (1 - |a_2|^2)/3$ is given for normalized convex univalent functions.

If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent in $|z| < 1$, it is well known that $|a_2^2 - a_3| \leq 1$. If f maps $|z| < 1$ onto a convex domain, the above inequality can be improved to

$$(1) \quad |a_2^2 - a_3| \leq (1 - |a_2|^2)/3.$$

This inequality follows from a result of J. A. Hummel [1, p. 1388, (11)] by using the fact that $g(z) = zf'(z)$ is starlike if f is convex. From (1), it follows that $|a_2^2 - a_3| \leq 1/3$. This inequality has appeared in [2], [4], and [5].

Hummel's proof uses variational techniques. The purpose of this note is to give an elementary proof of (1). Let $h(z) = 1 + zf''(z)/f'(z)$. Let $\Phi(z) = [1 - h(z)]/[1 + h(z)]$. Then $|\Phi(z)| < 1$ and $\Phi(z) = -a_2 z + 3(a_2^2 - a_3)z^2 + \cdots$. From [3, p. 172, Exercise 9], it follows that $|3(a_2^2 - a_3)| \leq 1 - |a_2|^2$. The proof is finished. Note that (1) is sharp for all permissible values of $|a_2|$: If $0 \leq \lambda \leq 1$ and $|z| < 1$, let

$$f_\lambda(z) = \int_0^z \left(\frac{1+\tau}{1-\tau} \right)^\lambda \frac{1}{1-\tau^2} d\tau = z + \lambda z^2 + \frac{1}{3}(2\lambda^2 + 1)z^3 + \cdots$$

If $\lambda < 1$, then $zf'_\lambda(z)$ is starlike with two slits. Since $zf'_1(z)$ is the Koebe function, all the f_λ are univalent and convex in $|z| < 1$. Their coefficients give equality in (1).

REFERENCES

1. J. A. Hummel, *The coefficient regions of starlike functions*, Pacific J. Math. 7 (1957), 1381–1389. MR 20 #1780.
2. F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. 20 (1969), 8–12. MR 38 #1249.

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3. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR 13, 640.
4. M. S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. 16 (1969), 97–101. MR 39 #5785.
5. J. Szynal, *Some remarks on coefficients inequality for α -convex functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), 917–919. MR 47 #454.

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