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## ON $L^1$ CONVERGENCE OF CERTAIN COSINE SUMS

BY JOHN W. GARRETT<sup>1</sup> AND ČASLAV V. STANOJEVIĆ

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**Abstract.** It is shown that to a certain cosine series  $f$ , a Rees-Stanojević cosine sum  $g_n$  can be associated such that  $g_n$  converges to  $f$  pointwise, and a necessary and sufficient condition for  $L^1$  convergence of  $g_n$  to  $f$  is given. As a corollary to that result we have a generalization of the classical result of this kind. Other corollaries are given concerning the well-known integrability conditions.

This paper gives an analogue for modified cosine sums of the classical result concerning  $L^1$  convergence of a Fourier sine series. Rees and Stanojević [1] introduced these cosine sums that approximate their pointwise limit "better" than the classical cosine series since they converge in the  $L^1$  metric space to their limit when the classical cosine series may not.

**LEMMA 1.** Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  where  $f_n(x) = \frac{1}{2}a(0) + \sum_{k=1}^n a(k)\cos kx$ ,  $\lim_{n \rightarrow \infty} a(n) = 0$ , and  $\sum_{k=0}^{\infty} |\Delta a(k)| < \infty$ . Let  $g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a(k) + \sum_{k=1}^n \sum_{j=k}^n \Delta a(j)\cos kx$ . Then  $\lim_{n \rightarrow \infty} g_n(x) = f(x)$ .

**THEOREM 1.** Let  $f$ ,  $f_n$ , and  $g_n$  be as defined in Lemma 1. Then  $g_n$  converges to  $f$  in the  $L^1$  metric if and only if given  $\epsilon > 0$  there exists  $\delta(\epsilon) > 0$  such that  $\int_0^\delta |\sum_{k=n+1}^{\infty} \Delta a(k)D_k(x)| < \epsilon$  for all  $n \geq 0$ , where  $D_k(x)$  is the Dirichlet kernel.

**COROLLARY 1.** Let  $f_n$  and  $f$  be as defined in Lemma 1. If for  $\epsilon > 0$  there exists  $\delta(\epsilon) > 0$  such that  $\int_0^\delta |\sum_{k=n}^{\infty} \Delta a(k)D_k(x)| < \epsilon$  for all  $n \geq 0$  then  $f_n$  converges to  $f$  in the  $L^1$  metric if and only if  $\lim_{n \rightarrow \infty} a(n) \log n = 0$ .

**COROLLARY 2.** Let  $f$  and  $g_n$  be as defined in Lemma 1. If  $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n+1) < \infty$ , then  $g_n$  converges to  $f$  in the  $L^1$  metric.

**COROLLARY 3.** Let  $f$  and  $g_n$  be as defined in Lemma 1. If  $\sum_{k=1}^{\infty} |\Delta a(k)| \log k < \infty$ , then  $g_n$  converges to  $f$  in the  $L^1$  metric.

**COROLLARY 4.** Let  $f$  and  $g_n$  be as defined in Lemma 1. If  $a(n) = b(n) + c(n)$  where  $\lim_{n \rightarrow \infty} b(n) = \lim_{n \rightarrow \infty} c(n) = 0$ ,  $\sum_{n=1}^{\infty} |\Delta b(n)| \log n < \infty$ , and  $\sum_{n=1}^{\infty} |\Delta^2 c(n)|(n+1) < \infty$ , then  $g_n$  converges to  $f$  in the  $L^1$  metric.

AMS (MOS) subject classifications (1970). Primary 42A20, 42A32.

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<sup>1</sup>Portions of these results appear in a doctoral thesis of John W. Garrett at the University of Missouri-Rolla in 1974.

COROLLARY 5. *Let  $f$  and  $g_n$  be as defined in Lemma 1. If  $a(n) = \alpha(n)\beta(n)$  where  $\sum_{n=1}^{\infty} |\Delta\alpha(n)| < \infty$ ,  $|\beta(n)| \leq M$ ,  $\sum_{n=1}^{\infty} |\Delta^2\beta(n)|(n+1) < \infty$ , and  $\sum_{n=1}^{\infty} |\beta(n)\Delta\alpha(n)| \log n < \infty$ , then  $g_n$  converges to  $f$  in the  $L^1$  metric.*

Proofs and details of these results will appear elsewhere.

#### REFERENCE

1. C. S. Rees and Č. V. Stanojević, *Necessary and sufficient conditions for integrability of certain cosine sums*, J. Math. Anal. Appl. 43 (1973), 579–586. MR 48 #794.

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