

01 Nov 2005

A GRASP for Unitary Space-Time Codes

Adam Panagos

Kurt Louis Kosbar

Missouri University of Science and Technology, kosbar@mst.edu

Follow this and additional works at: https://scholarsmine.mst.edu/ele_comeng_facwork

 Part of the [Electrical and Computer Engineering Commons](#)

Recommended Citation

A. Panagos and K. L. Kosbar, "A GRASP for Unitary Space-Time Codes," *Proceedings of the 2005 IEEE Global Telecommunications Conference (2005, St. Louis, MO)*, vol. 6, pp. 3310-3313, Institute of Electrical and Electronics Engineers (IEEE), Nov 2005.

The definitive version is available at <https://doi.org/10.1109/GLOCOM.2005.1578387>

This Article - Conference proceedings is brought to you for free and open access by Scholars' Mine. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Research & Creative Works by an authorized administrator of Scholars' Mine. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact scholarsmine@mst.edu.

A GRASP for Unitary Space-Time Codes

Adam Panagos
 Department of Electrical and
 Computer Engineering
 University of Missouri-Rolla
 Rolla, MO 65401
 panagos@umr.edu

Kurt Kosbar
 Department of Electrical and
 Computer Engineering
 University of Missouri-Rolla
 Rolla, MO 65401
 kosbar@umr.edu

Abstract—Unitary space-time codes perform well at high signal-to-noise ratios on MIMO channels even when the propagation coefficients between transmitter and receiver are unknown. One method of constructing unitary space-time constellations uses a random search to find signal constellations that minimize the maximum pairwise correlation between transmitted signals. The work presented here uses a greedy randomized adaptive search procedure (GRASP) for finding good unitary space-time code constellations. Simulation results show that, on average, this technique finds codes with better correlation properties than the random search method. This new search procedure was also used to find signal constellations with better correlation properties than those previously obtained with the random search technique.

I. INTRODUCTION

Pioneering work by Foschini, Gans [1], and Telatar [2] on the wireless multiple-input multiple-output (MIMO) communication channel showed the use of multiple antenna elements can significantly increase channel capacity. Achieving these promising theoretical results is the goal of space-time coding.

Unitary space-time codes [3],[4] have been shown to perform well on MIMO channels at high signal-to-noise ratios or when the block fading length of the channel T is much larger than the number of transmitting antennas M . A systematic design procedure for constructing unitary space-time code constellations has been presented in [4]. The design procedure starts with the first signal in the constellation and generates the remaining signals by successive rotations of the initial signal. This construction technique is attractive since it only requires the initial signal Φ_1 and rotation matrices Θ_k , $k = 1, \dots, K$ to be stored in order to generate large constellations.

While the systematic construction method makes it simple to construct an arbitrary signal constellation for a specific number of transmit and receive antennas, using this technique to design a constellation that yields a small probability of error can be computationally expensive. The computational complexity lies in choosing the proper rotation matrices Θ_k $k = 1, \dots, K$ that generate a constellation that minimizes the maximum pairwise correlation between signals in the constellation. The maximum pairwise correlation between signals is the design metric δ which is defined as:

$$\delta = \min_{\Theta_k} \max_{i \neq j} \|\Phi_i^\dagger \Phi_j\| \quad (1)$$

where signals Φ_i and Φ_j are two of the L unique constellation points. The norm operator $\|\cdot\|$ is defined as:

$$\|\Phi_i^\dagger \Phi_j\|^2 = \frac{1}{M} \text{tr} \left\{ (\Phi_i^\dagger \Phi_j)^\dagger (\Phi_i^\dagger \Phi_j) \right\} \quad (2)$$

where M is the number of receive antennas and $\text{tr}(\cdot)$ is the trace operation.

The systematic construction method restricts the form of Θ_k so the collection of possible Θ_k is finite, but very large in many practical cases. Performing an exhaustive search over Θ_k for a globally minimum value of δ may not be practical. Because of this, the systematic design technique uses a random search to find Θ_k that yield a minimum value of δ .

In this work, a greedy randomized adaptive search procedure (GRASP) [5] is used as an alternative search technique for finding the rotation matrices Θ_k that yield a minimum value of δ . Simulation results show that the GRASP is able to find signal constellations on average that have a smaller δ .

The following section provides an overview of the systematic construction technique for generating unitary space-time constellations [4]. Section 3 provides a description of GRASP [5], how it has been applied to finding optimum unitary space-time constellations, and how it differs from the random search method of [4]. Section 4 discusses the performance of the new search procedure and simulation results show that on average, the GRASP finds unitary space-time constellations with better correlation properties. A comparison of the codes found using the random search method and newly found codes using GRASP have been tabulated in Section 5. Finally, the work presented here is summarized and final comments are made in the conclusion of Section 6.

II. UNITARY SPACE-TIME CODES

The K -indexed systematically generated unitary space-time signal $S_{l_1 \dots l_K}$ is:

$$S_{l_1 \dots l_K} = \sqrt{T} \Phi_{l_1 \dots l_K} \quad (3)$$

where the integer T is the block fading length of the channel, the total of number of signals in the constellation is $L = \prod_{k=1}^K L_k$, the indices l_1, \dots, l_K are integers such that $1 \leq l_i \leq L_i$ for $k = 1 \dots K$, and $\Phi_{l_1 \dots l_K}$ is a $T \times M$ unitary matrix where M is the number of transmit antennas.

The unitary matrices $\Phi_{l_1 \dots l_K}$ are generated using:

$$\Phi_{l_1 l_2 \dots l_K} = \Theta_1^{l_1-1} \Theta_2^{l_2-1} \dots \Theta_K^{l_K-1} \Phi_1 \quad (4)$$

where $\Phi_1 \equiv \Phi_{11\dots 1}$ is the the “starting matrix” and Θ_k for $k = 1 \dots K$ are $T \times T$ diagonal matrices defined as:

$$\Theta_k = \text{diag} \left\{ \exp \left(i \frac{2\pi}{L_k} u_{k1} \right), \dots, \exp \left(i \frac{2\pi}{L_k} u_{kT} \right) \right\} \quad (5)$$

where u_{kt} for $k = 1 \dots K$ and $t = 1 \dots T$ are integers such that $0 \leq u_{kt} \leq L_k - 1$.

The matrices $\Phi_{l_1 \dots l_K}$ are unitary matrices. These matrices are calculated from successive rotations of Φ_1 according to (4). Thus, we require that:

$$\Phi_1^\dagger \Phi_1 = I_M \quad (6)$$

where \dagger is the complex conjugate transpose operations and I_M is an $M \times M$ identity matrix. To ensure this constraint is met, the M columns of Φ_1 can be chosen as the first M columns of a $T \times T$ DFT matrix [4].

Given the number of transmit antennas M , the number of receive antennas N , the constellation index K , the starting matrix Φ_1 , and the number of signals $L = L_1 \cdot L_2 \dots L_K$, the designer is free to choose the integer elements u_{kt} for $k = 1 \dots K$, $t = 1 \dots T$. The collection of $\{u_{kt}\}$ is defined as the $K \times T$ parity check matrix U . Since the rotation matrices Θ_k are constructed from U , the design criteria δ for the K -indexed constellation can be re-written as:

$$\delta = \min_U \max_{\mathbf{l} \neq \mathbf{l}'} \|\Phi_1^\dagger \Phi_{\mathbf{l}}\| \quad (7)$$

where we have defined the vectors $\mathbf{l} = l_1 \dots l_L$ and $\mathbf{l}' = l'_1 \dots l'_L$ to simplify notation and used prime (') to indicate that \mathbf{l} and \mathbf{l}' are different. Choosing U appropriately is a important design decision as the choice of U can have a profound impact on the unitary space-time code's performance in terms of minimizing transmission errors.

III. THE GRASP

A GRASP is a general algorithm that has been successfully applied to a variety of problems such as set covering, quadratic assignment, and random graph problems [5]. The GRASP technique is attractive since each iteration yields a solution to the current optimization problem. In general, increasing the number of iterations increases the probability of finding a good solution. The number of iterations can be chosen based on the amount of computation time that can be tolerated.

Each GRASP iteration consists of two phases. During the first phase, solutions to the given problem are constructed in a greedy and random manner. The set of best solutions from phase one is referred to as the restricted candidate list (RCL). The number of solutions in the RCL can be controlled through use of a cardinality restriction (i.e. the best $\beta = 10$ solution are placed on the RCL). In phase two, solutions from the RCL are selected at random and a local search is performed in the neighborhood of the selected solution. During the local

search phase the GRASP algorithm converges to a *locally* optimum solution in the neighborhood of the randomly chosen solution. This two phase process is repeated numerous times and the best found locally optimum solution found is kept as the best found *global* solution. Pseudo-code for a generic GRASP procedure can be seen in Table I [5]:

TABLE I
GENERIC GRASP PSEUDO-CODE

```

procedure GRASP()
  while GRASP stopping criteria not satisfied
    Solution = ConstructGreedyRandomizedSolution()
    LocalSearch(Solution)
    UpdateSolution(Solution,BestSolutionFound)
  end while
  return(BestSolutionFound)
end GRASP
    
```

Section 2 described how the choice of the parity check matrix U can have a significant impact on the performance of the unitary-space time constellation generated by the systematic construction technique. The size of the parity check matrix space is very large for most codes of practical consideration. For example, with $T = 8$, $K = 1$ and $L = 16$, the number of possible parity check matrices that can be chosen is $8^{16} \approx 10^{14}$. For constellations with more points, the parity check matrix space is even larger. An exhaustive search for the optimum U is thus computationally expensive. Some avoid the prohibitive cost of an exhaustive search by employing a random search to find parity check matrices U that yield as small a δ as possible [4].

The search technique proposed in this work for finding unitary space-time constellations with small δ uses the GRASP framework and thus makes several changes to the random search technique. The simulation results of the next section show that on average this new search technique is able to find parity check matrices that yield smaller δ than the original random search method. The differences between the random search method and this new technique are as follows:

- The random search technique restricted the possible parity check matrices to be of systematic form. Thus, the first $K \times K$ block of the parity check matrix U was a $K \times K$ identity matrix. The $K \times (T - K)$ portion of U has been previously denoted as U' . The parity check matrices found using the GRASP technique are not restricted to systematic form.
- The elements of the parity check matrix are no longer restricted to be integers, but can be any real value. A simple example below shows that in general, a smaller value of δ can be achieved if real values of u_{kt} are used. Except for increasing the cardinality of the parity check matrix space from a finite value to an infinite value, using real values for the elements of U does not violate any power, bandwidth, or physical constraints of the problem. We note that using real valued parity check

matrix entries may introduce new implementation issues regarding numerical precision, rounding, etc. We do not consider implementation issues in this work, but simply focus on finding the best possible codes.

- After generating a random value for U , the Nelder-Mead [6] search technique is used to find the local minimum of δ in the neighborhood of U .

The use of real values for $\{u_{kt}\}$ is an important change as real values can in general allow smaller values of δ to be obtained. The following simple example demonstrates this.

Consider the following parameters: $L = 2, q = 2, T = 3, M = 1$, and $K = 1$. The systematic construction method restricts u_{kt} to be integers in the range $0 \leq u_{kt} \leq q$. Thus, in the systematic construction technique, u_{kt} can only take on the values one and zero. The eight possible values for U and the corresponding values for delta have been tabulated in Table II. We see that $\delta = 0.333$ is the best value of δ that can be obtained for this set of system parameters using integer elements for the parity check matrix U .

TABLE II
POSSIBLE VALUES OF δ FOR A SIMPLE CODE.

U	δ
[0 0 0]	1.000
[0 0 1]	0.333
[0 1 0]	0.333
[0 1 1]	0.333
[1 0 0]	0.333
[1 0 1]	0.333
[1 1 0]	0.333
[1 1 1]	1.000

If u_{kt} are allowed to take on real values, a value of $\delta < 1 \times 10^{-3}$ can be achieved with $U \approx [0.58, -0.08, -0.75]$. This simple example shows that it may be possible to reduce δ by using real values for u_{kt} . A plot of the design metric surface δ and the corresponding contour plot for this simple example is shown in Fig. 1. This plot was generated by holding u_{13} fixed at $u_{13} = -0.75$ and allowing u_{11} and u_{12} to range from -1 to 1. At each location (u_{11}, u_{12}) , the value of δ was calculated yielding the surfaces and contours shown. Note that one of the minimum values of $\delta < 1 \times 10^{-3}$ occurs at $U \approx [0.58, -0.08, -0.75]$ as already stated. This figure shows that using real values for u_{kt} allow us to reach points on the design metric surface δ that are lower and and unreachable when u_{kt} are restricted to integers.

A pseudo-code version of the GRASP algorithm used in this work is given in Table III. The simulation results presented below used the following parameters: For each GRASP iteration performed, $N = 500$ parity check matrices were randomly generated. The RCL was constructed by selecting the parity check matrices with the $\beta = 10$ smallest values of δ . The search for a locally optimum parity check matrix in Step 5 was performed using the Nelder-Mead simplex method, a common multivariable optimization algorithm. The local search was stopped when consecutive iterations of the minimization algorithm yielded changes in δ less than 0.5

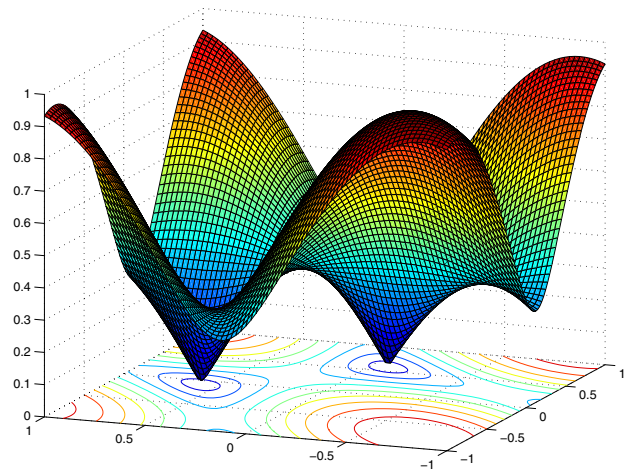


Fig. 1. δ surface and contour for the simple code

percent.

TABLE III
UNITARY SPACE-TIME CODE GRASP PSEUDO-CODE

```

For n = 1 to Number of GRASP Iterations
  1. Randomly generate N parity check matrices  $U_n, n = 1, \dots, N$ 
  2. Calculate  $\delta_n$  on each  $U_n$ 
  3. Create RCL by selecting best  $\beta$  parity check matrices  $U_n$ 
  4. Randomly select element  $U_{select}$  from RCL
  5. Perform search for locally optimum  $\delta_{loc,opt}$  in neighborhood of  $U_{select}$ 
  6. Store  $\delta_{loc,opt}$  in array  $\Delta$  and associated  $U$  in array  $\mathcal{U}$ 
End
Calculate  $\delta_{final} = \min(\Delta)$ 
Parity check matrix associated with  $\delta_{final}$  is  $U_{final}$ 
    
```

IV. NEW SEARCH TECHNIQUE PERFORMANCE

The performance of the proposed search technique was evaluated by running numerous searches for a variety of different codes. As described above, for each GRASP iteration a locally optimum value of δ was found. The distribution of δ obtained using the GRASP was then compared to the distribution of δ obtained using the random search method of [4].

As an example of the difference in the distribution on δ consider Fig. 2. This figure shows the delta distribution obtained using a random search and the GRASP for Code 06 of [4]. Note that the mean value and variance of δ obtained using the new search technique was considerably smaller than the mean and variance of δ obtained using the random search method. Plots of the distribution of δ for other codes show similar results. The results for the other codes have been summarized in Table IV. Thus, on average, the GRASP is able to find unitary-space time constellations with better δ than the previously proposed random search technique.

TABLE IV

COMPARISON OF EMPIRICAL $E[\delta]$ AND $\text{Var}[\delta]$ FOR THE GRASP AND RANDOM SEARCH METHODS.

Code	$E[\delta_{Grasp}]$	$\text{Var}[\delta_{Grasp}]$	$E[\delta_{random}]$	$\text{Var}[\delta_{random}]$
02	0.338	4.52×10^{-4}	0.572	0.0133
03	0.453	9.33×10^{-4}	0.702	0.00954
04	0.588	2.88×10^{-4}	0.731	0.00493
05	0.589	7.05×10^{-4}	0.783	0.00513
06	0.665	4.99×10^{-4}	0.809	0.00276
07	0.715	4.38×10^{-4}	0.857	0.00330
08	0.765	3.79×10^{-4}	0.869	0.00139

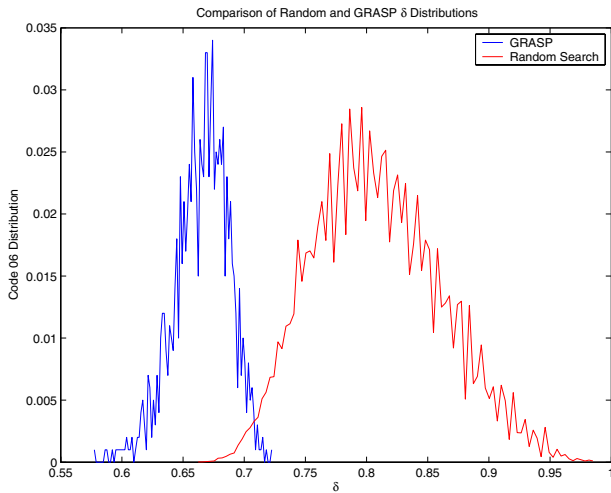


Fig. 2. δ distribution for Code 06

V. NEW BEST KNOWN CODES

The GRASP proposed here was able to find codes with better δ than those found with the random search technique. These δ for the newly found best codes have been tabulated in Table V along with the previously best found value of δ . The corresponding parity check matrices U for these codes have been tabulated in Table VI.

TABLE V

COMPARISON OF BEST FOUND CODES.

Code No.	δ_{GRASP}	δ_{random}
02	0.291	0.306
03	0.279	0.354
04	0.521	0.534
05	0.483	0.559
06	0.577	0.643
07	0.600	0.696
08	0.697	0.749

VI. CONCLUSION

A greedy randomized adaptive search procedure (GRASP) has been proposed for searching for optimum unitary space-time code constellations. This new procedure was able to find unitary space-time constellations with better design metrics (i.e. δ) than those previously published. Also, simulation results show that this technique generally yields distributions

TABLE VI

ASSOCIATED PARITY CHECK MATRICES

Code No.	$\approx U_{GRASP}$
02	[5.6, 1.4, 2.5, 7.2, 9.8, 4.5, 0.65, 10.6]
03	[2.6, 0.22, 4.1, 2.3, 1.2, 1.7, 1.8, 0.69] [1.1, 2.4, 3.2, 0.72, 3.8, 2.7, 4.2, 3.4] [3.5, 4.5, 1.8, 3.1, 2.8, 1.1, 3.0, 2.3]
04	[25.4, 20.6, 12.5, 71.03, 21.5, 37.4, 82.1, 56.3]
05	[1.7, 3.1, 0.0, 2.5, 3.4, 2.6, 2.9, 0.0] [3.4, 2.9, 3.5, 2.2, 1.1, 1.2, 2.6, 1.1] [3.0, 1.3, 1.0, 0.83, 2.1, 0.82, 1.0, 2.2] [3.0, 1.3, 2.0, 0.92, 2.8, 0.0, 1.3, 3.2]
06	[2.9, 18.4, 0.41, 1.1, 14.4, 8.0, 6.2, 0.8] [1.1, 14.6, 14.6, 15.9, 4.6, 11.7, 7.1, 15.3]
07	[2.9, 3.2, 4.5, 2.0, 3.9, 1.0, 0.99, 1.3] [2.1, 1.1, 1.2, 2.0, 1.0, 5.4, 3.9, 5.6] [5.3, 2.2, 4.0, 5.4, 3.9, 1.7, 2.1, 2.1] [0.81, 2.2, 5.1, 5.3, 4.6, 0.0, 0.0, 0.0]
08	[8.5, 23.6, 33.1, 43.7, 7.9, 18.3, 10.6, 4.8] [25.7, 1.3, 11.7, 12.0, 25.3, 36.6, 32.9, 3.4]

on the design metric δ with smaller mean and variance than the random search method originally used. Thus, a GRASP is well suited for the problem of finding good unitary space-time constellations. The authors are currently investigating ways to modify the basic GRASP framework presented here to increase search speed, reduce algorithmic complexity, and give fundamental insights into the problem of finding good codes. A complexity comparison between this new method and other search techniques is also in progress.

REFERENCES

- [1] G. Foschini and M. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, no. 3, pp. 311–335, Mar. 1998.
- [2] E. Telatar, "Capacity of multi-antenna gaussian channels," *AT&T Bell Laboratories Internal Tech. Memo.*, June 1995.
- [3] B. Hochwald and T. Marzetta, "Unitary space-time modulation for multiple-antenna communications in rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 46, pp. 543–564, Mar. 2000.
- [4] B. Hochwald, T. Marzetta, T. Richardson, W. Sweldens, and R. Urbanke, "Systematic design of unitary space-time constellations," *IEEE Trans. Inform. Theory*, vol. 46, no. 6, pp. 1962–1973, Sept. 2000.
- [5] T. Feo and M. Resende, "Greedy randomized adaptive search procedures," *Journal of Global Optimization*, vol. 6, pp. 109–133, 1995.
- [6] J. Nelder and R. Mead, "A simplex method for function minimization," *Comput. J.*, vol. 7, pp. 308–313, 1965.