Doppler Shift in Frequency in the Transport of Electromagnetic Waves through an Underdense Plasma

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Doppler Shift in Frequency in the Transport of Electromagnetic Waves through an Underdense Plasma*

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In an earlier publication, the validity of the radiation transport theory was studied for the calculation of multiple scattering of electromagnetic waves by a turbulent plasma. In the present paper, we extend the transport theory to include a description of the Doppler shift in frequency caused by electron motion.

1. INTRODUCTION

In Part I of this series[1] the classical radiation transport equation was derived from Maxwell's equations for the study of scattering of electromagnetic waves by a turbulent plasma. In Part II[2] some techniques for using the transport equation were discussed. In both of these papers the Doppler shift in frequency caused by the motion of the scattering electrons was neglected. In the present paper we extend the transport theory to include any frequency shift of the scattered waves.

An exhaustive analysis of the relation between a wave equation and the corresponding classical transport approximation has yet to be made. The first such analysis seems to have been given by Foldy,[3] who discussed the scattering of scalar waves by a set of uncorrelated point scatterers, obtaining a transport equation. The quantum theory of scattering by a "weakly bound medium" was related to a classical transport theory by Watson.[4] It was an adaptation of the methods used in this work to Maxwell's equations which was given in I. A different approach was used by Barabanenkov and Finkel'berg,[5] who derived a transport equation from the scalar wave equation using a "Bethe–Salpeter" type of equation.

In Sec. II, we summarize the results derived in this paper. These lead to a radiation transport equation of conventional form,[6] the scattering kernel being explicitly expressed in terms of plasma density fluctuations. The reader who is not interested in the details of the derivation will probably find the account in Sec. II adequate for using the transport equation.

2. SUMMARY OF RESULTS

The phenomena which we wish to describe are illustrated in Fig. 1. A plasma of finite extent is illuminated by an electromagnetic wave emitted by a distant source S and propagating in the direction $\hat{\mathbf{k}}$. The intensity of the waves scattered by the plasma is measured with a receiver R, also a great distance away. (The restriction to a distant source and receiver is, of course, not required for a derivation of the transport equation.)

Several assumptions concerning the plasma were introduced in I. We shall accept these here and, in addition, explicitly suppose the plasma electrons to have nonrelativistic energies. The assumed turbulence properties of the plasma will be reviewed later in this section. The nonrelativistic assumption will be expressed by the inequality

$$(kR_c)(v_e/c) \ll 1,$$

which we will call NR. Here $k/2\pi$ is the wave number of the radiation, R_c a measure of the distance over which plasma motions are correlated, v_e the mean speed of the plasma electrons, and c the speed of light.

As in I, we suppose the plasma to be underdense (Assumption B3) and that $kR_S \gg 1$ (Assumption B4), where R_S is the "size" of the plasma. Assumption B4 allows us to ignore diffraction scattering from the entire plasma (in all but a small cone with axis parallel to $\hat{\mathbf{k}}$).

In the classical theory of radiation transport, the flow of radiant energy at a point $\mathbf{x}$ per unit area, per unit time, and traveling in the direction $\hat{\mathbf{p}}$ is

$$I(\mathbf{x}, \hat{\mathbf{p}}, \omega)\, d\Omega_{\hat{p}}\, d\omega. \tag{2.1}$$

The notation here implies that the radiation has an angular frequency ω, within the interval $d\omega$, and is confined to propagation directions lying within the solid angle $d\Omega_{\hat{p}}$.

For waves which have some degree of polarization, it is necessary to generalize (2.1). This was done by Chandrasekhar[6] and, in a similar manner, in I. To do this, we shall follow the notation of I and introduce the two unit vectors $\hat{\mathbf{e}}_{\hat{p}}(i)$, $i = 1, 2$, for a plane

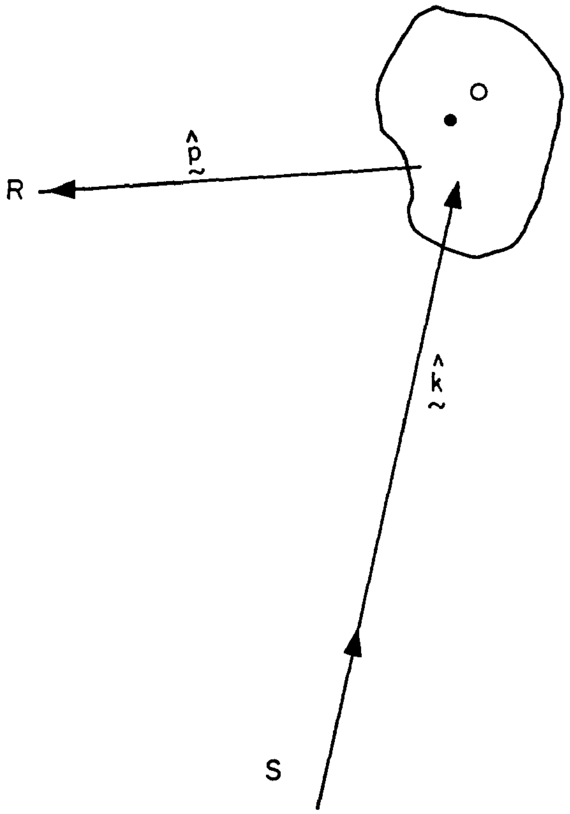

FIG. 1. Illustration of scattering from a plasma.

electromagnetic wave traveling in the direction $\hat{\mathbf{p}}$. The electric field vector for such a wave is of the form

$$\mathbf{E}_{\hat{p}} = [E_{\hat{p}}(1)\hat{\mathbf{e}}_{\hat{p}}(1) + E_{\hat{p}}(2)\hat{\mathbf{e}}_{\hat{p}}(2)]e^{-i\omega t}. \quad (2.2)$$

The unit vectors $\hat{\mathbf{e}}_{\hat{p}}(i)$ are defined in terms of $\hat{\mathbf{k}}$, the direction of propagation of the incident wave before entering the plasma.[7] These are

$$\hat{\mathbf{e}}_{\hat{p}}(2) = C(\hat{\mathbf{p}})\hat{\mathbf{p}} \times \hat{\mathbf{k}},$$
$$\hat{\mathbf{e}}_{\hat{p}}(1) = \hat{\mathbf{e}}_{\hat{p}}(2) \times \hat{\mathbf{k}}, \quad (2.3)$$

where

$$C(\hat{\mathbf{p}}) = (|\hat{\mathbf{p}} \times \hat{\mathbf{k}}|)^{-1}. \quad (2.3')$$

To define the polarization vectors for the incident wave, we orient the z axis of a rectangular coordinate system to be parallel to $\hat{\mathbf{k}}$ and choose $\hat{\mathbf{e}}_{\hat{k}}(1)$ and $\hat{\mathbf{e}}_{\hat{k}}(2)$ to be parallel to the x and y axes, respectively. For backscatter we define[8]

$$\hat{\mathbf{e}}_{-\hat{k}}(1) = \hat{\mathbf{e}}_{\hat{k}}(1),$$
$$\hat{\mathbf{e}}_{-\hat{k}}(2) = -\hat{\mathbf{e}}_{\hat{k}}(2). \quad (2.4)$$

The electric field at any point can be represented as a sum of waves of the form (2.2). If we fix our attention on a single "bundle" of wavelets propagating within $d\Omega_{\hat{p}}$ and $d\omega$, we may define the intensity as

$$I_{ij}(\mathbf{x}, \hat{\mathbf{p}}, \omega) = \text{const} \times \langle E^*_{\hat{p}}(i)E_{\hat{p}}(j)\rangle, \quad (2.5)$$

$i, j = 1, 2$. Here "$\langle\cdots\rangle$" represents an ensemble (or statistical) average over the plasma (and any source) fluctuations. The "constant" in Eq. (2.5) is defined by the following condition. We suppose that a filter at $\mathbf{x}$ passes only the component of $\mathbf{E}$ parallel to some direction $\hat{\mathbf{e}}$. Then, the power per unit area passed by the filter, corresponding to propagation within $d\Omega_{\hat{p}}$ and frequency within $d\omega$, is

$$\hat{\mathbf{e}} \cdot \left[\sum_{i,j=1}^{2} \hat{\mathbf{e}}_{\hat{p}}(i)I_{ij}(\mathbf{x}, \hat{\mathbf{p}}, \omega)\hat{\mathbf{e}}_{\hat{p}}(j)\right] \cdot \hat{\mathbf{e}}\, d\Omega_{\hat{p}}\, d\omega. \quad (2.6)$$

We suppose the statistical properties of the plasma to be represented as a stationary random process. If the plasma contains N free electrons with coordinates $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_N$, we take the probability that electron 1 is at $\mathbf{z}_1$ within d^3z_1 at time t_1, etc., to be

$$P_N(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2; \cdots; \mathbf{z}_N, t_N)\, d^3z_1 \cdots d^3z_N. \quad (2.7)$$

The statement that this is a stationary distribution function is equivalent to[9]

$$P_N(\mathbf{z}_1, t_1 + \tau; \mathbf{z}_2, t_2 + \tau; \cdots; \mathbf{z}_N, t_N + \tau)$$
$$= P_N(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2; \cdots; \mathbf{z}_N, t_N). \quad (2.8)$$

We further suppose that from P_N we can define a hierarchy of distribution functions as follows:

$$P_1(\mathbf{z}_1) = \int P_N\, d^3z_2 \cdots d^3z_N, \quad (2.9a)$$

$$P_2(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2) = \int P_N\, d^3z_3 \cdots d^3z_N, \quad (2.9b)$$

etc. Here

$$P_2(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2) = P_2(\mathbf{z}_1, 0; \mathbf{z}_2, t_2 - t_1),$$

etc.

Following the notation of I, we assume that P_2, $P_3, \cdots$ may be developed in terms of correlation functions. Thus, for example,

$$P_2(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2) = P_1(\mathbf{z}_1)P_1(\mathbf{z}_2)[1 + g(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2)]. \quad (2.10)$$

Here the "pair correlation function" g is considered to vanish for $|\mathbf{z}_1 - \mathbf{z}_2| \gg R_c$, the "correlation range," or for $|t_1 - t_2| \gg t_c$, the "correlation time." Again, we write P_3 in the form

$$P_3(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2; \mathbf{z}_3, t_3)$$
$$= P_1(\mathbf{z}_1)P_1(\mathbf{z}_2)P_1(\mathbf{z}_3)$$
$$\times [1 + g(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2) + g(\mathbf{z}_2, t_2; \mathbf{z}_3, t_3)$$
$$+ g(\mathbf{z}_3, t_3; \mathbf{z}_1, t_1) + g_3(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2; \mathbf{z}_3, t_3)]. \quad (2.11)$$

The "triplet correlation function" g_3 is assumed to vanish when any pair of the three coordinates is separated by a distance large compared to R_c or any pair of times, by an interval large compared to t_c.

Continuing as above, we can express the probability functions $P_4, P_5, \cdots$ in terms of correlation functions.[10] The n-particle correlation function

$$g_n(\mathbf{z}_1, t_1; \cdots; \mathbf{z}_n, t_n)$$

vanishes unless all n coordinates lie within a volume characterized by the linear dimension R_c and all n times within an interval of order t_c.

In the absence of significant effects from external magnetic fields and/or Coriolis forces, time-reversal invariance implies several symmetry relations for the P's and g's.[11–13] For the pair correlation we have, for example,

$$g(\mathbf{z}_1, t; \mathbf{z}_2, 0) = g(\mathbf{z}_1, -t; \mathbf{z}_2, 0). \quad (2.12a)$$

Because we have assumed a stationary random process, we may conclude that

$$\begin{aligned} g(\mathbf{z}_1, t; \mathbf{z}_2, 0) &= g(\mathbf{z}_1, 0; \mathbf{z}_2, -t) \\ &= g(\mathbf{z}_1, 0; \mathbf{z}_2, t), \end{aligned} \quad (2.12b)$$

using (2.12a). On setting $t_2 - t_1 \equiv \tau$, we obtain

$$\begin{aligned} g(\mathbf{z}_1, t_1; \mathbf{z}_2, t_2) &\equiv g(\mathbf{z}_1, \mathbf{z}_2; \tau) \\ &= g(\mathbf{z}_1, \mathbf{z}_2; -\tau). \end{aligned} \quad (2.13)$$

We finally assume, following I, that

$$\begin{aligned} g(\mathbf{z}_1, \mathbf{z}_2; \tau) &= g(\mathbf{z}_1; |\mathbf{z}_1 - \mathbf{z}_2|; \tau) \\ &\cong g(\mathbf{z}_2; |\mathbf{z}_1 - \mathbf{z}_2|; \tau). \end{aligned} \quad (2.14)$$

[The assumption (2.14) is not required for the derivation of the transport equation. It does permit us to write the scattering kernel (2.19) in "prettier" form, however.]

The mean plasma electron density at a point $\mathbf{z}$ is

$$\rho(\mathbf{z}) = NP_1(\mathbf{z}). \quad (2.15)$$

The electron collision frequency at $\mathbf{z}$ will be written as $\nu_c(\mathbf{z})$ and the plasma frequency as

$$\omega_p(\mathbf{z}) = [4\pi e^2 \rho(\mathbf{z})/m]^{\frac{1}{2}}. \quad (2.16)$$

The refractive index $n(\mathbf{z})$ of the plasma was discussed in I. The first approximation to this was written as n_1 and is given by the familiar expression

$$n_1^2(\mathbf{z}) = 1 - \omega_p^2[\omega(\omega + i\nu_c)]^{-1}. \quad (2.17a)$$

We shall, as in I, suppose the imaginary part of $n(\mathbf{z})$ to be negligible for propagation over distances comparable to R_c. This permits us to take

$$n_1^2(\mathbf{z}) \cong 1 - \omega_p^2(\omega^2 + \nu_c^2)^{-1} \quad (2.17b)$$

in Eqs. (2.20) and (2.22) below.

The absorption length $l_c(\mathbf{z})$ caused by electron collisions is expressed as

$$\frac{1}{l_c(\mathbf{z})} \cong \frac{\omega_p^2}{(\omega^2 + \nu_c^2)} \frac{\nu_c}{c}, \quad (2.18)$$

where c is the speed of light.

We now define the *scattering kernel* $\mathbf{M}$ for scattering a wave from the direction $\hat{\mathbf{p}}'$ to direction $\hat{\mathbf{p}}$ as

$$(ij|\, M(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega)\, |sr) \equiv \sigma_g(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega)(ij|\, m\, |sr), \quad (2.19a)$$

where

$$(ij|\, m\, |sr) = [\hat{\mathbf{e}}_{\hat{p}}(i) \cdot \hat{\mathbf{e}}_{\hat{p}'}(s)][\hat{\mathbf{e}}_{\hat{p}}(j) \cdot \hat{\mathbf{e}}_{\hat{p}'}(r)] \quad (2.19b)$$

and

$$\begin{aligned} \sigma_g(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega) = &\left(\frac{r_0^2}{1 + (\nu_c/\omega)^2}\right)[\rho^2(\mathbf{z})] \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} \\ &\times \int d^3R g(\mathbf{z}; R, \tau) \\ &\times \exp\,[in_1(\mathbf{z})k(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \cdot \mathbf{R}]. \end{aligned} \quad (2.20)$$

For later reference we observe that because of the time reversal invariance property (2.12a) σ_g is *even* in Ω.

The absorption length $l_t(\mathbf{z})$ for scattering is defined by the equation

$$\frac{1}{l_t(\mathbf{z})} = \frac{1}{2} \int d\Omega_{\hat{p}'} \sigma_g(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')[1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2], \quad (2.21)$$

where

$$\begin{aligned} \sigma_g(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') &\equiv \int_{-\infty}^{\infty} d\Omega \sigma_g(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega) \\ &= \left(\frac{r_0^2}{1 + \nu_c^2/\omega^2}\right) \rho^2(\mathbf{z}) \int d^3R g(\mathbf{z}; R) \\ &\quad \times \exp\,[in_1 k(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \cdot \mathbf{R}] \end{aligned} \quad (2.22)$$

and

$$g(\mathbf{z}; R) \equiv g(\mathbf{z}; R, 0). \quad (2.23)$$

An elementary calculation yields

$$\sum_{s=1}^{2} \int_{-\infty}^{\infty} d\Omega \int d\Omega_{\hat{p}'} (ij|\, M(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega)\, |ss) = \frac{\delta_{ij}}{l_t}. \quad (2.24)$$

The net absorption length $l(\mathbf{z})$ is defined, finally, as

$$l^{-1} = l_t^{-1} + l_c^{-1}. \quad (2.25a)$$

We note that this is equivalent to the equation

$$l^{-1} = 2k \,\mathrm{Im}\, n, \quad (2.25b)$$

where n is the refractive index given to the order of accuracy obtained in I.

The transport equation for I_{ij} (to be derived in Sec. V) is

$$\hat{\mathbf{p}} \cdot \nabla \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega) + \frac{1}{l(\mathbf{x})} \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega) = \int_0^\infty d\omega' \int d\Omega_{\hat{\mathbf{p}}'} \mathbf{M}(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \omega - \omega') \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}', \omega'). \quad (2.26)$$

Here we have written I_{ij} as a column matrix with four elements and $(ij|\, M\, |sr)$ as a 4×4 square matrix (evaluated at the point $\mathbf{x}$, of course). The product $\mathbf{MI}$ is then the column matrix with elements

$$\sum_{s,r=1}^{2} (ij|\, M\, |sr) I_{sr}, \quad i, j = 1, 2.$$

We see from Eq. (2.20) that $\mathbf{M}$ will vanish for $|\omega - \omega'| \gg t_c^{-1}$. If $\mathbf{I}$ is nearly constant over a frequency range of order t_c^{-1}, we can rewrite Eq. (2.26) in the form

$$\hat{\mathbf{p}} \cdot \nabla \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega) + \frac{1}{l(\mathbf{x})} \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega) = \int d\Omega_{\hat{\mathbf{p}}'} \mathbf{M}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}', \omega). \quad (2.27)$$

Here

$$\mathbf{M}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \equiv \int_{-\infty}^{\infty} d\Omega \mathbf{M}(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \Omega). \quad (2.28)$$

Alternatively, if the radiation is confined to a sufficiently narrow frequency interval $\Delta\omega$, we can integrate Eq. (2.26) over frequency to obtain Eq. (2.27), as satisfied by the integrated intensity

$$\mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}) \equiv \int_0^\infty d\omega \mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega). \quad (2.29)$$

It was this equation which was obtained in I.

The fundamental assumption required to derive the classical transport equation (2.26) is that

$$R_c \ll l_t, \quad (2.30\text{a})$$

where (we recall) R_c is the correlation length. When R_c may be taken as k^{-1}, we may rewrite (2.30a) as the condition that

$$\zeta(\omega_p^4/\omega^4) \ll 1, \quad (2.30\text{b})$$

where

$$\zeta = \delta\rho^2/\rho^2, \quad (2.31)$$

with $\delta\rho^2$ the mean-square electron density fluctuation.

In the derivation of Eq. (2.26), it was also assumed that the paths of geometrical optics for rays propagating with the refractive index $n(\mathbf{z})$ could be approximated by straight lines. More generally, Eq. (2.26) must be integrated along curved ray paths.

3. THE POWER SPECTRUM

We consider an electric field variable $E(t)$ defined over the "long" time interval $-\frac{1}{2}T < t < \frac{1}{2}T$ and vanishing outside this interval. In representing a scattered wave, E will depend parameterically on the electron coordinates $\mathbf{z}_1, \cdots, \mathbf{z}_N$ and on any random variables characterizing the source. It will be convenient to use a complex representation for E, so the "power density" is

$$\mathcal{T}_0 = (8\pi)^{-1} \langle E^*(t) E(t) \rangle, \quad (3.1)$$

in a suitable system of units.[14] Here the average $\langle \cdots \rangle$ represents an average over both plasma electron coordinates and over source fluctuations. That is,

$$\langle E^*(t) E(t) \rangle = \int P_N \, d^3 z_1 \cdots d^3 z_N \langle E^*(t) E(t) \rangle_S, \quad (3.2)$$

where $\langle \cdots \rangle_S$ represents an average over source fluctuations only. We extend the assumption (2.8) that we are dealing with a stationary random process to include the source. Thus, for example,

$$\langle E^*(t) E(t) \rangle = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt \langle E^*(t) E(t) \rangle. \quad (3.3)$$

The field $E(t)$ is expressed in terms of its Fourier transform $\hat{E}(\omega)$ as

$$E(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \hat{E}(\omega) e^{-i\omega t} \, d\omega. \quad (3.4)$$

The *power spectrum* of $E(t)$ is then

$$\mathcal{T}(\omega) = \frac{1}{T} \left\langle \frac{|\hat{E}(\omega)|^2}{8\pi} \right\rangle = \frac{1}{8\pi} \int \langle E^*(t) E(t + \tau) \rangle \frac{e^{i\omega\tau}}{2\pi} \, d\tau, \quad (3.5)$$

normalized to

$$\int_{-\infty}^{\infty} \mathcal{T}(\omega) \, d\omega = \mathcal{T}_0. \quad (3.6)$$

It should be noted that we are here defining the power spectrum over the interval $-\infty < \omega < +\infty$. We shall see that our transport equation is even in ω, so $\mathbf{I}$ may be defined on the interval $0 < \omega < \infty$.

The incident plane wave emitted by the distant source (see Fig. 1) is assumed, for the present, to be plane polarized[15] and of the form

$$\mathbf{E}_I(\mathbf{r}, t) = \hat{\mathbf{e}}_{\mathbf{k}}(1) E_I(\mathbf{r}, t),$$

$$E_I(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int \hat{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \, d\omega. \quad (3.7)$$

The power spectrum of the incident wave is

$$\mathcal{T}_I(\omega) = T^{-1} \langle |\hat{E}_0(\omega)|^2 / 8\pi \rangle = \mathcal{T}_I(-\omega), \quad (3.8)$$

which must be even in ω if it is to correspond to a physical wave. The incident intensity, representing flow of power per unit area, is then (here c is again the speed of light)

$$I^0(\omega) = 2c\mathfrak{I}_I(\omega), \quad 0 < \omega < \infty. \tag{3.9}$$

The total intensity is then

$$I^0 \equiv \int_0^\infty I^0(\omega)\, d\omega. \tag{3.10}$$

4. THE BORN APPROXIMATION

It is instructive to first calculate the scattered power in the Born approximation.[16] The scattered waves at a point $\mathbf{r}$ far from the plasma can then be written in the form [see Eqs. (2.3)]

$$\mathbf{E}_{sc}(\mathbf{r}, t) = \sum_{j=1}^{2} \hat{\mathbf{e}}_{\hat{\mathbf{p}}}(j) E_{sc}(j, t),$$

$$E_{sc}(j, t) = \sum_{\alpha=1}^{N} G^0_{r\alpha} f_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}) E_I(\mathbf{z}_\alpha, t). \tag{4.1}$$

Here E_I is the incident field (3.7) and $\mathbf{z}_\alpha \equiv \mathbf{z}_\alpha(t_\alpha)$, $\alpha = 1, 2, \cdots, N$, is an electron coordinate evaluated at the retarded time

$$t_\alpha = t - R_\alpha/c, \tag{4.2}$$

where

$$\mathbf{R}_\alpha = \mathbf{r} - \mathbf{z}_\alpha(t_\alpha) \tag{4.3}$$

and $\hat{\mathbf{p}} = \hat{\mathbf{r}}$.

For a plane wave having wave number $k/2\pi$ and angular frequency $\omega = kc$,

$$G^0_{r\alpha} = \frac{e^{ikR_\alpha}}{R_\alpha}, \tag{4.4}$$

and

$$f_{ji}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) = -r_0\left(1 + i\frac{\nu_c}{\omega}\right)^{-1} \hat{\mathbf{e}}_{\hat{\mathbf{p}}}(j)\cdot\hat{\mathbf{e}}_{\hat{\mathbf{k}}}(i) \tag{4.5}$$

is the Thomson amplitude (here r_0 is the classical electron radius). Since E_I contains a spectrum of plane waves, we interpret (4.1) as follows:

$$G^0_{r\alpha} f_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}) E_I(\mathbf{z}_\alpha, t) = \int \frac{d\omega}{(2\pi)^{\frac{1}{2}}} \frac{e^{i\bar{k}R_\alpha}}{R_\alpha} f_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) \hat{E}_0(\omega) e^{i(\mathbf{k}\cdot\mathbf{z}_\alpha - \omega t)}. \tag{4.6}$$

Here $\bar{k} = \bar{k}(k)$ is the wave number after scattering for an incident wave number k. Now, $|\bar{k} - k| = O(kv_e/c) \ll k$ by assumption NR made at the beginning of Sec. II. We shall interpret this to mean, for example, that

$$f_{ji}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) \cong f_{ji}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \bar{\omega}), \tag{4.7}$$

where $\bar{\omega} = \bar{k}c$.

On setting $t' \equiv t + \tau$, using Eqs. (4.1) and (4.6), and writing $R_\alpha \cong r - \hat{\mathbf{p}}\cdot\mathbf{z}_\alpha$, etc., we obtain

$$\begin{aligned}\langle E^*_{sc}(j, t) E_{sc}(l, t')\rangle &= \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt \langle E^*_{sc}(j, t) E_{sc}(l, t')\rangle \\ &= \sum_{\alpha,\beta=1}^{N} \frac{1}{r^2}\int_{-\infty}^{\infty} d\omega f^*_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) f_{l1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) \\ &\quad \times \langle T^{-1} |\hat{E}_0(\omega)|^2 \exp(-i\omega\tau) \\ &\quad \times \exp\{i[(\mathbf{z}_\beta(t_\beta) - \mathbf{z}_\alpha(t'_\alpha))\cdot(\bar{k}\hat{\mathbf{p}} - \mathbf{k})]\}\rangle. \end{aligned} \tag{4.8}$$

According to Eq. (3.2) the average $\langle\cdots\rangle$ here implies the integration

$$\int d^3z_\alpha\, d^3z_\beta P_2(\mathbf{z}_\alpha, t'_\alpha; \mathbf{z}_\beta, t_\beta)\cdots. \tag{4.9}$$

From Eq. (2.10) we see that the term *not* involving the pair correlation function does not involve the times t'_α and t_β, which could be taken to be *any* two times in the interval $-\frac{1}{2}T < t < \frac{1}{2}T$. Also, for this term we have $\bar{k} = k$, corresponding to coherent scattering.

For the other term, involving the pair correlation function, we have $|\mathbf{z}_\alpha - \mathbf{z}_\beta| \simeq O(R_c)$. Since the distribution function is stationary, we may set

$$\begin{aligned}\mathbf{z}_\alpha(t'_\alpha) - \mathbf{z}_\beta(t_\beta) &= \mathbf{z}_\alpha(t'_\alpha + r/c) - \mathbf{z}_\beta(t_\beta + r/c) \\ &= \mathbf{z}_\alpha(t') - \mathbf{z}_\beta(t) + O((v_e/c)R_c)\end{aligned} \tag{4.10}$$

within the average in Eq. (4.8). The term of $O((v_e/c)R_ck)$ may be dropped in the exponential, using assumption NR. Also, since $\bar{k} - k \cong (kv_e)/c$, we may use assumption NR to set $\bar{k} = k$ in (4.8). Finally, then, we may write this equation in the form

$$\begin{aligned}\langle E^*_{sc}(j, t) E_{sc}(l, t')\rangle &= \sum_{\alpha,\beta=1}^{N} \frac{1}{r^2}\int_{-\infty}^{\infty} d\omega' f^*_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega') f_{l1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega')[8\pi\mathfrak{I}_I(\omega')] \\ &\quad \times e^{-i\omega'\tau}\int P_1(\mathbf{x})P_1(\mathbf{x}')[1 + g(\mathbf{x}, \mathbf{x}'; \tau)] \\ &\quad \times \exp[i(\mathbf{k}' - \mathbf{p}')\cdot(\mathbf{x} - \mathbf{x}')]\, d^3x\, d^3x', \end{aligned} \tag{4.11}$$

where $\hat{\mathbf{p}}' = k'\hat{\mathbf{r}}$.

The power spectrum of the scattered waves is then

$$\begin{aligned}\mathfrak{I}_{jl}(\omega) &= \frac{1}{8\pi}\int \langle E^*_{sc}(j, t) E_{sc}(l, t')\rangle \frac{e^{i\omega\tau}}{2\pi}\, d\tau \\ &= \frac{1}{r^2} f^*_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) f_{l1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) \int d\omega' \mathfrak{I}_I(\omega') \int \frac{d\tau}{2\pi} \\ &\quad \times \int d^3x\, d^3x' \rho(\mathbf{x})\rho(\mathbf{x}')[1 + g(\mathbf{x}, \mathbf{x}'; \tau)] \\ &\quad \times \exp[i(\omega - \omega')\tau] \exp[i(\mathbf{k} - \mathbf{p})\cdot(\mathbf{x} - \mathbf{x}')]. \end{aligned} \tag{4.12}$$

We have here used the relation (4.7) to remove the scattering amplitudes from the ω' integrand.

The *coherent* scattering in Eq. (4.12) is given by the term that does not involve g. This is immediately seen to reduce to

$$\mathcal{I}_{jl}(\omega)\big|_{\text{coh}} = \frac{1}{r^2} f^*_{j1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega) f_{l1}(\hat{\mathbf{p}}, \hat{\mathbf{k}}, \omega)\mathcal{I}_I(\omega) \times \left| \int d^3x \exp[i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}]\rho(\mathbf{x})\right|^2, \quad (4.13)$$

where $\mathbf{p} = k\hat{\mathbf{r}}$.

The remaining part of (4.12) represents the *incoherent* scattered power. This may be written in the form

$$\mathcal{I}_{jl}(\omega)\big|_{\text{inc}} = \frac{1}{r^2}\int d\omega' \,(jl|\, M(\hat{\mathbf{p}}, \hat{\mathbf{k}}; \omega - \omega')\,|11)\,\mathcal{I}_I(\omega'), \quad (4.14)$$

where $\mathbf{M}$ is defined by Eqs. (2.19) and (2.20) and the refractive index n_1 is replaced by unity.

To derive the transport equation, we must consider a sequence of scatterings, just like the single one just described. In I it was shown that all coherent scatterings result in propagation with the refractive index n. The incoherent scatterings lead to the transport equation.

A sequence of coherent scatterings will not lead to a frequency shift. On the other hand, a long sequence of incoherent scatterings may lead to a large frequency shift in the wave. For each single scattering in such a sequence, we can continue to assume that the frequency shift $(\bar{\omega} - \omega) = O(\omega v_e/c)$ is small, because of the assumption NR. In particular, we can continue to use the relation (4.7), where ω and $\bar{\omega}$ are the respective frequencies *before* and *after* a given single scattering.

5. DERIVATION OF THE TRANSPORT EQUATION

The derivation of the transport equation, as given in I, needs only minor modifications to take account of the frequency shift. In this section we shall, therefore, rely heavily on the development given in I.

Following the discussion given in I, we write a particular component of the scattered electric field vector in the form

$$E(\mathbf{r}) = \sum_{n=1}^{\infty} \sum_{\alpha_1,\alpha_2,\cdots,\alpha_n} Q_n(\mathbf{r}; \mathbf{z}_{\alpha_1}, \cdots, \mathbf{z}_{\alpha_n}). \quad (5.1)$$

Here Q_n represents the contribution from a wave multiply scattered by electrons at $\mathbf{z}_{\alpha_1}, \cdots, \mathbf{z}_{\alpha_n}$ and the sum is over all electrons and numbers of scatterings.

To find the scattered power, we must evaluate such quantities as

$$\mathcal{I}_{nm} \equiv \langle Q_m^*(\mathbf{r}; \mathbf{z}_{\beta_1}, \cdots, \mathbf{z}_{\beta_m}) Q_n(\mathbf{r}; \mathbf{z}_{\alpha_1}, \cdots, \mathbf{z}_{\alpha_n})\rangle = \left\langle \int P_{m+n} Q_m^* Q_n \, d^3z_{\beta_1}, \cdots, d^3z_{\alpha_n} \right\rangle_S, \quad (5.2)$$

using the notation of Eq. (3.2). We suppose the probability function P_{m+n} to be decomposed into a cluster expansion of correlated coordinates, as in Eqs. (2.10) and (2.11). For each term of this expansion, each coordinate is a member of a correlated cluster of coordinates. First, a given coordinate $\mathbf{z}_\alpha$ may be *uncorrelated* with another coordinate. If not *uncorrelated*, $\mathbf{z}_\alpha$ is correlated with other members of the set $\mathbf{z}_{\beta_1}, \cdots, \mathbf{z}_{\alpha_n}$ in (5.2).

Let us suppose that $\mathbf{z}_\alpha$ belongs to the correlated cluster set $\mathbf{z}_{\alpha_c}, \cdots, \mathbf{z}_{\alpha_d}$, which consist of only $\mathbf{z}_\alpha$. In this case the integral over $\mathbf{z}_{\alpha_c}, \cdots, \mathbf{z}_{\alpha_d}$ involves only Q_n. This was called a "coherent part" of the average in I. Such "coherent part" averages may clearly be performed on each factor of $E(\mathbf{r})$ before squaring. It was shown in I that the effect of the "coherent part" averages is to give the plasma a refractive index. This result may be taken unchanged for our present analysis.

To see this, we note that the introduction of the time-dependent correlation functions does not modify the expressions obtained in I for the refractive index. This is obvious [because of the stationary property (2.8)] for scatterings which are uncorrelated. Scatterings which are correlated are separated by distances of the order of R_c. During the time R_c/c required for propagation across a correlated cluster, a typical electron will have moved a distance $(R_c v_e)/c$. The resulting change of phase in the exponentials is, therefore, of order

$$(kR_c)v_e/c \ll 1, \quad (5.3)$$

by assumption NR, and can be neglected.

The resulting equations for the multiply scattered waves are [see Eqs. (I.3.31), (I.3.32), (I.3.33)]

$$\mathbf{E}(\mathbf{z}_\alpha, t) = \mathbf{E}_c(\mathbf{z}_\alpha, t) + \sum_{\beta(\neq\alpha)=1}^{N} \sum_{j=1}^{2} \hat{\mathbf{e}}_{\alpha\beta}(j) E_{\alpha\beta}(j, t), \quad (5.4)$$

$$E_{\alpha\beta}(i, t) = G_{\alpha\beta} f_{i1}(\alpha\beta, \beta 0) E_c(\mathbf{z}_\beta) + \sum_{\sigma(\neq\beta)=1}^{N} \sum_{j=1}^{2} G_{\alpha\beta} f_{ij}(\alpha\beta, \beta\sigma) E_{\beta\sigma}(j, t). \quad (5.5)$$

Here

$$\hat{\mathbf{e}}_{\alpha\beta}(j) \equiv \hat{\mathbf{e}}_{\hat{\mathbf{q}}}(j), \quad (5.6)$$

where $\hat{\mathbf{q}}$ is the unit vector parallel to $(\mathbf{z}_\alpha - \mathbf{z}_\beta)$. The quantity $\mathbf{E}_c(\mathbf{z}_\alpha, t)$ represents the *coherent wave* [see

Eqs. (3.7)]

$$\mathbf{E}_c(\mathbf{z}_\alpha, t) = \hat{\mathbf{e}}_{\mathbf{k}}(1)E_c(\mathbf{z}_\alpha, t),$$

$$E_c(\mathbf{z}_\alpha, t) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int d\omega \hat{E}_0(\omega)e^{i(kS_\alpha-\omega t)}, \quad (5.7)$$

where S_α is the eikonal for the coherent wave [Eq. (I.3.34)]

$$S_\alpha \equiv S(\mathbf{z}_\alpha) = \int_{-\infty}^{\mathbf{z}_\alpha}[n(\mathbf{x}) - 1]\, ds + \hat{\mathbf{k}}\cdot\mathbf{z}_\alpha. \quad (5.8)$$

Here $n(\mathbf{x})$ is the refractive index and the constant of integration has been chosen to be consistent with Eq. (3.7). The Thomson amplitude (4.5) for scattering has been rewritten in Eqs. (5.5) to indicate scattering from the direction of $(\mathbf{z}_\beta - \mathbf{z}_\sigma)$ to that of $(\mathbf{z}_\alpha - \mathbf{z}_\beta)$, etc. For a monochromatic wave, the Green's function [see Eq. (I.3.29)] is

$$G_{\alpha\beta} = \frac{e^{ikS_{\alpha\beta}}}{R_{\alpha\beta}}, \quad (5.9)$$

$$S_{\alpha\beta} = \int_{\mathbf{z}_\beta}^{\mathbf{z}_\alpha} n(\mathbf{x})\, ds, \quad (5.10)$$

the integral being taken along the straight line path from $\mathbf{z}_\beta$ to $\mathbf{z}_\alpha$ and

$$\mathbf{R}_{\alpha\beta} \equiv \mathbf{z}_\alpha - \mathbf{z}_\beta. \quad (5.11)$$

We have used the notation of Eq. (4.6) on the right-hand side of Eq. (5.5), writing

$$G_{\alpha\beta}f_{ij}(\alpha\beta, \beta\sigma)E_{\beta\sigma}(j, t) = \int \frac{d\omega}{(2\pi)^{\frac{1}{2}}}\frac{e^{ikS_{\alpha\beta}}}{R_{\alpha\beta}} f_{ij}(\alpha\beta, \beta\sigma; \omega)\hat{E}_{\beta\sigma}(j, \omega)e^{-i\omega t} \quad (5.12)$$

in terms of the Fourier transform $\hat{E}_{\beta\sigma}$ of $E_{\beta\sigma}$.

The argument given, following Eq. (4.6), lets us set $\bar{k} = k$ in the exponential in (5.12).

The "coherent part" averages in the expressions (5.2) permitted us to derive the multiple scattering equations (5.4) and (5.5). On performing the remaining averages, after using Eqs. (5.4) and (5.5), we must omit "coherent part" averages. This means that every coordinate $\mathbf{z}_{\alpha_i}$ must be now correlated with at least one of the $\mathbf{z}_\beta$ in (5.2).

Continuing to follow I, we define [a generalization of Eq. (I.5.6)]

$$U_{ij}(\alpha, \beta; \omega) \equiv \frac{1}{2\pi}\int d\tau e^{+i\omega\tau}\int d^3z_\gamma\, d^3z_\sigma \delta[\tfrac{1}{2}(\mathbf{z}_\gamma + \mathbf{z}_\sigma) - \mathbf{z}_\beta] \times \rho(\mathbf{z}_\gamma)\rho(\mathbf{z}_\sigma)g(\mathbf{z}_\gamma, \mathbf{z}_\sigma; \tau)\frac{1}{8\pi}\langle E^*_{\alpha\gamma}(i, t)E_{\alpha\sigma}(j, t + \tau)\rangle_{\alpha\gamma\sigma}. \quad (5.13)$$

Here the notation $\langle\cdots\rangle_{\alpha\gamma\sigma}$ means an average over all coordinates except for $\mathbf{z}_\alpha$, $\mathbf{z}_\gamma$, and $\mathbf{z}_\sigma$, which are held fixed.[17]

We now follow the derivation of Eq. (I.5.11). Equations (5.5) are substituted into the right-hand side of Eq. (5.13). Equations (I.5.9) and (I.5.10) are used to write $E_{\beta'\sigma}$ in terms of $E_{\beta\sigma}$, etc., for $|\mathbf{z}_\beta - \mathbf{z}_{\beta'}| = O(R_c)$. There finally results

$$\begin{aligned}U_{ij}(\alpha, \beta; \omega) &= (2\pi)^{-1}\int d\tau e^{i\omega\tau}\int d^3z_\gamma\, d^3z_{\gamma'}\delta(\tfrac{1}{2}(\mathbf{z}_\gamma + \mathbf{z}_{\gamma'}) - \mathbf{z}_\beta)\\ &\times \rho(\mathbf{z}_\gamma)\rho(\mathbf{z}_{\gamma'})g(\mathbf{z}_\gamma, \mathbf{z}_{\gamma'}; \tau)\,|G_{\alpha\beta}|^2\\ &\times \Big(f^*_{i1}(\alpha\beta, \beta 0)f_{j1}(\alpha\beta, \beta 0)(8\pi)^{-1}\\ &\times \langle E^*_c(\mathbf{z}_\beta, t)E_c(\mathbf{z}_\beta, t + \tau)\rangle_S\\ &\times \exp\,[in_1(\mathbf{z}_\beta)(\mathbf{k}_{\alpha\beta} - \mathbf{k})\cdot(\mathbf{R}_{\gamma\beta} - \mathbf{R}_{\gamma'\beta})]\\ &+ \sum_{s,s'=1}^{2}\int d^3z_\sigma\, d^3z_{\sigma'}\rho(\mathbf{z}_\sigma)\rho(\mathbf{z}_{\sigma'})g(\mathbf{z}_\sigma, \mathbf{z}_{\sigma'}; \tau)\\ &\times [f^*_{is}(\alpha\beta, \beta\sigma)f_{js'}(\alpha\beta, \beta\sigma)]\\ &\times \exp\,[in_1(\mathbf{z}_\beta)(\mathbf{k}_{\alpha\beta} - \mathbf{k}_{\beta\sigma})\cdot\mathbf{R}_{\gamma\gamma'}](8\pi)^{-1}\\ &\times \langle E^*_{\beta\sigma}(s, t)E_{\beta\sigma'}(s', t + \tau)\rangle_{\beta\sigma\sigma'} + \text{cross terms}\Big).\end{aligned} \quad (5.14)$$

To simplify the first term above, we write

$$\begin{aligned}(2\pi)^{-1}\int d\tau e^{+i\omega\tau}(8\pi)^{-1}\langle E^*_c(\mathbf{z}_\beta, t)E_c(\mathbf{z}_\beta, t + \tau)\rangle_S &= [(2c)^{-1}I^0(\omega)]e^{ik(S_\beta - S_\beta^*)}\\ &= [(2c)^{-1}I_c(\mathbf{z}_\beta, \omega)],\end{aligned} \quad (5.15)$$

where [see Eq. (3.9)],

$$I_c(\mathbf{z}_\beta, \omega) = I^0(\omega)\exp\left(-\int^{\mathbf{z}_\beta}\frac{ds}{l(\mathbf{x})}\right) \quad (5.16)$$

and the integral is taken along the straight line parallel to $\hat{\mathbf{k}}$.

On making use of the assumption that $R_c/l \ll 1$, we may neglect the cross terms in Eq. (5.14) and express the second term in terms of U. In so doing, we make use of our conclusion of Sec. IV that the change in frequency on a single scattering may be neglected in the Thomson amplitude and in the exponentials.

To simplify the second term in Eq. (5.14), we write it in the form

$$\begin{aligned}&\int d\tau \frac{e^{i\omega\tau}}{2\pi}\int d^3z_\gamma\, d^3z_{\gamma'}\delta(\tfrac{1}{2}(\mathbf{z}_\gamma + \mathbf{z}_{\gamma'}) - \mathbf{z}_\beta)\\ &\quad\times \rho(\mathbf{z}_\gamma)\rho(\mathbf{z}_{\gamma'})g(\mathbf{z}_\gamma, \mathbf{z}_{\gamma'}; \tau)\,|G_{\alpha\beta}|^2\\ &\quad\times \sum_{s,s'=1}^{2}\int d\omega' \frac{e^{-i\omega'\tau}}{2\pi}\int d\tau' e^{i\omega'\tau'}\\ &\quad\times \int d^3z_\sigma\, d^3z_{\sigma'}\int d^3z_\lambda \delta(\tfrac{1}{2}(\mathbf{z}_\sigma + \mathbf{z}_{\sigma'}) - \mathbf{z}_\lambda)\rho(\mathbf{z}_\sigma)\rho(\mathbf{z}_{\sigma'})\\ &\quad\times g(\mathbf{z}_\sigma, \mathbf{z}_{\sigma'}; \tau')[f^*_{is}(\alpha\beta, \beta\sigma)f_{js'}(\alpha\beta, \beta\sigma)]\\ &\quad\times \exp\,[in_1(\mathbf{z}_\beta)(\mathbf{k}_{\alpha\beta} - \mathbf{k}_{\beta\lambda})\cdot\mathbf{R}_{\gamma\gamma'}]\\ &\quad\times \langle(8\pi)^{-1}E^*_{\beta\sigma}(s, t)E_{\beta\sigma'}(s', t + \tau')\rangle_{\beta\sigma\sigma'}\\ &= \sum_{s,s'=1}^{2}\int d^3z_\lambda \int d\omega'\,|G_{\alpha\beta}|^2\,(ij|\,M(\alpha\beta, \beta\lambda; \omega - \omega')\,|ss')\\ &\quad\times U_{ss'}(\beta, \lambda; \omega').\end{aligned} \quad (5.17)$$

In writing the exponential involving $\mathbf{R}_{\gamma\gamma'}$, we have replaced $\mathbf{k}_{\beta\sigma}$ in Eq. (5.14) by $\mathbf{k}_{\beta\lambda}$. This is permissible since we assume $l \gg R_c$. The quantity $\mathbf{M}$ in Eq. (5.17) is defined by Eqs. (2.19), with the obvious notational change of indicating directions of propagation as $\beta\lambda$ and $\alpha\beta$.

The results (5.15) and (5.17) permit us to write (5.14) in the form

$$U_{ij}(\alpha, \beta; \omega) = \int d\omega' |G_{\alpha\beta}|^2 \Big((ij| M(\alpha\beta, \beta 0; \omega - \omega') |11) \frac{1}{2c} I_c(\mathbf{z}_\beta, \omega') + \sum_{s,s'=1}^{2} \int d^3 z_\sigma (ij| M(\alpha\beta, \beta\sigma; \omega - \omega') |ss') \times U_{ss'}(\beta, \sigma; \omega') \Big). \quad (5.18)$$

Since $\mathbf{M}$ is even in $(\omega - \omega')$ [see remark following Eq. (2.20)] and I_c is even in ω', it follows that $\mathbf{U}$ is even in ω. This lets us define the intensity I_{ij} for $\omega > 0$ with the equation

$$I_{ij}(\mathbf{z}_\alpha, \hat{\mathbf{p}}, \omega) = I_c(\mathbf{z}_\alpha, \omega)\delta_{i1}\delta_{j1}\delta_{\hat{\mathbf{k}},\hat{\mathbf{p}}} + 2c \int_{-\hat{\mathbf{p}}} R_{\alpha\beta}^2 \, dR_{\alpha\beta} U_{ij}(\alpha, \beta; \omega). \quad (5.19)$$

The δ function here is defined by the condition that

$$\int d\Omega_{\hat{\mathbf{p}}} f(\hat{\mathbf{p}}) \delta_{\hat{\mathbf{k}},\hat{\mathbf{p}}} = f(\hat{\mathbf{k}})$$

for a function $f(\hat{\mathbf{p}})$ which is regular at $\hat{\mathbf{k}} = \hat{\mathbf{p}}$. The integration in Eq. (5.19) is performed over $\mathbf{z}_\beta$ along the semi-infinite straight line beginning at $\mathbf{z}_\alpha$ and directed parallel to $-\hat{\mathbf{p}}$.

Using Eq. (5.19), we can express (5.18) in terms of I_{ij}. If we write

$$|G_{\alpha\beta}|^2 = \frac{1}{R_{\alpha\beta}^2} \exp\left(-\int_{\mathbf{z}_\beta}^{\mathbf{z}_\alpha} \frac{ds}{l(\mathbf{x})} \right),$$

we obtain [in the matrix notation of Eq. (2.26)]

$$\mathbf{I}(\mathbf{x}, \hat{\mathbf{p}}, \omega) = \mathbf{I}_c(\mathbf{x}, \omega)\delta_{\hat{\mathbf{p}},\hat{\mathbf{k}}} + \int_{-\hat{\mathbf{p}}} ds(\mathbf{x}) \exp\left(-\int_{\mathbf{z}}^{\mathbf{x}} \frac{ds'}{l(\mathbf{x}')} \right) \times \int_0^\infty d\omega' \int d\Omega_{\hat{\mathbf{p}}'} \mathbf{M}(\hat{\mathbf{p}}, \hat{\mathbf{p}}'; \omega - \omega') \mathbf{I}(\mathbf{z}, \hat{\mathbf{p}}', \omega'), \quad (5.20)$$

where now $\omega > 0$ and

$$\mathbf{I}_c(\mathbf{x}, \omega) = \mathbf{I}^0(\omega) \exp\left(-\int^{\mathbf{x}} \frac{ds}{l} \right). \quad (5.21)$$

The path integral in Eq. (5.20) extends along the straight line from $\mathbf{z}$ to ∞ in the direction $-\hat{\mathbf{p}}$.

Differentiation of Eq. (5.20) along a ray path leads to Eq. (2.26). Equation (5.20) is evidently valid for arbitrary incident polarization, to be specified by the choice of $\mathbf{I}^0(\omega)$.[8]

6. RADAR BACKSCATTER

It was pointed out in I that the transport equation is not valid for backscatter. The reason for this is illustrated in Fig. 2. To each ray path defined by a particular sequence of multiple scatterings there corresponds a path obtained by reversing all propagation vectors. These pairs of paths can interfere coherently, and this is not included in the transport equation. As was shown in I, this effect can be accounted for, however, by choosing a certain linear combination of solutions of the transport equation. The specific expression for backscatter was given in Eqs. (I.7.7) and (I.7.10).

When there is a frequency shift, Eqs. (I.7.5a) and (I.7.5b) are modified. These now read, respectively,

$$Q_n(i, s) = \sum_{j_1, \cdots, j_{n-1}} \frac{e^{ik_{n+1}S_{rn}}}{r} \times f_{ij_{n-1}}(-\hat{\mathbf{k}}, \hat{\mathbf{l}}_{n-1}) \cdots f_{j_1 s}(\hat{\mathbf{l}}_1, \hat{\mathbf{k}}) e^{ik_1 S_{1r}}, \quad (6.1a)$$

$$\tilde{Q}_n(i, s) = \sum_{j_1, \cdots, j_{n-1}} \frac{e^{ik_{n+1}S_{r1}}}{r} \times f_{ij_1}(-\hat{\mathbf{k}}, -\hat{\mathbf{l}}_1) \cdots f_{j_{n-1}s}(-\hat{\mathbf{l}}_{n-1}, \hat{\mathbf{k}}) e^{ik_1 S_{nr}}. \quad (6.1b)$$

Here k_1 is the incident wave number, k_2 that after the first scattering, $\cdots$, and k_{n+1} that after the nth scattering.

For Q_n and $\tilde{Q}_n$ to interfere coherently, the frequency (wave-number) spread must be small enough that the phase differences $[S_{ll-1}(k_l - k_{n-l+2})]$ are small compared to unity. The criterion for this is that

$$(\Delta\omega l)/c \ll 1, \quad (6.2)$$

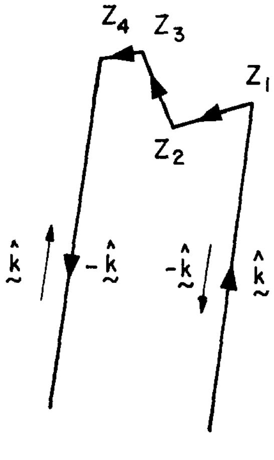

FIG. 2. Illustration of backscattering.

where $\Delta\omega$ is the total spread in frequency because of scattering. If condition (6.2) is satisfied, the *expressions* (I.7.7) and (I.7.10) *may be used.*

On the other hand, when

$$(\Delta\omega l)/c \gg 1, \tag{6.3}$$

Q_n and $\tilde{Q}_n$ will not interfere. Then, the transport equation (2.26) [or (5.20)] does tend to be valid for backscatter, without the special correction of Eqs. (I.7.7) and (I.7.10).

In intermediate cases it is not anticipated that the transport equation will be applicable to the calculation of backscatter.

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[6] See, for example, S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, London, 1950).

[7] Any fixed unit vector $\hat{\mathbf{k}}$ would do.

[8] The representation of polarization is discussed in II in some detail.

[9] As a practical condition, we require that (2.8) be valid for $\tau \lesssim O(\Delta\omega^{-1})$, where $\Delta\omega$ is the smallest frequency interval of interest in the intensity spectrum.

[10] This was described in more detail in I.

[11] See, for example, M. Lax, Rev. Mod. Phys. **32**, 25 (1960), or H. B. Callen, *Thermodynamics* (John Wiley & Sons, New York, 1960).

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[14] Since Eq. (2.26) is homogeneous in the intensity **I**, the actual system of units used is not significant. We are here using unrationalized Gaussian units, as in I.

[15] We shall later extend the result to include arbitrary polarization of the incident wave, as was done in I.

[16] See, for example, H. G. Booker, J. Geophys. Res. **64**, 2164 (1959), and A. D. Wheelon, J. Res. Natl. Bur. Std., D, **63**, 205 (1959).

[17] This notation was described in more detail in I.