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Abstract— **We propose a construction technique for unitary space-time codes that use Givens rotation matrices. These constellations have a desirable weak group property that leads to reduced construction and decoding complexity. The newly constructed constellations have the best known diversity product and diversity sum for a wide range of constellation sizes and number of transmit antennas.**

I. Introduction

Unitary space-time codes are useful for multiple-input multiple-output (MIMO) fading channels. They are particularly well suited for fast fading channels since channel state information (CSI) is not required at either the transmitter or receiver. These codes were first introduced by Hochwald and Marzetta [1] and a systematic construction technique was later proposed by Hochwald *et al* [2].

The goal of unitary constellation design is to construct a set of L unitary matrices, denoted $\{\Phi_i\}$ for $i = 0, \ldots, L-1$ with specific properties. For low signal-to-noise ratio (SNR), the design metric of interest is the *diversity sum*, denoted δ, and for high SNR the design metric is the *diversity product*, denoted ζ. Calculating the diversity sum or diversity product generally requires L^2 calculations. Calculating the design metric for large L can be computationally burdensome. However, if the constellation construction technique has certain properties, the diversity sum or product can be obtained with only $L-1$ calculations. The original systematic construction technique had this desirable property.

Since the original systematic construction technique was proposed a variety of other construction techniques have been investigated. The goal of these techniques was to improve the diversity product or diversity sum of the unitary constellations. Parametric codes and bounds on the optimal values of the diversity product and diversity sum were introduced by Liang and Xia [3]. Simple rotation matrices were used by Shan *et al* [4] to generate codes with improved diversity products, but were restricted to systems with an even number of transmit antennas. This work was extended by Soh *et al* [5] to an arbitrary number of transmitting antennas and improved constellations were once again reported. Bruhat decomposition was used to construct unitary constellations for an even number of antennas by Konishi [6]. This work was later extended to include odd antennas by Niyomsataya *et al* [7]. Coherent space-time codes were mapped to the Grassman manifold using an exponential mapping to construct non-coherent codes in [8]. Recently, a geometrical interpretation of unitary space-time codes and numerical techniques have been considered by Han and Rosenthal [9]. We note that the codes of Han and Rosenthal presented in [9] and on their website are often not the best known codes. However, some of their theoretical work is useful and will be used here. Also, the large database of diversity product and diversity sum results they have tabulated is quite useful, even though other techniques may be able to achieve better results.

The work presented here extends work originally reported by Panagos *et al* [10]. In this previous work, Givens rotation matrices were used to construct unitary constellations with improved diversity product. However, this previous construction technique required L^2 operations to calculate the diversity product. When used in conjunction with a greedy search algorithm [11], the polynomial complexity in L made it computationally expensive to search for large constellations. In this work we also use Givens rotation matrices as generator matrices for constellation construction. However, we modify the construction to ensure the resulting constellation has a weak-group property [9]. The weak-group property of the constellation is important as only L operations are required for calculating either the diversity product or diversity sum.

The next section provides background regarding weak group codes, specifies the newly proposed unitary space-time constellation construction technique, and compares this technique to the other recently proposed method [10] that also uses Givens rotation matrices. Section II discusses a reduced complexity decoding technique that can be used on these codes due to their weak group structure. A comparison of the codes found using this technique to the other construction methods mentioned in the introduction is given in Section III. In almost all cases examined these new codes have the best diversity product and diversity sum known.

II. Weak Group Codes: Background and New Construction

In this section we review several results regarding weak group codes originally derived and stated by Han and Rosenthal [9]. We then use these results to propose a weak group code structure based on Givens rotation matrices. We also compare this new construction technique to another recently proposed construction technique that also used Givens rotation matrices [10]. This comparison shows that the new technique of this paper has significantly reduced construction

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and decoding complexity, without suffering any performance degradation.

A. Background

Definition 2.1: Let $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_j$ be unitary matrices. The matrices $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_j$ are equivalent if there exists a unitary matrix U such that $\mathbf{\Phi}_i = U\mathbf{\Phi}_j U^{-1}$ or $\mathbf{\Phi}_i = U\mathbf{\Phi}_j^{-1}U^{-1}$.

Definition 2.2: Let $\mathcal{C} = \{\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_L\}$ be a unitary constellation of L signals. The constellation $\mathcal{C}$ has a *weak group structure* if for any two distinct elements $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_j$, the quantity $\mathbf{\Phi}_i^{-1}\mathbf{\Phi}_j$ is equivalent to some $\mathbf{\Phi}_k$.

Lemma 2.3: Let $\mathcal{C} = \{I, \mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_{L-1}\}$ be a unitary constellation with a weak group structure. Computing the diversity product or diversity sum requires only $L-1$ distance computations.

Theorem 2.4: Let $\mathcal{C} = \{\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_L\}$ be a unitary constellation of L signals. If $\mathcal{C}$ has a weak group structure then $\mathcal{C}$ takes one of the following forms:

$$\{I, A, A^2, \ldots, A^{L-1}\} \text{ or } \{I, AB, A^2B^2, \ldots, A^{L-1}B^{L-1}\} \tag{1}$$

Based on the result of this theorem, we are interested in constructing unitary space-time constellations of one of two forms. We note that this theorem is only necessary, i.e. if a constellation has a weak group structure than it is in one of two forms.

B. New Construction

Based on the insight provided by Theorem 2.4 we propose unitary space-time signals of the following form

$$\mathbf{\Phi}_l = (\mathbf{G}_{\text{prod}})^l \cdot \mathbf{D}^l \tag{2}$$

where

$$\mathbf{G}_{\text{prod}} \triangleq \mathbf{G}_1\mathbf{G}_2 \cdots \mathbf{G}_{\frac{M(M-1)}{2}}, \tag{3}$$

for $l = 0, \ldots, L-1$.

The $\mathbf{G}_j$ for $j = 1, 2, \ldots, \frac{M(M-1)}{2}$ are Givens rotation matrices defined as [12]

$$\mathbf{G}(i,k,\theta) = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c_j & \cdots & s_j & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s_j & \cdots & c_j & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}, \tag{4}$$

where $c_j \triangleq \cos(\theta_j)$ and $s_j \triangleq \sin(\theta_j)$. For simplicity we have suppressed the (θ) notation and defined $\mathbf{G}_i \equiv \mathbf{G}_i(\theta_i)$.

These plane rotation matrices are modifications of the appropriately sized identity matrix and are formed by placing c_j at coordinates (i,i) and (k,k), s_j at coordinate (i,k), and $-s_j$ at coordinate (k,i). These matrices can be used to rotate each individual (i,k) coordinate plane by θ radians.

The matrix $\mathbf{D}$ is a diagonal unitary matrix defined as

$$\mathbf{D} = \begin{pmatrix} e^{j\phi_1} & 0 & \cdots & 0 \\ 0 & e^{j\phi_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\phi_M} \end{pmatrix} \tag{5}$$

As a specific example, the Givens rotation matrices for an $M = 4$ system are of the form

$$\mathbf{G}_1 = \begin{pmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{G}_2 = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G}_3 = \begin{pmatrix} c_3 & 0 & 0 & s_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & 0 & 0 & c_3 \end{pmatrix}, \mathbf{G}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_4 & s_4 & 0 \\ 0 & -s_4 & c_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_5 & 0 & s_5 \\ 0 & 0 & 1 & 0 \\ 0 & -s_5 & 0 & c_5 \end{pmatrix}, \mathbf{G}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_6 & s_6 \\ 0 & 0 & -s_6 & c_6 \end{pmatrix}.$$

This constellation construction techniques has M values of ϕ associated with the diagonal matrix $\mathbf{D}$ and $\frac{M(M-1)}{2}$ values of θ associated with the rotation matrices $\mathbf{G}_i$. Thus, a total of $M + \frac{M(M-1)}{2} = \frac{M^2+M}{2}$ parameters are available for parameterizing the unitary constellation $\mathcal{C}$.

Lemma 2.5: The construction technique of Equation 2 yields a weak group constellation.

Proof: We must show that for distinct matrices $\mathbf{\Phi_i}$ and $\mathbf{\Phi_j}$ that $\mathbf{\Phi_i^{-1}\Phi_j}$ is equivalent to some other $\mathbf{\Phi_k}$. Without loss of generality assume $i < j$. Let $k \triangleq j - i$. We have $\mathbf{\Phi}_i = \mathbf{G}_{\text{prod}}^i\mathbf{D}^i$ and $\mathbf{\Phi}_j = \mathbf{G}_{\text{prod}}^j\mathbf{D}^j$. Thus

$$\mathbf{\Phi}_i^{-1} = (\mathbf{G}_{\text{prod}}^i\mathbf{D}^i)^{-1} = \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{-i} \tag{6}$$

and

$$\begin{aligned} \mathbf{\Phi_i^{-1}\Phi_j} &= \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{-i}\mathbf{G}_{\text{prod}}^{j}\mathbf{D}^j \\ &= \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{j-i}\mathbf{D}^j \\ &= \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{j-i}\mathbf{D}^j\mathbf{D}^{-i}\mathbf{D}^i \\ &= \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{j-i}\mathbf{D}^{j-i}\mathbf{D}^i \\ &= \mathbf{D}^{-i}\mathbf{G}_{\text{prod}}^{k}\mathbf{D}^{k}\mathbf{D}^i \\ &= \mathbf{D}^{-i}\mathbf{\Phi}_k\mathbf{D}^i \\ &= U\mathbf{\Phi}_k U^{-1} \end{aligned}$$

where $U \triangleq \mathbf{D}^{-i}$ is a unitary matrix. Thus, $\mathbf{\Phi_i^{-1}\Phi_j}$ is equivalent to $\mathbf{\Phi_k}$ by Definition 2.1 and the constellation forms a weak group by Definition 2.2. ■

C. Construction Technique Comparison

The authors of this paper have recently proposed another unitary space-time code constellation construction technique based on Givens rotation matrices [10]. The construction rule for the previous technique was

$$\mathbf{\Phi}_l = (\mathbf{G}^1_{\text{prod}})^l \cdot \mathbf{D}^l \cdot (\mathbf{G}^2_{\text{prod}})^l. \tag{7}$$

This previous technique uses the generator matrices $\mathbf{G}^1_{\text{prod}}, \mathbf{D}$, and $\mathbf{G}^2_{\text{prod}}$. Due to the additional generator matrix the constellations formed using this method do not form a weak group. When calculating the diversity product or diversity sum for these constellations, $\frac{L(L-1)}{2}$ distances are required. For the newly proposed constellations that have the weak group property, only $L-1$ distance calculations are required as stated in Lemma 2.3. For constellations with large numbers of signals (i.e. high rate codes), the computational savings due to the weak group property of the constellation are significant.

The number of parameters used to construct the constellation have also been reduced. The previous parameterization used M^2 parameters. The new parameterization uses only $\frac{M^2+M}{2}$.

Because of these differences the speed at which searches can be performed has increased significantly. The search technique used is based on our original work in [11]. Parameters are randomly generated, the constellation is constructed, and the diversity product or sum is calculated. This process is repeated numerous times and the parameters corresponding to the best found constellation are saved. A local search is then performed in the neighborhood of the best known parameters using gradient based techniques. Only requiring $L-1$ calculations instead of $\frac{L(L-1)}{2}$ makes it possible to quickly search over significantly larger constellations then was possible previously. The reduction in the number of parameters is also significant as the numerical routines for estimating gradients and ascending to local maximum values of the diversity product or diversity sum now take less time as well.

Even more promising are the diversity product and diversity sum results obtained using this technique. Thus far, the best known codes found using the construction of [10] have been replicated using this new technique. These results suggest that the reduced complexity of the search process due to the weak group property of the new constellations does *not* come at the expense of reduced constellation performance.

III. Reduced Complexity Decoding

The previous section discussed the significant reduction in code search complexity obtained due to the weak group property of the constellation.

In this section we discuss the reduction in decoding complexity due to the weak group structure of the constellation. We show how these weak group codes lend themselves to very efficient decoding using a sphere decoder.

Let X_τ be the received signal at time τ. For this first case we assume a single receive antenna (i.e. $N=1$). Rephrasing the following for cases $N>1$ is straightforward. The received vector at time τ is an $M \times 1$ vector. The elements of the received vector will be denoted $x_{\tau,m}$.

For our proposed construction technique with differential transmission, the maximum likelihood (ML) decoder solves the optimization problem

$$\hat{l} = \arg\min_l ||X_\tau - G^l_{\text{prod}} D^l X_{\tau-1}||^2_F \tag{8}$$

to determine which of the L signals was transmitted. For large constellations this exact ML decoding rule is prohibitively time consuming.

One can verify that the exact ML decoding rule is equivalent to [9]

$$||X_\tau - G^l_{\text{prod}} D^l X_{\tau-1}||^2_F = ||G^{-l}_{\text{prod}} X_\tau - D^l X_{\tau-1}||^2_F. \tag{9}$$

Also, since both G_{prod} and D are $M \times M$ unitary matrices, they can be written as

$$G_{\text{prod}} = U \text{diag}\left(e^{ig_1} \quad e^{ig_2} \quad \cdots e^{ig_M}\right) U^\dagger, \tag{10}$$

and

$$D = V \text{diag}\left(e^{id_1} \quad e^{id_2} \quad \cdots e^{id_M}\right) V^\dagger, \tag{11}$$

where U and V are also unitary matrices.

Thus, the ML decoding rule can be re-written as [9]

$$\begin{aligned}\hat{l} = \arg\min_l ||&U \text{diag}\left(e^{-ilg_1} \quad e^{-ilg_2} \quad \cdots e^{-ilg_M}\right) U^\dagger X_\tau \\ &- V \text{diag}\left(e^{ild_1} \quad e^{ild_2} \quad \cdots e^{ild_M}\right) V^\dagger X_{\tau-1}||^2_F.\end{aligned} \tag{12}$$

The original ML decoding rule has been written such that $X_\tau - G^l_{\text{prod}} D^l X_{\tau-1}$ is just a linear combination of trigonometric functions and the variable l. Following techniques suggested in [9], [13] (such as sphere decoding least-squares methods) will allow the ML solution to be obtained in polynomial time, an essential feature for high rate constellations. Thus, the new constellations constructed with this method not only have the best known diversity product and diversity sum, but also can be decoded efficiently.

IV. New Constellations

In this section we summarize the codes found with this new construction technique. To the authors knowledge, the best known diversity product and diversity sum results for constellations of size 2^k for $k = 3, 4, 5$ and 6 were most recently reported in [5]. Of the thirty results presented in this previous work, our new technique has found improved constellations in twenty-six cases. In three cases we have matched their results, and in only one case have we been unable to find a constellation as good as previously known. The comparison of our codes to these previous codes for both the diversity product and diversity sum design metrics can be found in Tables I and II.

For constellation sizes that aren't a power of two, Han [9] has tabulated a large number of diversity product and sum results on his website. In Figure 1 we compare the diversity

TABLE I

COMPARISON OF DIVERSITY PRODUCT

M	L	Our Code	Code in [5]	Code in [4]	Cyclic Code
3	8	0.700	0.647	NA	0.513
	16	0.600	0.565	NA	0.448
	32	0.468	0.459	NA	0.334
	64	0.425	0.416	NA	0.277
4	8	0.722	0.707	0.707	0.595
	16	0.627	0.615	0.615	0.545
	32	0.595	0.595	0.545	0.383
	64	0.480	0.437	0.406	0.340
5	8	0.710	0.670	NA	0.544
	16	0.605	0.601	NA	0.457
	32	0.555	0.549	NA	0.410
6	8	0.715	0.707	0.707	0.595
	16	0.622	0.603	0.595	0.507
	32	0.565	0.553	0.522	0.448
	64	0.494	0.507	0.450	0.379

TABLE II

COMPARISON OF DIVERSITY SUM

M	L	Our Code	Code in [5]	Code in [4]	Cyclic Code
3	8	0.745	0.707	NA	0.618
	16	0.715	0.673	NA	0.588
	32	0.707	0.658	NA	0.480
	64	0.614	0.609	NA	0.424
4	8	0.752	0.707	0.707	0.707
	16	0.717	0.707	0.707	0.707
	32	0.712	0.707	0.707	0.555
	64	0.707	0.707	0.567	0.523
5	8	0.752	0.707	NA	0.655
	16	0.713	0.687	NA	0.638
	32	0.692	0.681	NA	0.575
6	8	0.748	0.707	0.707	0.707
	16	0.720	0.707	0.707	0.625
	32	0.707	0.707	0.640	0.618
	64	0.707	0.662	0.633	0.567

product of our construction technique with his for $M = 3$ transmit antennas. We see that in every case the diversity product achieved using our technique is superior. Results are similar for $M > 3$ and can be found on our website (http://www.umr.edu/$\sim$ panagos).

V. CONCLUSION

A new construction technique for unitary space-time codes has been presented. The weak group property of the constellations constructed using this method lead to reduced complexity construction as only $L-1$ calculations are required to calculate the diversity product or diversity sum design metrics. As shown in Section III, the weak group structure also allows the codes to be decoded efficiently. The comparison of diversity product and diversity sum results in Section IV show these codes are currently the best known for almost all cases examined. We are currently performing simulations to quantify the symbol-error-rate improvement these codes achieve over the previously best known codes. These results will be a topic of a future paper.

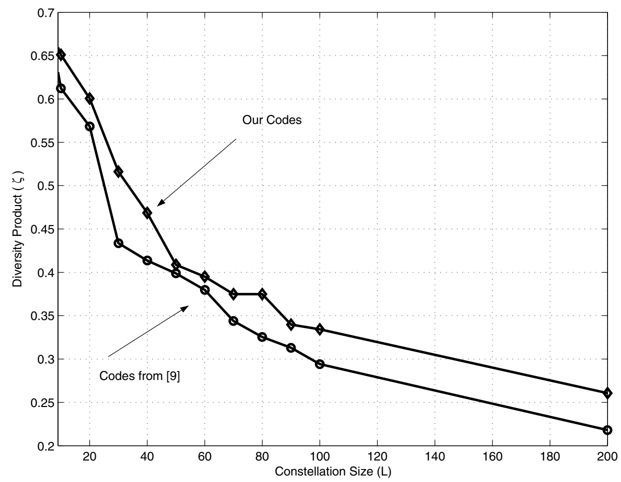

Fig. 1. Comparison of diversity product (ζ) for $M = 3$ transmit antennas and various constellation sizes, L. The recently proposed codes from [9] are marked with o's and our proposed codes are marked with $\diamond$'s. Our codes offer superior diversity product in every case.

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