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## ASYMPTOTIC ANALYSIS OF THE LINEARIZED NAVIER-STOKES EQUATIONS IN A CHANNEL

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**Abstract.** In this article we study and derive explicit formulas for the boundary layers occurring in the linearized channel flows in the limit of small viscosity. Our study is based on classical boundary layer techniques combined with a new global treatment of the pressure term.

**Introduction.** In the limit of large Reynolds numbers (or small viscosity), the solutions to the incompressible Navier-Stokes equations display a turbulent behavior whose understanding is still a major open problem in mathematical physics.

It seems that there are at least two important aspects of the problem; on the one hand the role played by the nonlinear terms which tend to mix the modes and propagate energy among them. On the other hand for physically realistic flows which involve boundaries, it is known that important phenomena occur at the boundary in a thin region called the boundary layer; in particular, the generation of vortices which propagate in the fluid and contribute to generate the motion.

Although much remains to be done, there has been recently some progress in the mathematical aspects of nonlinear dynamics, in relation with the point of view of attractors, based on the dynamical system approach to turbulence of Smale and Ruelle-Takens.

Concerning the mathematical study of boundary layers, a thorough study is available in the long article of M.I. Vishik and L.A. Lyusternik ([15]) and in the book of J.L. Lions ([6]); see also W. Eckhaus ([2]) and P. Lagerstrom ([5]) at the interface of mathematics and mechanics. All these references contain a thorough study of boundary layers; in particular, for time-independent problems. For time-dependent problems partial results appear in [5] and [6]; let us mention also the work of O. Oleinik ([8]) who studied the mathematical theory of the Prandtl's equations ([9]) independently of their relation to the Navier-Stokes equations and the work of P. Fife who studied the validity of Prandtl's equations in the stationary case with special type of pressure ([3]).

Although the full study of the convergence of the Navier-Stokes equations to the Euler equations is an outstanding problem which may still be out of reach, we would like in this and in forthcoming articles ([13, 14]) to develop some tools for the study of boundary layers in flows. An essential aspect of our approach as compared to the classical studies on boundary layers in the references above is that we treat the pressure in a global manner, and not only locally. The global treatment of the pressure is an essential aspect in the numerical treatment of the Navier-Stokes; it plays also an important role for a different problem in [12].

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The type of problems that we treat here is physically simple and perhaps physically irrelevant, but we feel that the techniques introduced here will be necessary for physically more relevant problems. We study the boundary layers for a channel flow, dropping furthermore the nonlinear terms of the equations; hence we study an evolution Stokes equation in a channel with two fixed boundaries and periodicity in the directions parallel to the walls. The simplifications due to the flat wall will be overcome in a subsequent article ([13]) where we will consider curved boundaries.

The article is organized as follows. In Section 1 we study the boundary layers for the heat equation first in space dimension one, then in space dimension two (or higher). Although these results may not be new, we did not find them available in the literature in the form in which we present and use them. Section 2 is devoted to the linearized channel flow. Section 3 concerns some results on the behavior of the solutions for large time and small viscosity parameter. Finally the Appendix contains the proof of technical results.

**1. Boundary layer for the heat equation.** In this section we shall study the corrector problem for the heat equation in a 2D periodic channel. This is one of the necessary ingredients for describing the corrector for channel flows. In Section 1.1 we shall study the 1D case, derive some explicit expressions for the solutions and for the boundary layers and then in Section 1.2 we present the corrector for the heat equation in the 2D channel.

**1.1. A preliminary result: The heat equation in space dimension one.** As indicated before we want here to derive suitable expressions of the one-dimensional singularly perturbed heat equation and its corrector. Although some form of these correctors may be found perhaps in the engineering literature, we did not find them available in the mathematical literature (see e.g. M.I. Vishik and L.A. Lyusternik ([15]), J.L. Lions ([6])) and to the best of our knowledge the mathematical results given in Section 1 are new.

We need to start with the classical heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = f, & 0 < y < 1, \quad t > 0, \\ u|_{t=0} = g, \quad u(0) = u(1) = 0. \end{cases} \quad (1.1)$$

for which we give an explicit formula of the solution of (1.1) using the heat kernel.

More precisely, let us define the extension operator  $\mathcal{T} : L^2(0, 1) \rightarrow L^2_{\text{loc}}(\mathbb{R}^1)$ ,

$$\begin{cases} (\mathcal{T}(g))(y) = \begin{cases} g(y), & y \in (0, 1), \\ -g(-y), & y \in (-1, 0), \end{cases} \quad g \in L^2(0, 1) \\ \text{and } \mathcal{T}(g) \text{ is extended periodically outside } (-1, 1), \text{ with period } 2. \end{cases}$$

We then define

$$\tilde{u}(t, y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} (\mathcal{T}(g))(x) dx, \quad (1.2)$$

and we have the following result:

**Lemma 1.1.** *Suppose  $g \in L^2(0, 1)$ . Then  $\tilde{u}|_{(0,1)}$  defined in (1.2) solves the 1D heat equation (1.1) with initial data  $u_0 = g$  and zero forcing term ( $f = 0$ ), in the sense that*

$$\tilde{u}(t, \cdot) \rightarrow \mathcal{T}(g) \quad \text{in } L^2_{loc}(\mathbb{R}^1), \quad \text{or} \quad \|\tilde{u}(t, \cdot) - g\|_{L^2(0,1)} \rightarrow 0 \quad \text{as } t \rightarrow 0, \tag{1.3}$$

and

$$\tilde{u}|_{(0,1)} \in H^1_0(0, 1), \quad \text{or} \quad \tilde{u}(t, 0) = \tilde{u}(t, 1) = 0, \quad \text{for all } t > 0.$$

For the sake of completeness we recall the proof of this well-known result in the Appendix.

Similarly by applying Duhamel’s principle we can show that the solution to (1.1) (with general  $u_0$  and  $f$ ) can be expressed as

$$\begin{aligned} u(t, y) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} (\mathcal{T}(u_0))(x) dx \\ &+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} (\mathcal{T}(f(s)))(x) dx ds. \end{aligned} \tag{1.4}$$

Now let us consider the boundary-layer problem for the singularly perturbed heat equation on  $(0, 1)$ :

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 u^\varepsilon}{\partial y^2} = f, & 0 < y < 1, \\ u^\varepsilon|_{t=0} = u_0, \quad u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0. \end{cases} \tag{1.5}$$

The corresponding “inviscid” equation ( $\varepsilon = 0$ ) reads

$$\frac{\partial u^0}{\partial t} = f, \quad u^0|_{t=0} = u_0, \tag{1.6}$$

whose solution is

$$u^0 = u_0 + \int_0^t f(s) ds.$$

We will assume as much regularity as needed from  $u_0$  and  $f$  but, however,  $u^0$  will not vanish in general at  $x = 0$  and  $1$  ( $u^0 \notin H^1_0(0, 1)$ ), unless both  $u_0$  and  $f$  do, which is not a necessary requirement for the data in (1.5).

Consequently, as  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon$  can not converge to  $u^0$  in  $L^2(0, T; H^1(0, 1))$  if either  $f$  or  $u^0$  lies outside  $H^1_0(0, 1)$ . Thus there is a boundary layer near  $y = 0$  and  $y = 1$  due to the disparity of the boundary conditions for  $u^\varepsilon$  and  $u^0$ . Therefore a corrector (boundary layer-type function)  $\theta^\varepsilon$  is needed to ensure the  $H^1$  convergence. A heuristic argument or an approximation to the equation satisfied by  $u^\varepsilon - u^0$  suggests that  $\theta^\varepsilon$  be the solution of

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \theta^\varepsilon}{\partial y^2} = 0, \quad \theta^\varepsilon|_{t=0} = 0, \quad \theta^\varepsilon|_{\partial\Omega} = -u^0|_{\partial\Omega}.$$

Using straightforward energy estimates we verify that  $\theta^\varepsilon$  is a corrector in the following sense:

$$|u^\varepsilon - (u^0 + \theta^\varepsilon)|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon, \quad |u^\varepsilon - (u^0 + \theta^\varepsilon)|_{L^2(0,T;H^1(0,1))} \leq \kappa\varepsilon^{1/2}. \tag{1.7}$$

Here and after  $\kappa$  is constant depending on  $u^0$  and  $T$  but not on  $\varepsilon$ ; it may be a different one at different places.

Our next step is to get a nice representation and a good approximation of  $\theta^\varepsilon$ . To do that, we transform the boundary conditions to the homogeneous ones by setting

$$\varphi^\varepsilon(t; y) = \theta^\varepsilon(t; y) + (1 - y)u^0(t; 0) + yu^0(t; 1). \tag{1.8}$$

Then  $\varphi^\varepsilon$  satisfies the equation

$$\begin{cases} \frac{\partial \varphi^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \varphi^\varepsilon}{\partial y^2} = (1 - y) \frac{\partial u^0(t; 0)}{\partial t} + y \frac{\partial u^0(t; 1)}{\partial t}, \\ \varphi^\varepsilon|_{t=0} = (1 - y)u^0(0; 0) + yu^0(0; 1); \quad \varphi^\varepsilon(t; 0) = \varphi^\varepsilon(t; 1) = 0. \end{cases} \tag{1.9}$$

According to formula (1.4) with  $t$  replaced by  $\varepsilon t$ , we see that  $\varphi^\varepsilon$  has the following representation:

$$\begin{aligned} \varphi^\varepsilon(t; y) &= I_1(t; y)u^0(0; 1) + I_0(t; y)u^0(0; 0) \\ &\quad + \int_0^t \{ I_1(t - s; y) \frac{\partial u^0}{\partial t}(s; 1) + I_0(t - s; y) \frac{\partial u^0}{\partial t}(s; 0) \} ds, \end{aligned} \tag{1.10}$$

where

$$I_j(t; y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} \mathcal{T}(g_j)(x) dx, \quad j = 0, 1,$$

and  $g_0(y) = 1 - y$ ,  $g_1(y) = y$ . By inserting (1.10) into (1.8) and utilizing the fact that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dx = 1, \quad \forall y \in \mathbb{R}, \forall t > 0, \tag{1.11}$$

we deduce that

$$\begin{aligned} \theta^\varepsilon(t, y) &= (I_1(t; y) - g_1(y)) u^0(0; 1) + (I_0(t; y) - g_0(y)) u^0(0; 0) \\ &\quad + \int_0^t \{ (I_1(t - s; y) - g_1(y)) \frac{\partial u^0}{\partial t}(s; 1) + (I_0(t - s; y) - g_0(y)) \frac{\partial u^0}{\partial t}(s; 0) \} ds. \end{aligned}$$

At this stage the expression of  $\theta^\varepsilon$  is not yet simple and the boundary-layer phenomenon is not obvious from the expression. Thus we will approximate  $\theta^\varepsilon$ , and we shall do this in two steps. In the first step we shall approximate the integral by a finite one introducing in this way an exponentially small error and in the second step we shall approximate the finite integral using the standard error function to get an explicit form for the boundary layer.

For the first approximation let us observe that

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx - \int_{-1}^2 \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx \right\|_{H^1(0,1)} \\ \leq 4e^{-\frac{1}{16\epsilon t}} \|g\|_{L^2(0,1)}. \end{aligned} \tag{1.12}$$

To prove (1.12) let

$$\begin{aligned} \Delta &= \text{LHS of (1.12) inside the norm} \\ &= \int_2^{\infty} \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx + \int_{-\infty}^{-1} \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx = \Delta_1 + \Delta_2. \end{aligned} \tag{1.13}$$

To estimate the  $L^2$  norm of  $\Delta_1$ , let  $h \in L^2(0, 1)$ ,

$$\begin{aligned} \int_0^1 \Delta_1(y)h(y) dy &= \int_2^{\infty} \frac{1}{\sqrt{4\pi\epsilon t}} \int_0^1 h(y)e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx dy \quad (\text{by Fubini's theorem}) \\ &\leq e^{-\frac{1}{8\epsilon t}} \int_2^{\infty} \int_0^1 \frac{|h(y)|}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{8\epsilon t}} |\mathcal{T}(g)(x)| dy dx \quad (\text{since } |y-x| \geq 1) \\ &\leq e^{-\frac{1}{8\epsilon t}} \int_{-\infty}^{\infty} \int_0^1 \frac{|h(y)|}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{8\epsilon t}} |\mathcal{T}(g)(x)| dy dx \leq \sqrt{2}e^{-\frac{1}{8\epsilon t}} \|h\|_{L^2(0,1)} \cdot \|g\|_{L^2(0,1)}. \end{aligned}$$

To estimate the  $H^1$  norm of  $\Delta_1$ , we notice that

$$\frac{\partial}{\partial y} \Delta_1(y) = \int_2^{\infty} -\frac{(y-x)}{2\epsilon t} \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx,$$

and hence

$$\begin{aligned} \int_0^1 \frac{\partial}{\partial y} \Delta_1(y)h(y) dy &= \frac{1}{2\epsilon t\sqrt{4\pi\epsilon t}} \int_2^{\infty} \int_0^1 (x-y)h(y)e^{-\frac{(y-x)^2}{4\epsilon t}} \mathcal{T}(g)(x) dx dy \\ (\text{since } |y-x| \geq 1) &\leq \frac{e^{-\frac{1}{8\epsilon t}}}{2\epsilon t} \int_2^{\infty} \int_0^1 \frac{|y-x|}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{8\epsilon t}} |h(y)| |\mathcal{T}(g)(x)| dy dx \\ &\leq \frac{e^{-\frac{1}{8\epsilon t}}}{2\epsilon t} \int_{-\infty}^{\infty} \int_0^1 \frac{|y-x|}{\sqrt{4\pi\epsilon t}} e^{-\frac{(y-x)^2}{8\epsilon t}} |h(y)| |\mathcal{T}(g)(x)| dy dx \\ &\leq (\text{change of variable } z = y-x) \\ &\leq \frac{e^{-\frac{1}{8\epsilon t}}}{2\epsilon t} \int_0^1 \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{4\pi\epsilon t}} e^{-\frac{z^2}{8\epsilon t}} |h(y)| |\mathcal{T}(g)(y-z)| dz dy \\ &\leq \frac{e^{-\frac{1}{8\epsilon t}}}{2\epsilon t} \|h\|_{L^2(0,1)} \|g\|_{L^2(0,1)} \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{4\pi\epsilon t}} e^{-\frac{z^2}{8\epsilon t}} dz \\ &\leq \frac{2}{\sqrt{\pi\epsilon t}} e^{-\frac{1}{8\epsilon t}} \|h\|_{L^2(0,1)} \|g\|_{L^2(0,1)}. \end{aligned}$$

Hence

$$\|\Delta_1\|_{H^1(0,1)} \leq 2e^{-\frac{1}{16\varepsilon t}} \|g\|_{L^2(0,1)}.$$

Similarly we show that

$$\|\Delta_2\|_{H^1(0,1)} \leq 2e^{-\frac{1}{16\varepsilon t}} \|g\|_{L^2(0,1)},$$

and (1.12) follows.

For the second approximation, we write temporarily  $I(y) = I_j(t; y)$  and we first consider the case  $j = 1$ ,  $g_1(y) = y$ , so that

$$\mathcal{T}(g_1)(y) = \begin{cases} y, & -1 \leq y \leq 1, \\ y - 2, & 1 \leq y \leq 2. \end{cases}$$

Thus

$$\begin{aligned} I(y) &= \int_{-1}^2 \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} x \, dx - 2 \int_1^2 \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} \, dx \\ &= \int_{-1}^2 \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} (x - y) \, dx + y \int_{-1}^2 \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} \, dx - 2 \int_1^2 \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} \, dx \\ &= I_1 + y \cdot I_2 - 2 \cdot I_3. \end{aligned}$$

Before we estimate  $I$ , we recall the definition of the standard error function, erf:

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{x^2}{2}} \, dx.$$

For  $I_1$ , we make the change of variable  $z = \frac{x-y}{\sqrt{4\varepsilon t}}$

$$\begin{aligned} I_1 &= \int_{\frac{-1-y}{\sqrt{4\varepsilon t}}}^{\frac{2-y}{\sqrt{4\varepsilon t}}} \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-z^2} \sqrt{4\varepsilon t} z \cdot \sqrt{4\varepsilon t} \, dz = \sqrt{\frac{\varepsilon t}{\pi}} \int_{(\frac{-1-y}{\sqrt{4\varepsilon t}})^2}^{(\frac{2-y}{\sqrt{4\varepsilon t}})^2} e^{-z^2} \, d(z^2) \\ &= \sqrt{\frac{\varepsilon t}{\pi}} (e^{-\frac{(1+y)^2}{4\varepsilon t}} - e^{-\frac{(2-y)^2}{4\varepsilon t}}). \end{aligned}$$

Thus

$$\begin{aligned} |I_1| &\leq 2\sqrt{\frac{\varepsilon t}{\pi}} e^{-\frac{1}{4\varepsilon t}}, & \text{for } y \in (0, 1), \\ \|I_1\|_{H^1(0,1)} &\leq \kappa(\sqrt{\varepsilon t} + 1)e^{-\frac{1}{8\varepsilon t}} \leq \kappa e^{-\frac{1}{8\varepsilon t}}, & \text{for } \varepsilon \leq 1 \text{ and } t \leq T. \end{aligned} \tag{1.14}$$

Here and after we denote by  $\kappa$  an absolute constant which may be different at different places. For  $I_2$ , we observe that

$$|I_2(y) - 1| = \int_{-\infty}^{-1} \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(x-y)^2}{4\varepsilon t}} \, dx + \int_2^{\infty} \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{(y-x)^2}{4\varepsilon t}} \, dx$$

and hence, from (1.12) with  $g = 1$  and (1.11)

$$\|I_2(y) - 1\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{16\epsilon t}}, \quad \|yI_2(y) - y\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{16\epsilon t}}. \tag{1.15}$$

For  $I_3$  proceed as follows:

$$\begin{aligned} I_3(y) &= \text{(change of variable } z = \frac{x-y}{\sqrt{2\epsilon t}}) \\ &= \int_{\frac{1-y}{\sqrt{2\epsilon t}}}^{\frac{2-y}{\sqrt{4\epsilon t}}} \frac{1}{\sqrt{4\pi\epsilon t}} e^{-\frac{z^2}{2}} \sqrt{2\epsilon t} dz = \frac{1}{\sqrt{2\pi}} \int_{\frac{1-y}{\sqrt{2\epsilon t}}}^{\frac{2-y}{\sqrt{2\epsilon t}}} e^{-\frac{z^2}{2}} dz \\ &= \operatorname{erf}\left(\frac{2-y}{\sqrt{2\epsilon t}}\right) - \operatorname{erf}\left(\frac{1-y}{\sqrt{2\epsilon t}}\right). \end{aligned}$$

Notice that

$$\operatorname{erf}\left(\frac{2-y}{\sqrt{2\epsilon t}}\right) - \frac{1}{2} = -\frac{1}{\sqrt{2\pi}} \int_{\frac{2-y}{\sqrt{2\epsilon t}}}^{\infty} e^{-\frac{z^2}{2}} dz.$$

Thus for  $y \in (0, 1)$

$$\left| \operatorname{erf}\left(\frac{2-y}{\sqrt{2\epsilon t}}\right) - \frac{1}{2} \right| \leq e^{-\frac{1}{8\epsilon t}} \frac{1}{\sqrt{2\pi}} \int_{\frac{2-y}{\sqrt{2\epsilon t}}}^{\infty} e^{-\frac{z^2}{4}} dz \leq \frac{\sqrt{2}}{2} e^{-\frac{1}{8\epsilon t}},$$

and

$$\left\| \operatorname{erf}\left(\frac{2-y}{\sqrt{2\epsilon t}}\right) - \frac{1}{2} \right\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{8\epsilon t}}$$

and therefore

$$\|I_3(y) - \frac{1}{2} + \operatorname{erf}\left(\frac{1-y}{\sqrt{2\epsilon t}}\right)\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{8\epsilon t}}. \tag{1.16}$$

Combining together (1.15), (1.16) and (1.14) we find

$$\|I_1(t; y) - g_1(y) - 1 + 2 \operatorname{erf}\left(\frac{1-y}{\sqrt{2\epsilon t}}\right)\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{16\epsilon t}}. \tag{1.17}$$

Similar results hold for  $I_0(t, y)$  when  $g = g_0(y) = 1 - y$ ,

$$\|I_0(t; y) - g_0(y) - 2 \operatorname{erf}\left(\frac{y}{\sqrt{2\epsilon t}}\right)\|_{H^1(0,1)} \leq \kappa e^{-\frac{1}{16\epsilon t}}.$$

Combining (1.17), (1.12) and (1.13) and setting

$$\begin{aligned} \tilde{\theta}^\epsilon(t; y) &= u^0(0; 1)(1 - 2 \operatorname{erf}\left(\frac{1-y}{\sqrt{2\epsilon t}}\right)) + u^0(0; 0)(1 - 2 \operatorname{erf}\left(\frac{y}{\sqrt{2\epsilon t}}\right)) \\ &+ \int_0^t (1 - 2 \operatorname{erf}\left(\frac{1-y}{\sqrt{2\epsilon(t-s)}}\right)) \frac{\partial u^0(s; 1)}{\partial t} ds + \int_0^t (1 - 2 \operatorname{erf}\left(\frac{y}{\sqrt{2\epsilon(t-s)}}\right)) \frac{\partial u^0(s; 0)}{\partial t} ds \end{aligned} \tag{1.18}$$



we conclude that  $\tilde{\theta}^\varepsilon$  is a corrector for the problems (1.5)–(1.6) in the sense that (1.7) holds with  $\theta^\varepsilon$  replaced by  $\tilde{\theta}^\varepsilon$ . To see this we simply notice that

$$\begin{aligned} \|\theta^\varepsilon(t, y) - \tilde{\theta}^\varepsilon(t, y)\|_{H^1(0,1)} &\leq \kappa e^{-\frac{1}{16\varepsilon t}} \cdot (|u^0(0; 1)| + |u^0(0; 0)|) \\ &+ \kappa \int_0^t e^{-\frac{1}{16\varepsilon(t-s)}} \left( \left| \frac{\partial u^0(s; 1)}{\partial t} \right| + \left| \frac{\partial u^0(s; 0)}{\partial t} \right| \right) ds \leq \kappa e^{-\frac{1}{16\varepsilon t}}, \end{aligned} \tag{1.19}$$

$\kappa$  depending only on  $T$ .

**1.2. The heat equation in space dimension two.** In this subsection we shall give an explicit form for a corrector to the heat equation in a 2D channel  $\Omega_\infty = \mathbb{R}^1 \times (0, 1)$ ,  $\Omega = (0, 2\pi) \times (0, 1)$ . The equations under investigation are

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon = f, & (x, y) \in \Omega_\infty, \\ u^\varepsilon|_{t=0} = u_0, & u^\varepsilon|_{y=0,1} = 0, \end{cases} \tag{1.20}$$

$u^\varepsilon, u_0, f$  periodic in  $x$  with period  $2\pi$ . In the limit  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon$  converges to the “nonviscous” limit  $u^0 = u_0 + \int_0^t f(s) ds$ , which is the solution of

$$\frac{\partial u^0}{\partial t} = f, \quad u^0|_{t=0} = u_0.$$

Concerning the convergence of  $u^\varepsilon$  to  $u^0$  we have the first result hereafter which is proved in the Appendix.

**Proposition 1.1.** *Suppose  $u_0$  and  $f$  are sufficiently smooth. Then there exists a constant  $\kappa$  depending on  $u_0, f, T$  but not on  $\varepsilon$  such that*

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon^{1/4}, \quad \|u^\varepsilon - u^0\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa \varepsilon^{3/4}.$$

Again there is a problem near the boundary  $y = 0, 1$  and the  $H^1(\Omega)$  convergence does not hold if  $f$  or  $u_0$  lie outside

$$\tilde{H}_0^1(\Omega) = \{v|v \in H_{loc}^1(\Omega_\infty), v|_{y=0,1} = 0; v \text{ periodic in } x \text{ with period } 2\pi\}.$$

As in the 1D case we notice that  $\theta^\varepsilon$  given by the following equation (1.21) would be a good candidate for the corrector (boundary layer function):

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \Delta \theta^\varepsilon = 0, \quad \theta^\varepsilon|_{t=0} = 0, \quad \theta^\varepsilon|_{y=0,1} = -u^0|_{y=0,1}. \tag{1.21}$$

Just as in the 1D case we may apply energy estimates on  $u^\varepsilon - (u^0 + \theta^\varepsilon)$  to obtain

$$\|u^\varepsilon - (u^0 + \theta^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon, \quad \|u^\varepsilon - (u^0 + \theta^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leq \kappa \varepsilon^{\frac{1}{2}}. \tag{1.22}$$

In order to simplify  $\theta^\varepsilon$  and to observe the boundary layer behavior we do the following approximation to  $\theta^\varepsilon$  based on the heuristic argument that the boundary layer is in the vertical direction so the horizontal derivative should be negligible as compared with the vertical one. Thus we may drop the horizontal ( $x$ ) derivative in equation (1.39) to get

$$\frac{\partial \tilde{\theta}^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2}{\partial y^2} \tilde{\theta}^\varepsilon = 0, \quad \tilde{\theta}^\varepsilon|_{t=0} = 0, \quad \tilde{\theta}^\varepsilon|_{y=0,1} = -u^0|_{y=0,1}. \tag{1.23}$$

The following lemma verifies our heuristic argument.

**Lemma 1.2.** *Suppose  $u_0$  and  $f$  are sufficiently smooth; then the function  $\tilde{\theta}^\varepsilon$  defined in (1.23) is a corrector (or an approximation to  $\theta^\varepsilon$ ) in the sense that*

$$\|u^\varepsilon - (u^0 + \tilde{\theta}^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon, \quad \|u^\varepsilon - (u^0 + \tilde{\theta}^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}.$$

$\kappa$  depending on  $u_0, f$  and  $T$ , but not on  $\varepsilon$ .

**Proof.** Since we have (1.22), we need only to verify that

$$\|\theta^\varepsilon - \tilde{\theta}\|_{L^2(0,T;L^2(\Omega))} \leq \kappa\varepsilon, \quad \|\theta^\varepsilon - \tilde{\theta}^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}. \tag{1.24}$$

Set  $w^\varepsilon = \theta^\varepsilon - \tilde{\theta}^\varepsilon$ , and observe that  $w^\varepsilon$  satisfies the equation

$$\frac{\partial w^\varepsilon}{\partial t} - \varepsilon\Delta w^\varepsilon = \varepsilon\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}, \quad w^\varepsilon|_{t=0} = 0; \quad w^\varepsilon|_{y=0,1} = 0.$$

We are left to prove that  $\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}$  is bounded (in  $L^2(0, T; L^2(\Omega))$ ) since straightforward energy estimates would imply the result (1.24) provided this is true.

Notice that  $\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}$  satisfies the following equation obtained by differentiating (1.23) in  $x$  twice,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}\right) - \varepsilon\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}\right) = 0, \quad \frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}|_{t=0} = 0, \quad \frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}|_{y=0,1} = -\frac{\partial^2 u^0}{\partial x^2}|_{y=0,1}.$$

Let

$$\varphi^\varepsilon = \frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2} + y\frac{\partial^2 u^0}{\partial x^2}(t; x, 1) + (1 - y)\frac{\partial^2 u^0}{\partial x^2}(t; x, 0). \tag{1.25}$$

Then  $\varphi^\varepsilon$  satisfies the equation

$$\begin{cases} \frac{\partial \varphi^\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 \varphi^\varepsilon}{\partial y^2} = y\frac{\partial^3 u^0(t; x, 1)}{\partial t\partial x^2} + (1 - y)\frac{\partial^3 u^0(t; x, 0)}{\partial t\partial x^2}, \\ \varphi^\varepsilon|_{t=0} = y\frac{\partial^2 u^0}{\partial x^2}(x, 1) + (1 - y)\frac{\partial^2 u^0}{\partial x^2}(x; 0), \quad \varphi^\varepsilon|_{y=0,1} = 0. \end{cases} \tag{1.26}$$

A simple energy estimate (multiply (1.26) by  $\varphi^\varepsilon$ ) yields  $|\varphi^\varepsilon|_{L^\infty(0,T;L^2(\Omega))} \leq \text{constant}$ , and hence by (1.25)

$$\left|\frac{\partial^2 \tilde{\theta}^\varepsilon}{\partial x^2}\right|_{L^\infty(0,T;L^2(\Omega))} \leq \text{constant}.$$

This ends the proof for Lemma 1.2.

Now we apply the result of Section 1.1., in particular (1.18), to derive an explicit approximation of  $\tilde{\theta}^\varepsilon$ .

**Proposition 1.2.** *Assume that  $u_0$  and  $f$  are sufficiently regular and define  $\tilde{\theta}^\varepsilon(t; x, y)$  to be*

$$\begin{aligned} \tilde{\theta}^\varepsilon(t, x, y) &= u^0(0; x, 1)(1 - 2 \operatorname{erf}(\frac{1-y}{\sqrt{2\varepsilon t}})) + u^0(0; x, 0)(1 - 2 \operatorname{erf}(\frac{y}{\sqrt{2\varepsilon t}})) \\ &+ \int_0^t (1 - 2 \operatorname{erf}(\frac{1-y}{\sqrt{2\varepsilon(t-s)}})) \frac{\partial u^0(s; x, 1)}{\partial t} ds \\ &+ \int_0^t (1 - 2 \operatorname{erf}(\frac{y}{\sqrt{2\varepsilon(t-s)}})) \frac{\partial u^0(s; x, 0)}{\partial t} ds. \end{aligned}$$

Then  $\tilde{\theta}^\varepsilon$  is a corrector for the problems (1.20) (or an approximate solution to (1.39)) in the sense that

$$\|u^\varepsilon - (u^0 + \tilde{\theta}^\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \leq \kappa\varepsilon, \quad \|u^2 - (u^0 + \tilde{\theta}^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}.$$

**Proof.** Thanks to Lemma 1.2, we need only to prove that

$$\|\tilde{\theta}^\varepsilon - \tilde{\theta}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon, \quad \|\tilde{\theta}^\varepsilon - \tilde{\theta}^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq \kappa\varepsilon^{\frac{1}{2}}. \tag{1.27}$$

By (1.19) with  $\theta^\varepsilon(t, y) = \tilde{\theta}^\varepsilon(t; x, y)$ ,  $\tilde{\theta}^\varepsilon(t, y) = \tilde{\theta}^\varepsilon(t; x, y)$  we have, for fixed  $x$ ,

$$\begin{aligned} \|\tilde{\theta}^\varepsilon(t; x, \cdot) - \tilde{\theta}^\varepsilon(t; x, \cdot)\|_{H^1(0,1)} &\leq \kappa e^{-\frac{1}{16\varepsilon t}} \{ |u^0(0, x, 1)| + |u^0(0; x, 0)| \\ &+ \int_0^t (|\frac{\partial u^0(s; x, 1)}{\partial t}| + |\frac{\partial u^0(s; x, 0)}{\partial t}|) ds \}. \end{aligned}$$

Integrating over  $x$  on  $[0, 2\pi]$  we deduce the first inequality in (1.27). To derive the other one, we observe that  $\partial\tilde{\theta}^\varepsilon/\partial x$  and  $\partial\tilde{\theta}^\varepsilon/\partial x$  satisfy the same kind of equations with corresponding initial and boundary data; thus (1.19) applies again which produces

$$\begin{aligned} \|\frac{\partial\theta^\varepsilon}{\partial x}(t; x, \cdot) - \frac{\partial\tilde{\theta}^\varepsilon}{\partial x}(t; x, \cdot)\|_{H^1(0,1)} &\leq \kappa e^{-\frac{1}{16\varepsilon t}} (|\frac{\partial u^0}{\partial x}(0; x, 1)| + |\frac{\partial u^0}{\partial x}(0; x, 0)| \\ &+ \int_0^t \{ |\frac{\partial^2 u^0}{\partial x \partial t}(s; x, 1)| + |\frac{\partial^2 u^0}{\partial x \partial t}(s; x, 0)| \} ds). \end{aligned}$$

Integrating over  $[0, 2\pi]$  in  $x$  we obtain the other inequality in (1.27). This completes the proof of Proposition 1.2.

**Remark 1.1.** Proposition 1.2 improves, for this geometry (Channel domain), a result in J.L. Lions ([6]) (see Theorem 5.1, page 292 and page 294). The case of general domain  $\Omega$  will be considered elsewhere ([13]).

We conclude this section with the following technical result on the corrector  $\theta^\varepsilon$  (and hence on  $\tilde{\theta}^\varepsilon$ ); this result is valid in dimensions 1 and 2.

**Lemma 1.3.** *There exists  $\kappa = \kappa(u_0, f, T)$ , such that the function  $\theta^\varepsilon$  defined by (1.39) satisfies*

$$|\theta^\varepsilon|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}, \quad |\theta^\varepsilon|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}. \tag{1.28}$$

Moreover, if  $u_0 = u^0|_{t=0}$  vanishes at  $y = 0, 1$ , then we have

$$\|\theta^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq \kappa\varepsilon^{-1/4}, \quad \|\theta^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq \kappa\varepsilon^{-3/4}. \tag{1.29}$$

**Proof.** Although this result could be derived from the estimates above on  $\theta^\varepsilon - \tilde{\theta}^\varepsilon$  and from the explicit expression of  $\tilde{\theta}^\varepsilon$ , we prove it here with Proposition 1.1.

Let  $v^\varepsilon = \theta^\varepsilon + u^0$ . Then  $v^\varepsilon$  satisfies the equation

$$\frac{\partial v^\varepsilon}{\partial t} - \varepsilon\Delta v^\varepsilon = \frac{\partial u^0}{\partial t} - \varepsilon\Delta u^0, \quad v^\varepsilon|_{t=0} = u^0|_{t=0} = u_0, \quad v^\varepsilon|_{y=0,1} = 0.$$

Proposition 1.1 applies and (1.28) follows.

To prove (1.29), notice that the  $H^1$  estimates in (1.29) follow by interpolation from the  $L^2$  and  $H^2$  estimates. Hence there remains to prove the  $H^2$  estimates. This is equivalent to showing that

$$|\varepsilon\Delta\theta^\varepsilon|_{L^\infty(0,T;L^2(\Omega))} = \left| \frac{\partial\theta^\varepsilon}{\partial t} \right|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}.$$

Under the assumption that  $u_0 \in H^2(\Omega)$  and that it vanishes at  $y = 0, 1$ , and by using a classical result on the behavior at  $t = 0$  of solutions of semilinear evolution equations in the simple linear case (see e.g. Temam, [12]), we have  $v^\varepsilon \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ . Thus at  $t = 0$

$$\frac{\partial v^\varepsilon}{\partial t} = \varepsilon\Delta v^\varepsilon + \frac{\partial u^0}{\partial t} - \varepsilon\Delta u^0 = f \quad \text{in } L^2(\Omega).$$

Then we observe that  $\psi^\varepsilon = \frac{\partial v^\varepsilon}{\partial t}$  satisfies

$$\frac{\partial\psi^\varepsilon}{\partial t} - \varepsilon\Delta\psi^\varepsilon = \frac{\partial f}{\partial t} - \varepsilon\Delta f, \quad \psi^\varepsilon|_{t=0} = f|_{t=0}, \quad \psi|_{y=0,1} = 0.$$

Hence we can apply to  $\psi^\varepsilon$  Proposition 1.1 in the form given in the Appendix, and we obtain

$$|\psi^\varepsilon - f|_{L^\infty(0,T;L^2(\Omega))} = \left| \frac{\partial\psi^\varepsilon}{\partial t} \right|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}.$$

The lemma is proved.

**2. Boundary layers for channel flows.**

**2.1. Description of the problem.** In this subsection we shall describe the problem under investigation; namely, the corrector problem for channel flows. In the subsequent subsections we shall address the problem of  $L^2$  and  $H^1$  convergence. We shall confine ourselves to the linearized Navier-Stokes equation since for the full equation even  $L^2$  convergence already requires technicalities beyond the scope of this paper.

In this section we shall study the following linearized Navier-Stokes equation:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = f, & (x, y) \in \Omega_\infty, \\ u^\varepsilon|_{t=0} = u_0, \quad \operatorname{div} u^\varepsilon = 0, \quad u^\varepsilon|_{y=0,1} = 0, \end{cases} \quad (2.1)$$

and the corresponding linearized Euler equation

$$\frac{\partial u^0}{\partial t} + \nabla p^0 = f, \quad u^0|_{t=0} = u_0, \quad \operatorname{div} u^0 = 0, \quad u^0|_{y=0,1} = 0. \quad (2.2)$$

It can be shown via a straightforward energy estimate that  $u^\varepsilon$  converges to  $u^0$  in  $(L^2(\Omega))^2$  as  $\varepsilon \rightarrow 0$  (existence and uniqueness for  $u^\varepsilon$  and  $u^0$  are easy; for  $u^0$  see below). Contrary to the case where  $\Omega$  has no boundary; i.e.,  $\Omega$  is the whole space or with periodic boundary conditions in both directions, see e.g. T. Kato ([4]) and the references therein, the  $H^1(\Omega)$  convergence is not true due to the boundary layer phenomenon, even the rate of convergence in  $(L^2(\Omega))^2$  is hindered by the mismatch of the boundary conditions. This necessitates a corrector  $\theta^\varepsilon$ . A naive choice of  $\theta^\varepsilon$  would be  $u^\varepsilon - u^0$ , but this would not provide us with any new information since we have to solve both the viscous and the inviscid equations to find this corrector. Thus this corrector bears two disadvantages: it is hard to calculate and it possesses no explicit formula. However, it does have an advantage; i.e.,  $u^\varepsilon - (u^0 + \theta^\varepsilon)$  converges strongly in  $H^1$ . We would favor a corrector which has a nice representation and displays explicit boundary layer behavior and is easy to calculate. As one could imagine the major difficulty of this problem lies in the divergence-free constraint and the presence of pressure. We shall employ two different techniques to attack the problem. The first one is to construct divergence-free functions with prescribed tangential velocity at the boundary. This method turns out to be successful to get the rate of convergence in  $(L^2(\Omega))^2$  spaces. The construction of divergence-free functions seems to be new and it might have applications in other areas. The second method we use is that of classical energy methods applied to the Navier-Stokes equation. There are some technical difficulties in estimating the pressure due to the fact that our corrector is not divergence free. At this stage we are only able to have a weak corrector as in J.L. Lions ([6, page 29]). We shall address the problem of strong convergence in a subsequent work.

Before we state our results, we introduce the spaces that we are going to use, which are standard in Navier-Stokes theory. Let

$$H = \{v \in (L^2_{\text{loc}}(\Omega_\infty))^2 : \operatorname{div} v = 0, v_2|_{y=0,1} = 0, v \text{ is periodic in } x \text{ with period } 2\pi\},$$

$$V = \{v \in (H^1_{\text{loc}}(\Omega_\infty))^2 : v \in H, v|_{y=0,1} = 0\}.$$

The space  $H$  is well defined and is a closed subspace of  $(L^2(\Omega))^2$  by the standard trace theorem for divergence-free functions.

Without loss of generality we may assume that

$$f \in L^\infty(0, T; H), \quad (2.3)$$

since  $(I - P)f$  can be absorbed in the pressure term where  $P$  is the Leray projection from  $(L^2(\Omega))^2$  to  $H$ . It is well known (R. Temam, [10]) that the operator  $P$  is continuous from  $(H^k(\Omega))^2$  to  $(H^k(\Omega))^2 \cap H$  for all  $k$ . Thus  $Pf$  is smooth as long as  $f$  is smooth enough.

Assuming (2.3), (2.2) is equivalent to

$$\frac{du^0}{dt} = f, \quad \text{for } t > 0 \text{ in } H, \quad u^0|_{t=0} = u_0,$$

so that  $u^0 = u_0 + \int_0^t f(s) ds$  and  $(p^0 = (I - P)f$  if  $f \in L^\infty(0, T; L^2(\Omega)^2)$ . Our result on  $L^2$  convergence is the following:

**Proposition 2.1.** *Assume that  $u_0 \in H$  and  $f \in L^\infty(0, T; H)$  are sufficiently regular. Then there exists a constant  $\kappa = \kappa(u_0, f, T)$  such that*

$$|u^\varepsilon - u^0|_{L^\infty(0, T; H)} \leq \kappa \varepsilon^{\frac{1}{4}},$$

where  $u^\varepsilon$  and  $u^0$  are given by (2.1) and (2.2).

Our result on  $H^1$  weak corrector is

**Theorem 2.1.** *Assume that  $u_0$  and  $f$  are sufficiently regular and that  $u^0 \in V$  and  $f \in L^\infty(0, T; H)$ . Let  $\theta^\varepsilon$  be the solution of*

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \Delta \theta^\varepsilon = 0, \quad \theta^\varepsilon|_{t=0} = 0, \quad \theta^\varepsilon|_{y=0,1} = -u^0|_{y=0,1}. \tag{2.4}$$

Then  $\theta^\varepsilon$  is a zero<sup>th</sup>-order corrector for the problems (2.1) and (2.2) in the sense that

$$\|u^\varepsilon - (u^0 + \theta^\varepsilon)\|_{L^\infty(0, T; (L^2(\Omega))^2)} \leq \kappa \varepsilon^{\frac{1}{2}}, \tag{2.5}$$

$$\|u^\varepsilon - (u^0 + \theta^\varepsilon)\|_{L^2(0, T; (H^1(\Omega))^2)} \leq \kappa, \tag{2.6}$$

$$\|p^\varepsilon - p^0\|_{L^2(0, T; H^1(\Omega))} \leq \kappa \varepsilon^{\frac{1}{4}},$$

$$u^\varepsilon - (u^0 + \theta^\varepsilon) \rightharpoonup 0 \text{ weakly in } L^2(0, T; (H^1(\Omega))^2), \tag{2.7}$$

where the generic constant  $\kappa$  depends on  $u_0, f$  and  $T$  but not on  $\varepsilon$  ( $p^0 = 0$ ). The same is true with  $\theta^\varepsilon$  replaced by  $\tilde{\theta}^\varepsilon$  explicitly given by (1.3)

**Remark 2.1.** We call  $\theta^\varepsilon, \tilde{\theta}^\varepsilon$  a weak zero<sup>th</sup>-order corrector, because of (2.6) and (2.7) which are the same as in J.L. Lions ([6]) (see Theorem 5.1, page 292 and (2.91)). We intend in a subsequent publication to derive strong zero<sup>th</sup>-order correctors as we did in Proposition 1.1 for the heat equation (see also Remark 1.1).

Note however the following consequence of (2.5), (2.6):

**Corollary 2.1.** *Under the assumptions of Theorem 2.1,*

$$\|u^\epsilon - (u^\theta + \theta^\epsilon)\|_{L^2(0,T;\tilde{H}^s(\Omega)^2)} \leq \kappa\epsilon^{0.5(1-s)},$$

for any  $s, 0 \leq s \leq 1$ , where  $\kappa$  depends  $u_0, f$  and  $T$  but not on  $\epsilon$  and  $s$ . The same is true with  $\theta^\epsilon$  replaced by  $\tilde{\theta}^\epsilon$  explicitly given by (1.3).

**2.2. Proof of  $L^2$  convergence.** In this subsection we shall prove Proposition 2.1. Before we prove the result, we need to introduce the following lemma on constructing divergence-free functions with given boundary tangential velocity. The details of the proof will be presented in the Appendix.

**Lemma 2.1.** *Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^2$  and let  $\varphi$  be a smooth function on  $\partial\Omega$  satisfying  $\varphi \cdot n = 0$  where  $n$  is the unit outer normal to  $\partial\Omega$ . Then there exists a constant  $\kappa = \kappa(\Omega)$  such that for each  $\epsilon > 0$ , there exists a smooth function  $\varphi^\epsilon \in H$  satisfying*

$$\varphi^\epsilon|_{\partial\Omega} = \varphi, \quad |\varphi^\epsilon|_H \leq \kappa\epsilon^{\frac{1}{4}}|\varphi|_{W^{1,\infty}(\partial\Omega)}, \quad \|\varphi^\epsilon\|_{H^1(\Omega)} \leq \kappa\epsilon^{-\frac{1}{4}}|\varphi|_{W^{2,\infty}(\partial\Omega)}, \quad (2.8)$$

and the mapping  $\varphi \rightarrow \varphi^\epsilon$  is linear continuous from  $W^{1,\infty}(\partial\Omega)$  into  $H$  and from  $W^{2,\infty}(\partial\Omega)$  into  $H^1(\Omega)$ .

**Proof.** See the Appendix.

**Remark 2.2.** Lemma 2.1 is valid also, with a similar proof, in the case of  $\Omega_\infty, \varphi$  given periodic on  $\partial\Omega_\infty$ . We use it in this context, but we will prove Lemma 2.1 in the more general form in which it is stated.

**Proof of Proposition 2.1.** Using the linearity of the equation we may separate the problem into two parts, where one part is the pure initial-value problem and the other corresponds to zero initial value but nonzero forcing term. Let  $u^\epsilon = u^{1\epsilon} + u^{2\epsilon}$ , where

$$\frac{\partial u^{1\epsilon}}{\partial t} - \epsilon\Delta u^{1\epsilon} + \nabla p^{1\epsilon} = 0, \quad u^{1\epsilon}|_{t=0} = u_0, \quad u^{1\epsilon} \in V,$$

$$\frac{\partial u^{10}}{\partial t} + \nabla p^{10} = 0, \quad u^{10}|_{t=0} = u_0, \quad u^{10} \in H.$$

It is easy to check that

$$u^{10} = u_0, \quad p^{10} = 0. \quad (2.9)$$

Thanks to Lemma 2.1, there exists  $\varphi^\epsilon$  (corresponding to  $\varphi = u^{10}|_{\partial\Omega} = u_0|_{\partial\Omega}$ ) such that (2.8) holds.

Consider the function  $w^{1\epsilon} = u^{1\epsilon} - (u^{10} - \varphi^\epsilon)$ . Then  $w^{1\epsilon} \in V$  and it satisfies the equation

$$\frac{\partial w^{1\epsilon}}{\partial t} - \epsilon\Delta w^{1\epsilon} + \nabla(p^{1\epsilon} - p^{10}) = \epsilon\Delta u^{10} - \epsilon\Delta\varphi^\epsilon, \quad w^{1\epsilon}|_{t=0} = \varphi^\epsilon, \quad w^{1\epsilon} \in V. \quad (2.10)$$

Multiply (2.10) by  $w^\varepsilon$ , integrate over  $\Omega$ , integrate by parts and apply the Cauchy-Schwarz inequality. This yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |w^{1\varepsilon}|_{L^2}^2 + \varepsilon |\nabla w^{1\varepsilon}|_{L^2}^2 \leq \varepsilon |\nabla u|_{L^2} \cdot |\nabla w^{1\varepsilon}|_{L^2} + \varepsilon |\nabla \varphi^\varepsilon|_{L^2} |\nabla w^{1\varepsilon}|_{L^2} \\ & \leq \frac{\varepsilon}{2} |\nabla w^{1\varepsilon}|_{L^2}^2 + \frac{\varepsilon}{2} (|\nabla u|_{L^2} + |\nabla \varphi^\varepsilon|_{L^2})^2 \leq (\text{thanks to (2.8) and (2.9)}) \\ & \leq \frac{\varepsilon}{2} |\nabla w^{1\varepsilon}|_{L^2}^2 + \kappa \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Applying Gronwall’s inequality, we find

$$\|w^{1\varepsilon}\|_{L^\infty(0,T;H)} \leq \kappa \varepsilon^{\frac{1}{4}}, \quad \|w^{1\varepsilon}\|_{L^2(0,T;V)} \leq \kappa \varepsilon^{-\frac{1}{4}}. \tag{2.11}$$

Similarly, we have

$$\begin{aligned} \frac{\partial u^{2\varepsilon}}{\partial t} - \varepsilon \Delta u^{2\varepsilon} + \nabla p^{2\varepsilon} &= f, \quad u^{2\varepsilon}|_{t=0} = 0, \quad u^{2\varepsilon} \in V, \\ \frac{\partial u^{20}}{\partial t} + \nabla p^{20} &= f, \quad u^{20}|_{t=0} = 0, \quad u^{20} \in H. \end{aligned}$$

It is easy to observe that

$$u^{20}(t) = \int_0^t f(s) ds, p^{20} = 0, \quad \text{since } f \in H. \tag{2.12}$$

We apply Lemma 2.1 to  $\int_0^t f(s) ds$  with the resulting function  $\varphi^\varepsilon$  denoted  $\varphi^{2\varepsilon}$ . We let  $w^{2\varepsilon} = u^{2\varepsilon} - (u^{20} - \varphi^{2\varepsilon})$ ; then  $w^{2\varepsilon}$  satisfies the equation

$$\frac{\partial w^{2\varepsilon}}{\partial t} - \varepsilon \Delta w^{2\varepsilon} + \nabla(p^{2\varepsilon} - p^{20}) = \frac{\partial \varphi^\varepsilon}{\partial t} + \varepsilon \Delta u^{20} - \varepsilon \Delta \varphi^{2\varepsilon}, \quad w^{2\varepsilon}|_{t=0} = 0, \quad w^{2\varepsilon} \in V.$$

Observe that if  $\tilde{\varphi}^\varepsilon(x)$  is the function corresponding to  $f|_{\partial\Omega}$  in Lemma 2.1, then the relation between  $\varphi^{2\varepsilon}$  and  $\tilde{\varphi}^\varepsilon$  is given by

$$\varphi^{2\varepsilon} = \int_0^t \tilde{\varphi}^\varepsilon(s) ds$$

and thus (2.12) can be rewritten as

$$\frac{\partial}{\partial t} w^{2\varepsilon} - \varepsilon \Delta w^{2\varepsilon} + \nabla(p^{2\varepsilon} - p^{20}) = \tilde{\varphi}^\varepsilon(t, x) + \varepsilon \Delta u^{20} - \varepsilon \int_0^t \Delta \tilde{\varphi}^\varepsilon(s) ds. \tag{2.13}$$

Multiply (2.13) by  $w^{2\varepsilon}$  and apply the usual integration by parts and Cauchy-Schwarz inequality to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |w^{2\varepsilon}|_{L^2}^2 + \varepsilon |\nabla w^{2\varepsilon}|_{L^2}^2 \\ & \leq |\tilde{\varphi}^\varepsilon|_{L^2} |w^{2\varepsilon}|_{L^2} + \varepsilon |\nabla u^{20}|_{L^2} |\nabla w^{2\varepsilon}|_{L^2} + \varepsilon \left( \int_0^t |\nabla \tilde{\varphi}^\varepsilon|_{L^2} ds \right) |\nabla w^{2\varepsilon}|_{L^2} \\ & \leq \frac{\varepsilon}{2} |\nabla w^{2\varepsilon}|_{L^2}^2 + \frac{1}{2} |w^{2\varepsilon}|_{L^2}^2 + \frac{1}{2} |\tilde{\varphi}^\varepsilon|_{L^2}^2 + \varepsilon |\nabla u^{20}|_{L^2}^2 + \varepsilon \left( \int_0^t |\nabla \tilde{\varphi}^\varepsilon|_{L^2} \right)^2 \\ & \leq (\text{thanks to (2.8) and (2.12)}) \leq \frac{\varepsilon}{2} |\nabla w^{2\varepsilon}|_{L^2}^2 + \frac{1}{2} |w^{2\varepsilon}|_{L^2}^2 + \kappa \varepsilon^{\frac{1}{2}}. \end{aligned}$$



This implies, thanks to Gronwall’s inequality,

$$\|w^{2\varepsilon}\|_{L^\infty(0,T;H)} \leq \kappa\varepsilon^{\frac{1}{4}}, \quad \|w^{2\varepsilon}\|_{L^2(0,T;V)} \leq \kappa\varepsilon^{-\frac{1}{4}}. \tag{2.14}$$

Combining (2.11) and (2.14) the result of the Proposition 2.1 follows immediately. This completes the proof of the proposition.

**Remark 2.3.** The estimates that we have here are somehow optimal. Indeed if we consider initial data of the form  $u_0 = (u_{01}(y), 0)$ ,  $f = (f_1(y), 0)$ , then the Stokes system reduces to the heat equation and we know by explicit calculations that  $\varepsilon^{\frac{1}{4}}$  is optimal in this case.

**Remark 2.4.** It is well known that the  $L^2$  convergence of the full Navier-Stokes equation to Euler’s equation is highly nontrivial. Using our Lemma 2.1 we may observe that even the  $L^2$  convergence is already related to the boundary-layer phenomenon. Indeed, using Lemma 2.1 and standard techniques on energy methods for the Navier-Stokes equation (see [9]) one may show that

$$\left\{ \begin{array}{l} u^\varepsilon \rightarrow u^0 \text{ in } L^\infty(0, T; H) \text{ if and only if there exist a constant } \kappa_1 > 0 \\ \text{and } \alpha > 0 \text{ such that } \lim_{\varepsilon \rightarrow 0} \int_0^T (\varepsilon |\nabla u^\varepsilon|_{L^2(\mathcal{O}_{\kappa_1\varepsilon}(\partial\Omega))}^2)^\alpha dt = 0, \end{array} \right.$$

where  $\mathcal{O}_{\kappa_1\varepsilon}(\partial\Omega)$  denotes the points in  $\Omega$  which are within a  $\kappa_1\varepsilon$  distance from  $\partial\Omega$ .

**2.3. Proof of Theorem 2.1.** In this subsection we shall prove the theorem stated in the previous section.

Before we prove our main result in this section, we need the following lemma which gives an estimate on the distance of  $\theta^\varepsilon$  to  $H$ .

**Lemma 2.2.** *The assumptions on  $u_0$  and  $f$  are as in Theorem 2.1 and  $\theta^\varepsilon$  is defined by (2.4). Then there exists a constant  $\kappa$  (independent of  $\varepsilon$ ) such that*

$$\|(I - P)\theta^\varepsilon\|_{L^\infty(0,T;(L^2(\Omega))^2)} \leq \kappa\varepsilon^{3/4}, \quad \|(I - P)\frac{\partial}{\partial t}\theta^\varepsilon\|_{L^\infty(0,T;(L^2(\Omega))^2)} \leq \kappa\varepsilon^{3/4}, \tag{2.15}$$

where  $P$  is the Leray projector; i.e., the orthogonal projection from  $(L^2(\Omega))^2$  onto  $H$ .

**Proof.** It is sufficient to prove the second inequality in (2.15) since  $\theta^\varepsilon|_{t=0} = 0$  and hence we can recover  $\theta^\varepsilon$  by integrating  $\frac{\partial\theta^\varepsilon}{\partial t}$ .

We know (see e.g. R. Temam, [10]) that

$$(I - P)\frac{\partial}{\partial t}\theta^\varepsilon = \nabla q^\varepsilon, \quad \int_\Omega q^\varepsilon = 0, \tag{2.16}$$

where  $q^\varepsilon$  is periodic in  $x$  and satisfies the equation

$$\Delta q^\varepsilon = \operatorname{div} \left( \frac{\partial}{\partial t} \theta^\varepsilon \right), \quad \text{in } \Omega_\infty, \quad \frac{\partial q^\varepsilon}{\partial y} \Big|_{y=0,1} = 0, \tag{2.31}$$

since by (2.4)  $\frac{\partial}{\partial t}\theta_2^\varepsilon|_{y=0,1} = -f_2|_{y=0,1} = 0$  and we take the inner product of (2.16) with  $n$ , the unit outer normal on  $\partial\Omega_\infty$ .

Thus, according to the Agmon-Douglis-Nirenberg elliptic regularity results we have

$$\|q^\varepsilon\|_{H^1(\Omega)} \leq c\|\operatorname{div}\left(\frac{\partial}{\partial t}\theta^\varepsilon\right)\|_{H^{-1}(\Omega)}. \tag{2.32}$$

Hence it remains to prove that, for all  $t \in [0, T]$ ,

$$\|\operatorname{div}\left(\frac{\partial}{\partial t}\theta^\varepsilon\right)\|_{H^{-1}(\Omega)} \leq \kappa\varepsilon^{3/4}. \tag{2.17}$$

Notice that, in (2.4), the second component of  $\theta^\varepsilon$  satisfies the homogeneous heat equation with zero initial data and zero boundary condition, since

$$u^0 = u_0 + \int_0^t f(s; x, y) ds \in H, \quad \text{as } u_0, f \in H.$$

Thus  $\theta_2^\varepsilon|_{y=0,1} = -u_2^0|_{y=0,1} = 0$ . Hence  $\theta_2^\varepsilon \equiv 0$ , and therefore

$$\operatorname{div}\left(\frac{\partial}{\partial t}\theta^\varepsilon\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\theta_1^\varepsilon\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\theta_1^\varepsilon\right). \tag{2.18}$$

We apply  $\frac{\partial^2}{\partial t\partial x}$  to (2.4);  $\frac{\partial}{\partial x}$  being a tangential derivative at  $y = 0, 1$  and since  $\Delta\theta^\varepsilon|_{t=0} = 0$ , we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\theta_1^\varepsilon}{\partial t\partial x}\right) - \varepsilon\Delta\left(\frac{\partial^2\theta_1^\varepsilon}{\partial t\partial x}\right) = 0, \quad \frac{\partial^2\theta_1^\varepsilon}{\partial t\partial x}|_{t=0} = 0, \quad \frac{\partial^2\theta_1^\varepsilon}{\partial t\partial x}|_{y=0,1} = -\frac{\partial^2u_1^0}{\partial t\partial x}|_{y=0,1}.$$

Let

$$\varphi^\varepsilon = \frac{\partial^2}{\partial t\partial x}\theta_1^\varepsilon + \frac{\partial^2u_1^0}{\partial t\partial x}.$$

Then  $\varphi^\varepsilon$  satisfies the equation

$$\frac{\partial\varphi^\varepsilon}{\partial t} - \varepsilon\Delta\varphi^\varepsilon = \frac{\partial^3u_1^0}{\partial t^2\partial x} - \varepsilon\Delta\frac{\partial^2u_1^0}{\partial t\partial x}, \quad \varphi^\varepsilon|_{t=0} = \frac{\partial^2u_1^0}{\partial t\partial x}|_{t=0}, \quad \varphi^\varepsilon|_{y=0,1} = 0. \tag{2.19}$$

Now we apply Proposition 1.1 from the Appendix to equation (2.19) and we obtain

$$\begin{aligned} & \left\| \varphi^\varepsilon - \left( \frac{\partial^2u_1^0}{\partial t\partial x} \Big|_{t=0} + \int_0^t \frac{\partial^3u_1^0}{\partial t^2\partial x} ds - \varepsilon \int_0^t \Delta \frac{\partial^2u_1^0}{\partial t\partial x} ds \right) \right\|_{L^\infty(0,T;H^{-1}(\Omega))} \\ &= \left\| \varphi^\varepsilon - \frac{\partial^2u_1^0}{\partial t\partial x} + \varepsilon \int_0^t \Delta \frac{\partial^2u_1^0}{\partial t\partial x} \right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}, \end{aligned}$$

which implies

$$\left\| \varphi^\varepsilon - \frac{\partial^2u_1^0}{\partial t\partial x} \right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}, \tag{2.20}$$

and therefore

$$\left\| \frac{\partial^2 \theta_1^\varepsilon}{\partial t \partial x} \right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa \varepsilon^{3/4},$$

or equivalently (by (2.18)) (2.17) holds. This completes the proof of Lemma 2.2.

Now we go to the proof of the main theorem in this section.

**Proof of the main result.** Multiply (2.1) by  $Au^\varepsilon$  and integrate over  $\Omega$ , where  $A$  is the Stokes operator on  $\Omega$  with the prescribed boundary conditions ( $Au^\varepsilon = -P\Delta u^\varepsilon$ ). Then,  $|\cdot|$  denoting the norm in  $H$  ( the  $L^2$  norm ),

$$\frac{1}{2} \frac{d}{dt} |A^{\frac{1}{2}} u^\varepsilon|^2 + \varepsilon |Au^\varepsilon|^2 = (f, Au^\varepsilon) \leq \frac{\varepsilon}{2} |Au^\varepsilon|^2 + \frac{|f|^2}{2\varepsilon},$$

which implies that

$$\begin{cases} \|\varepsilon^{\frac{1}{2}} A^{\frac{1}{2}} u^\varepsilon\|_{L^\infty(0,T;(L^2)^2)} \leq |f|_{L^2(0,T;(L^2(\Omega))^2)} + \|u_0\|_V, \\ \|\varepsilon Au^\varepsilon\|_{L^2(0,T;(L^2(\Omega))^2)} \leq |f|_{L^2(0,T;(L^2(\Omega))^2)} + \|u_0\|_V. \end{cases}$$

Thanks to the regularity results for the Stokes equation, we have

$$\begin{cases} \|\varepsilon^{\frac{1}{2}} u^\varepsilon\|_{L^\infty(0,T;V)} \leq |f|_{L^2(0,T;(L^2(\Omega))^2)} + |u_0|_V, \\ \|\varepsilon u^\varepsilon\|_{L^2(0,T;(H^2)^2)} \leq |f|_{L^2(0,T;(L^2(\Omega))^2)} + |u_0|_V. \end{cases} \tag{2.21}$$

Let  $w^\varepsilon = u^\varepsilon - (u^0 + \theta^\varepsilon)$ . Then  $w^\varepsilon$  satisfies the equation

$$\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + \nabla(p^\varepsilon - p^0) = \varepsilon \cdot g = \varepsilon \Delta u^0, \quad w^\varepsilon|_{t=0} = 0, \quad w^\varepsilon|_{y=0,1} = 0, \tag{2.22}$$

where  $p^0 = 0$  and  $g = \Delta u^0$ . We multiply (2.22) by  $Pw^\varepsilon$  and integrate over  $\Omega$ :

$$\frac{1}{2} \frac{d}{dt} |Pw^\varepsilon|^2 + \varepsilon |\nabla w^\varepsilon|^2 = -(\varepsilon \Delta w^\varepsilon, (I - P)w^\varepsilon) + (\varepsilon g, Pw^\varepsilon).$$

Notice that  $(I - P)w^\varepsilon = -(I - P)\theta^\varepsilon$ , and thus

$$\begin{aligned} |(\varepsilon \Delta w^\varepsilon, (I - P)w^\varepsilon)| &\leq |\varepsilon w^\varepsilon|_{H^2} |(I - P)\theta^\varepsilon| \leq (\text{thanks to Lemma 2.2}) \\ &\leq \kappa \varepsilon^{3/4} |\varepsilon w^\varepsilon|_{H^2} \leq (\text{thanks to (2.21)}) \leq \kappa \varepsilon^{3/4}. \end{aligned}$$

Notice also that

$$|(\varepsilon g, Pw^\varepsilon)| \leq \frac{1}{2} |Pw^\varepsilon|^2 + \frac{\varepsilon^2}{2} |u^0|_{H^2}^2.$$

Thus we can deduce, thanks to (2.21),

$$|Pw^\varepsilon|_{L^\infty(0,T;H)} \leq \kappa \varepsilon^{3/8}, \quad \|\varepsilon^{1/2} w^\varepsilon\|_{L^2(0,T;(H^1(\Omega))^2)} \leq \kappa \varepsilon^{3/8}.$$

Notice that

$$|w^\varepsilon| \leq |Pw^\varepsilon| + |(I - P)w^\varepsilon| = |Pw^\varepsilon| + |(I - P)\theta^\varepsilon| \leq \kappa\varepsilon^{3/8},$$

and thus,

$$|w^\varepsilon|_{L^\infty(0,T;(L^2(\Omega))^2)} \leq \kappa\varepsilon^{3/8}.$$

Now we multiply (2.22) by  $-\varepsilon P\Delta w^\varepsilon$  and integrate over  $\Omega$ . Notice that

$$\begin{aligned} \left(\frac{\partial w^\varepsilon}{\partial t}, -\varepsilon P\Delta w^\varepsilon\right) &= \left(\frac{\partial w^\varepsilon}{\partial t}, -\varepsilon\Delta w^\varepsilon\right) + \left(\frac{\partial w^\varepsilon}{\partial t}, \varepsilon(I - P)\Delta w^\varepsilon\right) \\ &= \frac{\varepsilon}{2} \frac{d}{dt} |\nabla w^\varepsilon|^2 + \left((I - P)\frac{\partial w^\varepsilon}{\partial t}, \varepsilon\Delta w^\varepsilon\right) \\ &= \frac{\varepsilon}{2} \frac{d}{dt} |\nabla w^\varepsilon|^2 + \left(-(I - P)\frac{\partial \theta^\varepsilon}{\partial t}, \varepsilon\Delta w^\varepsilon\right) \\ &\geq \text{(with (2.15)) } \geq \frac{\varepsilon}{2} \frac{d}{dt} |\nabla w^\varepsilon|^2 - \kappa\varepsilon^{3/4} |\varepsilon\Delta w^\varepsilon|. \end{aligned}$$

Thus we have

$$\varepsilon \frac{d}{dt} |\nabla w^\varepsilon|^2 + |\varepsilon P\Delta w^\varepsilon|^2 \leq c\varepsilon^{3/4} |\varepsilon\Delta w^\varepsilon| + c\varepsilon^2,$$

which implies

$$\|\varepsilon^{1/2} w^\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))^2)} \leq C\varepsilon^{3/8}, \quad \|\varepsilon P\Delta w^\varepsilon\|_{L^2(0,T;H)} \leq C\varepsilon^{3/8}.$$

Notice that

$$\begin{aligned} \|\varepsilon P\Delta u^\varepsilon\|_{L^2(0,T;H)} &\leq \|\varepsilon P\Delta w^\varepsilon\|_{L^2(0,T;H)} + \|\varepsilon P\Delta u^0\|_{L^2(0,T;H)} + \|\varepsilon P\Delta \theta^\varepsilon\|_{L^2(0,T;H)} \\ &\leq C\varepsilon^{3/8} + \|\varepsilon \theta^\varepsilon\|_{L^2(0,T;(H^2(\Omega))^2)} \leq C\varepsilon^{3/8} + \left\|\frac{\partial \theta^\varepsilon}{\partial t}\right\|_{L^2(0,T;(L^2(\Omega))^2)} \\ &\leq \text{(thanks to Lemma 2.3)} \leq C\varepsilon^{3/8} + C\varepsilon^{1/4} \leq C\varepsilon^{1/4}, \end{aligned}$$

hence

$$\|\varepsilon \Delta u^\varepsilon\|_{L^2(0,T;(L^2(\Omega))^2)} \leq C\varepsilon^{1/4}, \tag{2.23}$$

and thus

$$\begin{aligned} &\|\varepsilon \Delta w^\varepsilon\|_{L^2(0,T;(L^2)^2)} \\ &\leq \|\varepsilon \Delta u^\varepsilon\|_{L^2(0,T;(L^2)^2)} + \|\varepsilon \Delta u^0\|_{L^2(0,T;(L^2)^2)} + \|\varepsilon \Delta \theta^\varepsilon\|_{L^2(0,T;(L^2)^2)} \leq C\varepsilon^{1/4}. \end{aligned}$$

Next, we consider the equation satisfied by  $p^\varepsilon - p^0$ . Since  $p^0 = 0$  (we have assumed  $f \in H$ ),  $p^\varepsilon - p^0 = p^\varepsilon$  and it satisfies the equation (by applying  $\text{div}$  to (2.1))

$$\Delta p^\varepsilon = \text{div } f = 0, \quad \frac{\partial p^\varepsilon}{\partial y} \Big|_{y=0,1} = \pm \varepsilon \frac{\partial^2 u^\varepsilon}{\partial y^2} \Big|_{y=0,1}.$$

Thus

$$\begin{aligned} \|p^\varepsilon - p^0\|_{L^2(0,T;H^1(\Omega))} &= \|\nabla p^\varepsilon\|_{L^2(0,T;(L^2(\Omega))^2)} \leq c\|\varepsilon\Delta u^\varepsilon\|_{L^2(0,T;(H^{-1/2}(\partial\Omega_\infty \cap \partial\Omega)^2))} \\ &\leq (\text{thanks to the trace theorem for divergence free functions}) \\ &\leq C\|\varepsilon\Delta u^\varepsilon\|_{L^2(0,T;(L^2(\Omega))^2)} \leq C\varepsilon^{1/4} \quad (\text{by (2.23)}). \end{aligned} \tag{2.24}$$

Finally we multiply (2.22) by  $w^\varepsilon$  and integrate over  $\Omega$ :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |w^\varepsilon|^2 + \varepsilon |\nabla w^\varepsilon|^2 &\leq \varepsilon |g^\varepsilon| |w^\varepsilon| + |\nabla(p^\varepsilon - p^0)| |(I - P)w^\varepsilon| \\ &\leq (\text{thanks to Lemma 2.2 and (2.24)}) \leq \frac{1}{2} |w^\varepsilon|^2 + C\varepsilon^{1/4+3/4}. \end{aligned}$$

This implies

$$|w^\varepsilon|_{L^\infty(0,T;(L^2(\Omega))^2)} \leq C\varepsilon^{1/2}, \quad |w^\varepsilon|_{L^2(0,T;(H^1(\Omega))^2)} \leq C. \tag{2.25}$$

It remains to show the weak convergence of  $w^\varepsilon$  and this follows from the fact that each subsequence of  $w^\varepsilon$  has a weakly convergent subsequence in  $L^2(0, T; (H^1(\Omega))^2)$  and they all strongly converges to zero in  $L^2(0, T; (L^2(\Omega))^2)$ . Thus the whole sequence weakly converges to zero. This completes the proof of the main result.

**Remark 2.5.** We can slightly improve the result (2.25). Indeed we can prove that

$$|w^\varepsilon|_{L^\infty(0,T;(H^1(\Omega))^2)} \leq \text{constant}$$

with the extra cost of more regularity assumptions on  $u^0$  and  $f$  of course.

To see this, we notice that

$$\begin{aligned} \left(\frac{\partial w^\varepsilon}{\partial t}, \varepsilon P \Delta w^\varepsilon\right) &= \frac{1}{2} \frac{d}{dt} |\nabla w^\varepsilon|^2 + (\varepsilon \Delta w^\varepsilon, (I - P) \frac{\partial}{\partial t} w^\varepsilon) \\ &\geq (\text{thanks to Lemma 2.1 and (2.20)}) \geq \frac{1}{2} \frac{d}{dt} |\nabla w^\varepsilon|^2 - C\varepsilon. \end{aligned}$$

Thus instead of (2.25) we obtain

$$\varepsilon \frac{d}{dt} |\nabla w^\varepsilon|^2 + |\varepsilon P \Delta w^\varepsilon|^2 \leq \kappa \varepsilon,$$

which implies our remark.

**3. Large-time behavior.** In this section we want to describe the behavior of the solution of the channel-flow equation (2.1) for time  $t$  large and  $\varepsilon$  small. In fact our framework will be an abstract one so that the results in this section apply as well to the heat equation in dimension one or more as in Section 1.

**3.1. Position of the problem.** The problem that we address here is the following.

The boundary layer solutions that we have produced in Sections 1 and 2 are of the form

$$\varphi_\varepsilon(y, t) = 1 - 2 \operatorname{erf}\left(\frac{y}{\sqrt{2\varepsilon t}}\right) = 1 - \frac{2}{\sqrt{2\pi}} \int_0^{\frac{y}{\sqrt{2\varepsilon t}}} e^{-\frac{x^2}{2}} dx, \tag{3.1}$$

and a similar function with  $y$  replaced by  $1 - y$ . For fixed  $y$  and  $t$ , the expression  $\varphi_\varepsilon$  in (3.1) is equal to 1 at  $y = 0$ , is positive at  $y > 0$  and equal to 0 at  $y = +\infty$ . Thus  $\varphi_\varepsilon(y, t) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , for all  $t > 0$  fixed, for all  $y > 0$ , with  $\varphi_\varepsilon(0, t) = 1$ , which corresponds to the expected boundary-layer structure, and  $\varphi$  is not too small as long as  $0 < y < \mathcal{O}(\sqrt{\varepsilon})$ ,  $\sqrt{\varepsilon}$  being called the thickness of the boundary layer.

Now if  $t$  varies as well, the thickness of the boundary layer at  $\varepsilon, t$  fixed is  $\mathcal{O}(\sqrt{\varepsilon t})$ , so that for  $t$  large,  $t \gg 1/\varepsilon$ , the boundary layer region corresponding to (3.1) pervades the whole region  $0 < y < 1$  and it is not, anymore, located near  $y = 0$ . Therefore the asymptotic expansions that we have produced are valid only on a fixed interval of time,  $0 < t < T$  as stated in Theorems. They are not valid anymore for large  $t$ , in particular  $t \gg 1/\varepsilon$ . We are not able to describe what happens for  $t = \mathcal{O}(1/\varepsilon)$ , but we will derive some useful information valid for  $t \gg 1/\varepsilon$ .

Another preliminary observation is useful here. All equations that we have considered can be written as

$$\frac{du_\varepsilon}{dt} + \varepsilon Au_\varepsilon = f, \tag{3.2}$$

$$u_\varepsilon(0) = u_0, \tag{3.3}$$

where  $A = -\Delta$  with the appropriate boundary conditions in Section 1 (Dirichlet boundary conditions at  $y = 0, 1$ , space periodicity in  $x$ ); in Section 2,  $A$  is the Stokes operator corresponding to the same boundary conditions.

In the limit  $\varepsilon \rightarrow 0$ ,  $u_\varepsilon$  converges in  $L^2(0, T; H)$ , to  $u^0$ , where

$$\frac{du^0}{dt} = f, \tag{3.4}$$

$u^0(0) = u_0$ , so that

$$u^0(t) = u_0 + \int_0^t f(s) ds = u_0 + tf.$$

In this section we assume, for simplicity that  $f$  is independent of time,  $f(t) = f \in H$ , for all  $t > 0$ .

Now we observe the following: for fixed  $\varepsilon$ , the solution  $u_\varepsilon(t)$  of (3.1), (3.2) converges, as  $t \rightarrow \infty$ ; i.e.,  $\varphi_\varepsilon = \frac{1}{\varepsilon} A^{-1} f$ . Similarly, the solution of (3.3), (3.4) becomes infinite as  $t \rightarrow \infty$ :

$$t|f|_H - |u_0|_H \leq |u^0(t)|_H \leq t|f|_H + |u_0|_H; \quad \text{i.e., } |u^0(t)|_H \sim t|f|_H, \quad \text{as } t \rightarrow \infty.$$

Hence for  $t \gg 1/\varepsilon$ ,  $\varepsilon$  small and  $t$  large, we expect the solution  $u_\varepsilon$  of (3.1), (3.2) to be large in norm, and we are going to give some indications on this behavior.

**3.2. The main estimate.** We assume that  $u_0 \in H$  (and  $f \in H$  as usual). Letting  $v_\varepsilon = u_\varepsilon - \frac{1}{\varepsilon}\varphi$ , where  $\varphi = A^{-1}f$ , we have

$$\frac{dv_\varepsilon}{dt} + \varepsilon Av_\varepsilon = 0, \quad v_\varepsilon(0) = u_0 - \frac{1}{\varepsilon}\varphi. \tag{3.5}$$

Consider the eigenvector and eigenvalues of  $A$ ,

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A), \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \rightarrow \infty \text{ as } j \rightarrow \infty,$$

where  $D(A)$  is the domain of  $A$  in  $H$  ( $D(A) = A^{-1}H$ ). They constitute an orthonormal basis of  $H$ , an orthogonal basis of  $V$  and  $D(A)$ . Any  $v \in H$  can be expanded as

$$v = \sum_{k=1}^{\infty} v_k w_k \quad \text{and}$$

$$|v|_H = \left(\sum_{k=1}^{\infty} v_k^2\right)^{\frac{1}{2}}, \quad \|v\|_V = \left(\sum_{k=1}^{\infty} \lambda_k v_k^2\right)^{\frac{1}{2}}, \quad \|v\|_{D(A)} = |Av|_H = \left(\sum_{k=1}^{\infty} \lambda_k^2 v_k^2\right)^{\frac{1}{2}}.$$

The solution of (3.5) can be written

$$v_\varepsilon = u_\varepsilon - \frac{1}{\varepsilon}\varphi = e^{-\varepsilon At}u_0 - \frac{1}{\varepsilon}e^{-\varepsilon At}\varphi. \tag{3.6}$$

Here  $e^{-sA}u_0$  and  $e^{-sA}\varphi$  stand for

$$\sum_{k=1}^{\infty} e^{-s\lambda_k} u_{0k} w_k, \quad \forall u_0 = \sum_{k=1}^{\infty} u_{0k} w_k, \quad \text{and} \quad \sum_{k=1}^{\infty} e^{-s\lambda_k} \varphi_k w_k, \quad \forall \varphi = \sum_{k=1}^{\infty} \varphi_k w_k.$$

Now we want to estimate the terms in the right-hand side of (3.6) and show under what conditions they are small, so that we conclude that  $u_\varepsilon = \frac{1}{\varepsilon}\varphi + \text{s.o.t.}$ <sup>1</sup> We have

$$\begin{aligned} \left\| \frac{1}{\varepsilon} e^{-\varepsilon At} \varphi \right\|_V^2 &= \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \lambda_k e^{-2\varepsilon t \lambda_k} \varphi_k^2 \leq \frac{1}{\varepsilon^2} e^{-2\varepsilon t \lambda_1} \sum_{k=1}^{\infty} \lambda_k \varphi_k^2 \\ &= \frac{1}{\varepsilon^2} e^{-2\varepsilon t \lambda_1} \|\varphi\|_V^2 = \frac{1}{\varepsilon^2} e^{-2\varepsilon t \lambda_1} |A^{-1}f|_H^2 \leq \frac{c}{\varepsilon^2} e^{-2\varepsilon t \lambda_1} |f|_H^2. \end{aligned} \tag{3.7}$$

The estimate (3.7) is sharp in some sense since, for e.g.  $f = f_{k_0} w_{k_0}$ , we have

$$\left\| \frac{1}{\varepsilon} e^{-\varepsilon At} \varphi \right\|_V = \left\| \frac{1}{\varepsilon} e^{-\varepsilon At} A^{-1} f \right\|_V = \frac{1}{\varepsilon \lambda_{k_0}} e^{-\varepsilon t \lambda_{k_0}} |f|_H$$

and for  $f = f_1 w_1$ ,

$$\left\| \frac{1}{\varepsilon} e^{-\varepsilon At} \varphi \right\|_V = \frac{1}{\varepsilon \lambda_1} e^{-\varepsilon t \lambda_1} |f|_H.$$

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<sup>1</sup>Smaller order terms

Similarly, we write

$$\|e^{-\varepsilon At}u_0\|_V^2 = \sum_{k=1}^{\infty} \lambda_k e^{-2\varepsilon t \lambda_k} u_{0k}^2.$$

Setting  $g(s) = se^{-s}$ , we observe that  $g(s) \leq g(1) = e^{-1}$ , for all  $s > 0$ , and we write

$$\lambda_k e^{-2\varepsilon t \lambda_k} = \frac{1}{\varepsilon t} e^{-\varepsilon t \lambda_k} g(\varepsilon t \lambda_k) \leq \frac{1}{\varepsilon t} e^{-\varepsilon t \lambda_1}.$$

Therefore

$$\|e^{-\varepsilon At}u_0\| \leq \frac{c}{\sqrt{\varepsilon t}} e^{-\varepsilon t \lambda_1/2} |u_0|_H. \tag{3.8}$$

If  $u_0 \in V$  instead of  $u_0 \in H$ , we write more simply  $\|e^{-\varepsilon At}u_0\| \leq e^{-\varepsilon t \lambda_1} \|u_0\|_V$ . For fixed small  $\delta$ , because of (3.6), (3.7) and (3.8), we can have

$$\|v_\varepsilon(t)\|_v = \|u_\varepsilon(t) - \frac{1}{\varepsilon} A^{-1}f\|_V \leq \delta,$$

provided

$$\frac{c}{\sqrt{\varepsilon t}} e^{-\varepsilon t \lambda_1/2} |u_0|_H \leq \frac{\delta}{2}, \tag{3.9}$$

and

$$\frac{c}{\varepsilon} e^{-\varepsilon t \lambda_1} |f|_H \leq \frac{\delta}{2}. \tag{3.10}$$

These inequalities can not be achieved if  $\varepsilon t$  is too small; so let us consider the case where  $\varepsilon t \geq 1$ ; i.e.,  $t > 1/\varepsilon$ .

Observing that  $(1/\sqrt{\varepsilon t})e^{\varepsilon \lambda_1 t/4}$  is bounded for  $t \geq 1/\varepsilon$ , it suffices, for (3.9), to have  $t \geq \frac{c}{\varepsilon} \log(\frac{|u_0|_H}{c\delta})$ ; similarly, (3.10) holds if  $t \geq \frac{c}{\varepsilon} \log(\frac{|f|_H}{\varepsilon\delta})$ .

Finally, we have the following result:

**Theorem 3.1.** *For  $\varepsilon \rightarrow 0$  and  $t \rightarrow \infty$ ,  $t \geq 1/\varepsilon$ , the solution  $u_\varepsilon$  of (3.1), (3.2) is of the form*

$$u_\varepsilon = \frac{1}{\varepsilon} A^{-1}f + I_1 + I_2,$$

with

$$\|I_1\| \leq \frac{c}{\sqrt{\varepsilon t}} e^{-\varepsilon t \lambda_1/2} |u_0|_H, \quad \|I_2\| \leq \frac{c}{\varepsilon} e^{-\varepsilon t \lambda_1} |f|_H,$$

for  $\delta > 0$  fixed,  $\|I_1\|_V + \|I_2\|_V \leq \delta$ , when  $t \geq \frac{c'}{\varepsilon} (\log(\frac{1}{\varepsilon}) + \log(\frac{1}{\delta}) + c)$ , for some suitable constants  $c, c'$ .

**Appendix.** In this appendix we give the proofs for the three technical results used in the main body of the article, namely Lemma 1.1, Proposition 1.1 and Lemma 2.1.

**Proof of Lemma 1.1.** It is easy to see that  $\tilde{u}$  given by (1.2) is well-defined and smooth  $C^\infty$  for  $t > 0$ .



Indeed, according to (1.2) and the Cauchy-Schwarz inequality

$$\begin{aligned} |\tilde{u}(t, y)| &\leq \sum_{k=-\infty}^{\infty} \left( \int_k^{k+1} \frac{1}{4\pi t} e^{-\frac{x^2}{4t}} \right)^{1/2} \|g\|_{L^2(0,1)} \sqrt{2} \\ &\leq 2\sqrt{2} \sum_{k \geq 0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{k^2}{4t}} \|g\|_{L^2(0,1)} \leq 2\sqrt{2} \|g\|_{L^2(0,1)} / (\sqrt{4\pi t} (1 - e^{-1/4t})). \end{aligned}$$

Taking formal derivatives in  $t$  and  $y$  of  $\tilde{u}$  in (1.2), applying the same kind of estimates and the Lebesgue dominated convergence theorem, we obtain that  $\tilde{u} \in C^\infty((0, \infty) \times \mathbb{R}^1)$  and that it satisfies the heat equation

$$\frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial y^2} = 0, \quad \text{for } t > 0.$$

It remains to prove that  $\tilde{u}$  satisfies the initial and boundary conditions.

Let us investigate the initial data first by considering the quantity  $\|\tilde{u}(t, y) - g(y)\|_{L^2(0,1)}$ . We first prove that

$$\|\tilde{u}(t, \cdot)\|_{L^2(0,1)} \leq 2\|g\|_{L^2(0,1)}. \tag{A.1}$$

To see this, let us notice that

$$\tilde{u}(t, y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} (\mathcal{T}(g))(y - x) dx$$

and thus for any  $h \in L^2_{\text{loc}}(\mathbb{R}^1)$

$$\begin{aligned} \int_0^1 \tilde{u}(t, y) h(y) dy &= \int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} (\mathcal{T}(g))(y - x) h(y) dx dy \\ (\text{thanks to Fubini's theorem}) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \left( \int_0^1 (\mathcal{T}(g))(y - x) h(y) dy \right) dx \\ &\leq \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \|\mathcal{T}(g)\|_{L^2(0,1)} \|h\|_{L^2(0,1)} dx \leq 2\|g\|_{L^2(0,1)} \|h\|_{L^2(0,1)}. \end{aligned}$$

This implies (A.1).

Now we prove (1.3) via the following argument. Consider  $h \in L^2_{\text{loc}}(\mathbb{R}^1)$  and write:

$$\begin{aligned} \int_0^1 (\tilde{u}(t, y) - g(y)) h(y) dy &= \int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} ((\mathcal{T}(g))(y - x) - \mathcal{T}(g)(y)) h(y) dx dy \tag{A.2} \\ &= (\text{with Fubini's theorem}) = \int_{-\infty}^{\infty} \int_0^1 \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} ((\mathcal{T}(g)(y - x)) - \mathcal{T}(g)(y)) h(y) dy dx \\ &\leq \int_{|x| > \delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} 3\|g\|_{L^2(0,1)} \|h\|_{L^2(0,1)} + \int_{|x| \leq \delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \int_0^1 (\mathcal{T}(g)(y - x) - \mathcal{T}(g)(y)) h(y) dy dx \\ &\leq 3\|g\|_{L^2(0,1)} \|h\|_{L^2(0,1)} \int_{|x| > \delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx + \|h\|_{L^2(0,1)} \sup_{|x| \leq \delta} \|\mathcal{T}(g)(y - x) - \mathcal{T}(g)(y)\|_{L^2(0,1)}. \end{aligned}$$

Therefore we obtain a bound on  $\|\tilde{u}(t, \cdot) - g(\cdot)\|_{L^2(0,1)}$  which converges to 0 as  $t \rightarrow 0$ . To check that the boundary conditions are satisfied, we observe that

$$\tilde{u}(t, 0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} (\mathcal{T}(g))(-x) dx = 0, \quad \text{since } \mathcal{T}(g) \text{ is odd.}$$

Similarly  $\tilde{u}(t, 1) = 0$  and this completes the proof of this lemma.

**Remark A.1.** We can refine the proof of (A.2) and obtain the convergence rate of  $\tilde{u}$  to  $g$  as  $t \rightarrow 0$ , which is of the order  $t^{1/4}$ . But this depends on the explicit expression of the solution given by (1.2). Thus we shall prefer a more general proof which will be given hereafter.

Now we prove Proposition 1.1 which gives the convergence rate for a singularity-perturbed heat equation. In fact we prove it in the slightly more general form hereafter.

**Proposition A.1.** For  $v_0, g$  smooth enough and  $v^\varepsilon$  being the solution of

$$\frac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon = g, \quad (x, y) \in \Omega_\infty = \mathbb{R}^1 \times (0, 1), \quad v^\varepsilon|_{t=0} = v_0, \quad v^\varepsilon|_{y=0,1} = 0,$$

there exists a constant  $\kappa = \kappa(v_0, g, T)$ , such that,

$$\|v^\varepsilon - (v_0 + \int_0^t g(s) ds)\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon^{1/4}, \quad \|v^\varepsilon - (v_0 + \int_0^t g(s) ds)\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa \varepsilon^{3/4}.$$

**Proof.** We observe that  $v^\varepsilon = v_1^\varepsilon + v_2^\varepsilon$ , where  $v_1^\varepsilon$  and  $v_2^\varepsilon$  solve the equations

$$\frac{\partial v_1^\varepsilon}{\partial t} - \varepsilon \Delta v_1^\varepsilon = 0, \quad v_1^\varepsilon|_{t=0} = v_0, \quad v_1^\varepsilon|_{y=0,1} = 0,$$

and

$$\frac{\partial v_2^\varepsilon}{\partial t} - \varepsilon \Delta v_2^\varepsilon = g, \quad v_2^\varepsilon|_{t=0} = 0, \quad v_2^\varepsilon|_{y=0,1} = 0,$$

respectively. Now we use introduce a cut-off function of the form

$$\begin{cases} \rho \in C^\infty[0, \infty), \quad \rho(0) = 1, \quad \text{supp } \rho \subset [0, 1/2], \\ \rho^\eta(y) = \rho(y/\eta^{1/2}), \quad y \geq 0, \\ \tilde{\rho}^\eta(y) = \rho^\eta(y) + \rho^\eta(1 - y), \quad 0 \leq y \leq 1. \end{cases}$$

It is easy to verify that, for  $\eta < 1$ , the following hold:

$$\begin{cases} |\tilde{\rho}^\eta|_{L^p(\Omega)} \leq 2\eta^{1/2p} |\rho|_{L^p(\Omega)}, \quad p \geq 1, \\ \left| \frac{\partial}{\partial y} \tilde{\rho}^\eta \right|_{L^p(\Omega)} \leq 2\eta^{\frac{1}{2p} - \frac{1}{2}} \left| \frac{\partial}{\partial y} \rho \right|_{L^p(\Omega)}, \\ |\tilde{\rho}^\eta|_{H^{-1}(0,1)} \leq 2\eta^{3/4} |\rho|_{L^2(\Omega)}. \end{cases}$$

Now we use the cut-off function  $\tilde{\rho}^\varepsilon$  to further decompose  $v_1^\varepsilon$ , using the linearity of the equation.

Let  $v_1^\varepsilon = v_{11}^\varepsilon + v_{12}^\varepsilon$  where  $v_{11}^\varepsilon$  and  $v_{12}^\varepsilon$  satisfy

$$\frac{\partial v_{11}^\varepsilon}{\partial t} - \varepsilon \Delta v_{11}^\varepsilon = 0, \quad v_{11}^\varepsilon|_{t=0} = \tilde{\rho}^\varepsilon \cdot v_0, \quad v_{11}^\varepsilon|_{y=0,1} = 0, \tag{A.3}$$

and

$$\frac{\partial v_{12}^\varepsilon}{\partial t} - \varepsilon \Delta v_{12}^\varepsilon = 0, \quad v_{12}^\varepsilon|_{t=0} = (1 - \tilde{\rho}^\varepsilon) \cdot v_0, \quad v_{12}^\varepsilon|_{y=0,1} = 0.$$

We multiply (A.3) by  $v_{11}^\varepsilon$  and integrate over  $\Omega$ ; we find

$$\frac{1}{2} \frac{d}{dt} |v_{11}^\varepsilon|_{L^2(\Omega)}^2 + \varepsilon |\nabla v_{11}^\varepsilon|_{L^2(\Omega)}^2 = 0$$

which implies

$$\|v_{11}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq |\tilde{\rho}^\varepsilon v_0|_{L^2} \leq \kappa |\tilde{\rho}^\varepsilon|_{L^2} \leq \kappa \varepsilon^{1/4}. \tag{A.4}$$

We then multiply (A.3) by  $A^{-1}v_{11}^\varepsilon$ , where  $-A$  is here the Laplacian operator with homogeneous boundary condition in the  $y$  direction and periodic boundary condition in the  $x$  direction; this yields

$$\frac{1}{2} \frac{d}{dt} |v_{11}^\varepsilon|_{H^{-1}}^2 + \varepsilon |v_{11}^\varepsilon|_{L^2}^2 = 0,$$

which implies

$$\|v_{11}^\varepsilon\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq |\tilde{\rho}^\varepsilon v_0|_{H^{-1}} \leq \kappa |\tilde{\rho}^\varepsilon|_{H^{-1}} \leq \kappa \varepsilon^{3/4}. \tag{A.5}$$

To estimate  $v_{12}^\varepsilon$ , notice that  $(1 - \tilde{\rho}^\varepsilon)v_0|_{y=0,1} = 0$ ; thus we can subtract  $(1 - \tilde{\rho}^\varepsilon)v_0$  from  $v_{12}^\varepsilon$  to obtain the following equation:

$$\begin{cases} \frac{\partial}{\partial t}(v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0) - \varepsilon\Delta(v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0) = \varepsilon\Delta((1 - \tilde{\rho}^\varepsilon)v_0) \\ v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0|_{t=0} = 0, \quad v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0|_{y=0,1} = 0. \end{cases} \tag{A.6}$$

We multiply (A.6) by  $v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0$  and integrate over  $\Omega$ :

$$\frac{1}{2} \frac{d}{dt} |v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0|_{L^2}^2 + \varepsilon |\nabla(v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0)|_{L^2(\Omega)}^2 \leq \frac{\varepsilon}{2} |\nabla(v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0)|_{L^2(\Omega)}^2 + \kappa\varepsilon^{1/2}.$$

This implies that

$$\|v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}. \tag{A.7}$$

Similarly we multiply (A.6) by  $A^{-1}(v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0)$  and obtain

$$\|v_{12}^\varepsilon - (1 - \tilde{\rho}^\varepsilon)v_0\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}. \tag{A.8}$$

Combining (A.8), (A.7), (A.4), (A.5) we conclude that

$$\|v_1^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}, \quad \|v_1^\varepsilon\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}. \tag{A.9}$$

By the same kind of techniques applied to  $v_2^\varepsilon$  we obtain

$$\|v_2^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa\varepsilon^{1/4}, \quad \|v_2^\varepsilon\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \kappa\varepsilon^{3/4}, \tag{A.10}$$

and the proposition follows from (A.9) and (A.10).

**Remark A.2.** This proposition can be generalized to smooth bounded domains in  $\mathbb{R}^n$  giving us a convergence rate for the singularly perturbed heat equation in general domains.

Now we present the proof of Lemma 2.1 which gives an explicit formula for constructing divergence-free functions with prescribed tangential velocity at the boundary and of boundary-layer type.

**Proof of Lemma 2.1.** Let  $\rho \in C^\infty[0, \infty)$  be an even cut-off function satisfying

$$\rho(0) = 1, \quad \text{supp } \rho \subset [-1, 1], \quad \int_0^1 \rho dx = 0, \quad |\rho(x)| \leq 1, \quad \forall x.$$

Since  $\Omega$  is smooth and bounded, there exists  $\delta_0 > 0$ , such that all unit normals from  $\partial\Omega$  do not intersect in the  $2\delta_0$  neighborhood of  $\partial\Omega$  (denoted  $O_{2\delta_0}(\partial\Omega)$ ). Let  $d(x)$  be the unique point on  $\partial\Omega$  such that the line segment joining  $x$  to  $d(x)$  is parallel to the normal at  $d(x)$  for  $x \in O_{2\delta_0}(\partial\Omega)$ . Let  $T_x$  denote the clockwise unit tangent vector for  $\partial\Omega$  at the point  $x$ . Since  $\Omega$  is smooth, we see that  $d(x)$ ,  $T_{d(x)}$ ,  $\text{dist}(x, d(x)) = \text{dist}(x, \partial\Omega)$  are all smooth functions.

Now for  $\varepsilon > 0$ , we define

$$\psi^\varepsilon(x) = \begin{cases} \varphi(d(x))T_{d(x)} \int_0^{\text{dist}(x,d(x))} \rho\left(\frac{s}{\varepsilon^{1/2}}\right) ds, & \text{if } x \in O_{\delta_0}(\partial\Omega), \\ 0 & \text{else.} \end{cases}$$

It is easy to verify that our function is smooth for  $\varepsilon \leq \delta_0$  by our choice of  $\delta_0, \rho$ . Next we set

$$\varphi^\varepsilon(x) = \text{curl } \psi^\varepsilon = \begin{pmatrix} \frac{\partial \psi^\varepsilon}{\partial x_2} \\ -\frac{\partial \psi^\varepsilon}{\partial x_1} \end{pmatrix}. \tag{A.11}$$

Obviously  $\operatorname{div} \varphi^\varepsilon = 0$ ,  $\varphi^\varepsilon$  is smooth and the mapping  $\varphi \rightarrow \varphi^\varepsilon$ . Thus we need only to show that the inequalities (2.8) are true and that  $\varphi^\varepsilon$  matches  $\varphi$  on the boundary.

Let  $x_0 = (x_{01}, x_{02})$  be a given point on  $\partial\Omega$ . Since our construction is translation and rotation invariant, we can assume that  $T_x$  is the positive  $x$  direction, without loss of generality. Thanks to the implicit function theorem, there exists a smooth function defined in a neighborhood of  $x_0$  such that

$$\begin{cases} (x_1, f(x_1)) \in \partial\Omega & \text{for } x_1 \text{ near } x_{01}, \\ f'(x_{01}) = 0, & x_{02} = f(x_{01}). \end{cases}$$

It is easy to check that

$$\frac{\partial}{\partial x_1} \operatorname{dist}(x, \partial\Omega)|_{x_0} = 0, \quad \frac{\partial}{\partial x_2} \operatorname{dist}(x, \partial\Omega)|_{x_0} = 1.$$

Thus

$$\begin{aligned} \frac{\partial \psi^\varepsilon}{\partial x_1} |_{x_0} &= \frac{\partial}{\partial x_1} (\varphi(d(x)) \cdot T_{d(x)}) |_{x_0} \int_0^{\operatorname{dist}(x_0, d(x_0))} \rho(s/\varepsilon^{1/2}) ds \\ &\quad + \varphi(d(x_0)) \cdot T_{d(x_0)} \rho \left( \frac{\operatorname{dist}(x_0, d(x_0))}{\varepsilon^{1/2}} \right) \frac{\partial}{\partial x_1} \operatorname{dist}(x, \partial\Omega)|_{x_0} = 0 \\ \frac{\partial \psi^\varepsilon}{\partial x_2} |_{x_0} &= \frac{\partial}{\partial x_2} (\varphi(d(x)) \cdot T_{d(x)}) |_{x_0} \int_0^{\operatorname{dist}(x_0, d(x_0))} \rho(s/\varepsilon^{1/2}) ds \\ &\quad + \varphi(d(x_0)) T_{d(x_0)} \rho \left( \frac{\operatorname{dist}(x_0, d(x_0))}{\varepsilon^{1/2}} \right) \frac{\partial}{\partial x_2} \operatorname{dist}(x, \partial\Omega)|_{x_0} = \varphi(x_0) \cdot T_{d(x_0)} \rho(0) = \varphi_1(x_0); \end{aligned} \tag{A.12}$$

i.e.,  $\varphi^\varepsilon$  matches  $\varphi$  at  $x_0$  and hence  $\varphi^\varepsilon \in H$ .

To show the bounds on  $\varphi^\varepsilon$  in the  $H$  and  $H^1$  norms we observe that for  $\varepsilon^{1/2} \leq \delta_0$  and  $x \in \Omega \setminus \mathcal{O}_{\delta_0}(\partial\Omega)$

$$\int_0^{\operatorname{dist}(x, d(x))} \rho(s/\varepsilon^{1/2}) ds = \frac{\varepsilon}{2} \int_0^\infty \rho(s) ds = 0 \quad \text{and thus } \psi^\varepsilon(x) = 0, \text{ which implies } \varphi^\varepsilon(x) = 0.$$

In fact (A.12) is true for  $x \in \mathcal{O}_{\varepsilon^{1/2}}(\partial\Omega)$ . Now by (A.11) we have

$$\begin{aligned} |\varphi^\varepsilon(x)|_{L^\infty(\Omega)} &\leq |\nabla \psi^\varepsilon|_{L^\infty(\Omega)} \\ &\leq |\nabla(\varphi \cdot T)|_{L^\infty(\partial\Omega)} \int_0^1 |\rho| ds \cdot \varepsilon^{1/2} + |\varphi \cdot T|_{L^\infty(\partial\Omega)} |\nabla \operatorname{dist}(x, d(x))|_{L^\infty(\mathcal{O}_{\delta_0}(\partial\Omega))} |\rho|_{L^\infty} \\ &\leq \kappa (|\nabla \varphi|_{L^\infty(\partial\Omega)} + |\varphi|_{L^\infty(\partial\Omega)}) = \kappa |\varphi|_{W^{1,\infty}(\partial\Omega)} \end{aligned}$$

and hence

$$|\varphi^\varepsilon|_{L^2(\Omega)} = \left( \int_{\mathcal{O}_{\varepsilon^{1/2}}(\partial\Omega)} |\varphi^\varepsilon|^2 \right)^{1/2} dx \leq (\operatorname{measure}(\mathcal{O}_{\varepsilon^{1/2}}(\partial\Omega)))^{1/2} |\varphi^\varepsilon|_{L^\infty(\Omega)} \leq \kappa \varepsilon^{1/4} |\varphi|_{W^{1,\infty}(\partial\Omega)}.$$

Similarly

$$|\nabla \varphi^\varepsilon(x)|_{L^\infty(\Omega)} \leq |\nabla(\nabla \psi^\varepsilon)|_{L^\infty(\Omega)} \leq \kappa \varepsilon^{1/2} |\varphi|_{W^{2,\infty}(\partial\Omega)}.$$

Thus

$$|\varphi^\varepsilon|_{H^1(\Omega)} \leq (\operatorname{measure}(\mathcal{O}_{\varepsilon^{1/2}}(\partial\Omega)))^{1/2} (|\nabla \varphi^\varepsilon|_{L^\infty} + |\varphi^\varepsilon|_{L^\infty}) \leq \kappa \varepsilon^{-1/4} |\varphi|_{W^{2,\infty}(\partial\Omega)}$$

$$|\varphi^\varepsilon|_{H^{-1}(\Omega)} \leq \kappa |\psi^\varepsilon|_{L^2(\Omega)} \leq \kappa \varepsilon^{3/4} |\varphi|_{L^\infty(\partial\Omega)}.$$

This completes the proof of Lemma 2.1.

**Remark A.3.** 1. The lemma is stated and proved for general 2D bounded smooth domain. But the result is easily adjusted to the 2D channel with one direction periodic one direction Dirichlet as shall be used in this paper.

2. The lemma is basically an improvement of E. Hopf's construction of a divergence-free function with prescribed boundary velocity in the special case of tangential flow on the boundary (see [10]). We think this lemma could be applied in more contexts (see for example [7]).

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