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R. Temam

X. Wang

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Asymptotic Analysis of Oseen Equations for Small Viscosity

R. TEMAM
The Institute for Scientific Computing & Applied Mathematics, Indiana University
618 East Third Street, Bloomington, IN 47405, U.S.A.
and
Laboratoire d'Analyse Numérique, Université Paris-Sud
Bâtiment 425, 91405 Orsay, France

X. WANG
The Institute for Scientific Computing & Applied Mathematics, Indiana University
618 East Third Street, Bloomington, IN 47405, U.S.A.

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Abstract—In this article, we derive explicit asymptotic formulas for the solutions of Oseen's equations in space dimension two in a channel at large Reynolds number (small viscosity ε). These formulas exhibit typical boundary layers behaviors. Suitable correctors are defined to resolve the boundary obstacle and obtain convergence results valid up to the boundary. We study also the behavior of the boundary layer when simultaneously time and the Reynolds number tend to infinity in which case the boundary layer tends to pervade the whole domain.

Keywords—Asymptotic expansions, Boundary layer, Oseen's equations, Navier-Stokes equations, Correctors.

1. INTRODUCTION

As a preliminary step towards the understanding of the asymptotic behavior of the solutions to the Navier-Stokes equations in a bounded region equipped with nonslip boundary condition at small viscosity (large Reynolds number), we study here the asymptotic behavior of the Oseen equations which are derived from the Navier-Stokes equations by linearization around a constant flow $(U_\infty, 0)$. Considering these equations in space dimension two, in a channel we have

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + U_\infty D_1 u^\varepsilon + \nabla p^\varepsilon = f, \tag{1.1}$$

$$u^\varepsilon = u_0 \qquad \text{at } t = 0, \tag{1.2}$$

$$u^\varepsilon \in V \qquad \text{for } t > 0, \tag{1.3}$$

where

$$V = \left\{ v \in \left(H^1_{\text{loc}}(\Omega_\infty)\right)^2,\ \text{div } v = 0, v|_{\partial\Omega_\infty} = 0,\ v \text{ periodic in } x \text{ with period } 2\pi \right\}, \tag{1.4}$$

$$\Omega_\infty = \mathbb{R}^1 \times (0,1), \quad \Omega = (0, 2\pi) \times (0,1), \tag{1.5}$$

and D_1 denotes the derivative in the horizontal (x) direction.

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The corresponding inviscid equations (linearized Euler equations) read

$$\frac{\partial u^0}{\partial t} + U_\infty D_1 u^0 + \nabla p^0 = f, \tag{1.6}$$

$$u^0 = u_0 \qquad \text{at } t = 0, \tag{1.7}$$

$$u^0 \in H, \tag{1.8}$$

where

$$H = \text{closure of } V \text{ in } \left(L^2_{\text{loc}}(\Omega_\infty)\right)^2 = \left\{ v = (v_1, v_2) \in \left(L^2_{\text{loc}}(\Omega_\infty)\right)^2, \ \operatorname{div} v = 0, v_2|_{\partial\Omega_\infty} = 0, \ v_1 \text{ periodic in } x \text{ with period } 2\pi \right\}. \tag{1.9}$$

The convergence of u^ε to u^0 in some weak sense (say $L^2(0,T;L^2(\Omega)^2)$) is classical (see, e.g., [1,2]). However the convergence in stronger topology (say $L^2(0,T;H^1(\Omega)^2)$ or $L^\infty((0,T)\times\Omega)^2)$ is not obvious. In fact, this convergence is not true due to the disparity of the boundary conditions between u^ε and u^0, or the so-called boundary layer problem. Usually this difficulty is overcome by introducing a corrector (boundary layer type function) θ^ε and we obtain the convergence of $u^\varepsilon - (u^0 + \theta^\varepsilon)$ to 0 valid up to the boundary, e.g., in the strong topology of $L^2(0,T;H^1(\Omega)^2)$.

There is abundant literature about boundary layers in fluid mechanics (see, e.g., [3–10]). In the spirit of [11], and also [2,12], we derive here the boundary layer for the Oseen equations by constructing a corrector θ^ε (i.e., a function such that $u^\varepsilon - u^0 - \theta^\varepsilon \to 0$ strongly as $\varepsilon \to 0$), first in an abstract form, i.e., as the solution of an evolution equation, and then in an explicit form. The major difficulty here is the presence of pressure which is a global function of the velocity, and this makes localization efforts hard. This is present even in the linear stationary case (see, e.g., [2,13]).

To overcome this difficulty, we depart from several points of view from classical studies in boundary layers in fluid mechanics and in particular we have a functional analysis global treatment of the pressure term; we also consider a corrector which is not divergence free and which acts only on the tangential velocity: although we could produce a divergence free corrector, this appears to be of no avail. Finally we work with L^2 Sobolev type norms instead of uniform (L^∞) norms, which is useful since a corrector as θ^ε is small in the L^2 norm but not in the L^∞ norm. Notice that we consider here the Oseen equations in a nonconventional way. Usually they are introduced for low Reynolds number flows, while here we study them for large Reynolds number; hence they are of interest to us as a simplified model for the Navier-Stokes equations rather than for their usual physical relevance. In the process, we resolve a number of open questions raised in [2,12] and we also improve our previous result [11] when $U_\infty = 0$. It is worthwhile to point out two phenomenons which appear here and were not present in [11] when $U_\infty = 0$ and which give us a taste of the difficulties that might be encountered in the nonlinear case: the first is the appearance of a mixing of the boundary layers in the tangential directions due to the presence of the transport term; the second is some difference in the boundary layers depending on whether the driving force has a vanishing average over a period in the tangential direction.

2. THE MAIN RESULT

We now state our main result and make a few comments.

THEOREM. *Let $u_0, f \in H$ be sufficiently smooth. Then the solution u^ε of (1.1)–(1.3) has the following asymptotic expansion:*

$$u^\varepsilon(t,\cdot) - u^0(t,\cdot) - \theta^\varepsilon(t,\cdot) = \mathcal{O}\left(\varepsilon^{1-s/2}\right) \qquad \text{in } L^2\left(0,T;(H^s(\Omega))^2\right) \tag{2.1}$$

for $s \in [0,1]$. Here

$$\theta^\varepsilon(t;x,y) = u_0\left(x - U_\infty t, 1\right)\left(1 - 2\,\mathrm{erf}\,\frac{1-y}{\sqrt{2\varepsilon t}}\right) + u_0\left(x - U_\infty t, 0\right)\left(1 - 2\,\mathrm{erf}\left(\frac{y}{\sqrt{2\varepsilon t}}\right)\right)$$
$$+ \int_0^t \left(1 - 2\,\mathrm{erf}\left(\frac{1-y}{\sqrt{2\varepsilon(t-s)}}\right)\right)$$
$$\times \left\{\frac{\partial u^0\left(s; x - U_\infty(t-s), 1\right)}{\partial t} + U_\infty D_1 u^0\left(s; x - U_\infty(t-s), 1\right)\right\} ds \tag{2.2}$$
$$+ \int_0^t \left(1 - 2\,\mathrm{erf}\left(\frac{y}{\sqrt{2\varepsilon(t-s)}}\right)\right)$$
$$\times \left\{\frac{\partial u^0\left(s; x - U_\infty(t-s), 0\right)}{\partial t} + U_\infty D_1 u^0\left(s; x - U_\infty(t-s), 0\right)\right\} ds$$

and erf *is the standard error function, defined as*

$$\mathrm{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_0^y e^{-z^2/2}\, dz. \tag{2.3}$$

Moreover, we have the following estimate on the pressure:

$$\left\|p^\varepsilon - p^0\right\|_{L^2(0,T;H^1(\Omega))} \le \kappa \varepsilon^{1/2}, \tag{2.4}$$

where κ is a generic constant depending on the data but independent of ε.

The theorem is proved by choosing θ^ε to be the solution of

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \Delta \theta^\varepsilon + U_\infty D_1 \theta^\varepsilon = 0, \tag{2.5}$$
$$\theta^\varepsilon = 0 \qquad \text{at } t = 0, \tag{2.6}$$
$$\theta^\varepsilon = -u^0 \qquad \text{at } y = 0 \text{ or } 1, \tag{2.7}$$

then proving the convergence of $u^\varepsilon - u^0 - \theta^\varepsilon$ in $L^2(0,T;(H^1(\Omega))^2)$ with explicit rate depending on ε. Then we apply an asymptotic formula for θ^ε. The detailed proofs will appear elsewhere [14].

REMARK 2.1. An explicit boundary layer of thickness $\sqrt{\varepsilon t}$ can be observed from (2.1) which agrees with heuristic physical arguments (see, e.g., [4]). It is also observed that the transport term $U_\infty D_1 u^\varepsilon$ has the effect of mixing the boundary layers.

REMARK 2.2. When f is zero, the vorticity is transported by the inviscid equation. However we observe, by applying curl to (2.2), that vorticities are generated near the boundary and they are transported along the tangential direction of the boundary for the viscous equations.

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