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PHYSICA D

Time averaged energy dissipation rate for shear driven flows in $\mathbb{R}^n$

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Abstract

We drive an upper bound of the time averaged energy dissipation rate for boundary driven flows directly from the Navier–Stokes equations in $\mathbb{R}^n$. The upper bound is independent of the kinematic viscosity in accordance with Kolomogorov's scaling result.

Keywords: Navier–Stokes equations; Energy dissipation rate; Background flow

1. Introduction

One of the key quantities in the theory of incompressible Newtonian flows governed by the Navier–Stokes equations is the energy dissipation rate. Upper bounds at every instant of time yield estimates on the small length scales in the solutions, and the system's global attractor, if it exists, is also controlled by this quantity. Such upper bounds are available in space dimension two, but not in space dimension three due, perhaps, to the vortex stretching mechanism. We may still estimate the time averaged rate of viscous energy dissipation per unit mass, nevertheless. Such a quantity is of great interest in applications. In a nonequilibrium steady state the rate of energy dissipation must be balanced by the rate of work done by external forces to the systems. When these forces can be identified as physical drag forces, knowledge of the energy dissipation rate leads to estimates of the drag forces exerted by the fluid.

It is derived using scaling that at large Reynolds number the energy dissipation rate is independent of the viscosity (see e.g. [9]). But it was Doering and Constantin [5] who first established an upper bound, independent of the kinematic viscosity, for the energy dissipation rate in the case of shear flow in a channel, rigorously from the Navier–Stokes equations. They used the concept of "background flow" previously introduced by Hopf [7] for the study of stationary solutions and by Teman [13] to study the attractors for boundary driven flows. The result was generalized to time dependent shear flow in a channel by Marchiaro [10] and to nonconstant (velocity at the boundary depends on location as well) shear flow in channel and Couette–Taylor flows by Miranville and Wang [11]. See also the work of Busse [2] and Howard [8].

The purpose of this article is to derive upper bounds for the time averaged energy dissipation rate for boundary driven flows rigorously from the Navier–Stokes equations. The bounds that we obtain here are independent of the

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kinetic viscosity in accordance with Kolmogorov's scaling result. Our result is more general than that of Doering and Constantin [5] and Marchiaro [10] since we are able to treat more general geometry. Indeed we are able to treat any smooth bounded domain in $\mathbb{R}^3$ (in fact $\mathbb{R}^n$) and any assigned tangential velocity at the boundary.

We base our analysis in the following incompressible Navier–Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla_p = \nu \Delta u + f, \tag{1.1a}$$

$$\nabla \cdot u = 0, \tag{1.1b}$$

$$u = u_0 \quad \text{at } t = 0, \tag{1.1c}$$

$$u = \varphi \quad \text{on } \partial\Omega, \tag{1.1d}$$

where Ω is a bounded smooth domain $\mathbb{R}^n$ such that the boundary of Ω, $\partial\Omega$, takes the form $\partial\Omega = \bigcup_{j=1}^m \Gamma_j$, where each Γ_j is a connected component of $\partial\Omega$ which is also a smooth $n-1$-dimensional submanifold of $\mathbb{R}^n$ without boundary. We assume that Ω lies to one side of $\partial\Omega$.

The boundary conditions satisfy

$$\varphi \in W^{1,\infty}(\mathbb{R}^+, (W^{2,\infty}(\partial\Omega))^n), \tag{1.2a}$$

$$\varphi \cdot \vec{n} = 0 \quad \text{on } \partial\Omega, \tag{1.2b}$$

where $\vec{n}$ is the unit outer normal to Ω. This includes, but not restricted to, the shear driven channel flow considered by Doering and Constantin and the Couette–Taylor flow.

The time averaged energy dissipation rate (per unit volume) is defined as

$$\epsilon = \frac{\nu}{|\Omega|} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \int_\Omega |\nabla u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{1.3}$$

for any suitable weak solution u of (1) which satisfies

$$u \in C_w(\mathbb{R}^+, H) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{H}^1 \cap H), \tag{1.4a}$$

$$\begin{aligned} &\tfrac{1}{2}|u|^2(t) + \nu \int_0^t |\nabla u|^2(s)\, \mathrm{d}s \\ &\quad \le \tfrac{1}{2}|u_0|^2 - \int_0^t \left(u, \frac{\partial\tilde{\varphi}}{\partial t}\right)(s)\, \mathrm{d}s + (u(t), \tilde{\varphi}(t)) - (u_0, \tilde{\varphi}(0)) \\ &\qquad + \nu \int_0^t (\nabla u, \nabla\tilde{\varphi})(s)\, \mathrm{d}s + \int_0^t b(u, u, \tilde{\varphi})(s)\, \mathrm{d}s + \int_0^t (f, u)(s)\, \mathrm{d}s - \int_0^t (f, \tilde{\varphi})(s)\, \mathrm{d}s \end{aligned} \tag{1.4b}$$

for all $\tilde{\varphi} \in W^{1,\infty}(\mathbb{R}^+, (W^{1,\infty}(\Omega))^n \cap H)$ with $\tilde{\varphi}|_{\partial\Omega} = \varphi$, u solves (1.1) in the sense of distribution and with the given initial boundary conditions

$$u = u_0 \quad \text{at } t = 0, \tag{1.4c}$$

$$u|_{\partial\Omega} = \varphi, \tag{1.4d}$$

where we have used the standard notations (see e.g. [4] or [12])

$$H = \{v \in (L^2(\Omega))^n | \text{div} v = 0, v \cdot \vec{n}|_{\partial\Omega=0}\},$$

$$V = \mathbb{H}_0^1(\Omega) \cap H, \qquad b(u, v, w) = \int_\Omega \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

The existence of such suitable weak solutions under the current assumptions can be derived by modifying a classical result of Leray and Hopf in the case when $\varphi = 0$ by first homogenizing the boundary condition. We leave the details to the interested readers.

For the mathematical background for Navier–Stokes equations, the interested readers are referred to the book of Constantin and Foias [4], Teman [12] or the book of Doering and Gibbon [16] for a more applied approach.

The rest of the article is organized in the following way. In Section 2 we introduce a construction of divergence free functions with prescribed tangential velocity field at the boundary (the construction of "background flow"). Using this new construction we derive upper bounds for the time averaged energy dissipation rate in Section 3.

2. Construction of background flows

It is clear, thanks to (1.4), that one of the key steps in in deriving a good upper bound on ε is to find a suitable $\tilde{\varphi}$ (called a "background flow"). The fifth and sixth terms on the right-hand side of (1.4b) correspond to the diffusive and initial terms of the equations, respectively. Since we have no information on the exact solution we do not except cancellation between these two terms. Thus we shall construct $\tilde{\varphi}$ in such a way that both terms are small. For the fifth term to be small we tend to make the function $\tilde{\varphi}$ flat. For the sixth term to be small we tend to mske the support of $\tilde{\varphi}$ small. The final construction is a balance between these two tendencies and the requirements of the geometry. Indeed by finding an approximate solution to a suitable double obstacle minimization problem we obtain the following construction of background flows with prescribed tangential velocity at the boundary of Ω.

Lemma 1. Let $\varphi \in (W^{2,\infty}(\partial\Omega))^n$ be such that $\varphi \cdot \vec{n} = 0$ on $\partial\Omega$. Then there exist $\alpha < 1$ and κ, depending only on the shape of Ω (i.e. dimensionless constants depending on Ω only). such that for each $\beta < \alpha$, there exist a function $\varphi^\beta \in H$, satisfying

$$\varphi^\beta = \varphi \quad \text{on } \partial\Omega, \tag{2.1}$$

$$\| \varphi^\beta \|_{\mathbb{L}^p(\Omega)} \leq \kappa(p)\{(\beta^{1/p} + \beta^{1+1/p})L^{n/p} \| \varphi \|_{\mathbb{L}^\infty(\partial\Omega)} + \beta^{1+1/p} L^{n/p+1} \| \nabla^T \varphi \|_{\mathbb{L}^\infty(\partial\Omega)}\}, \tag{2.2}$$

$$\begin{aligned} \| \varphi^\beta \|_{W^{1,p}(\Omega)} \leq \kappa(p)\{ & (\beta^{1/p-1} + \beta^{1/p} + \beta^{1/p+1})L^{n/p-1} \| \varphi \|_{\mathbb{L}^\infty(\partial\Omega)} \\ & + (\beta^{1/p} + \beta^{1/p+1})L^{n/p} \| \nabla^T \varphi \|_{\mathbb{L}^\infty \partial\Omega} \\ & + \beta^{1/p+1} L^{n/p+1} \| \nabla^T \nabla^T \varphi \|_{\mathbb{L}^\infty(\partial\Omega)}\}, \end{aligned} \tag{2.3}$$

$$\| \varphi^\beta \|_{\mathbb{H}^{-1}(\Omega)} \leq \kappa \beta^{3/2} L^{n/2+1} \| \varphi \|_{\mathbb{L}^\infty(\partial\Omega)}, \tag{2.4}$$

where

$$L = \text{diameter of } \Omega, \tag{2.5}$$

and

$$\nabla^T = \text{tangential gradient operator on } \partial\Omega,$$

with the metric inherited from $\mathbb{R}^n$, and $p \in [1, \infty]$.

Proof. The construction is a generalization of the construction of the background flow with prescribed tangential velocity in space dimension two by Temam and Wang [14].

For dimensional reason, we construct such a background flow in a domain with unit diameter first, using local orthogonal curvilinear coordinates. The result on general domains is then derived using a dilation.

Let Ω_1 be the domain in $\mathbb{R}^n$ defined by

$$\Omega_1 = \frac{1}{L}\Omega. \tag{2.6}$$

Let Ψ be a given tangential vector field on $\partial\Omega_1$. By the choice of Ω_1, there exists a partition of unity of $\partial\Omega_1$,

$$\partial\Omega_1 = \bigcup_{j=1}^{M} \tilde{\Gamma}_j \tag{2.7}$$

and $\rho_j \in C^\infty(\bar{\Omega}_1)$, such that

$$\sum_{j=1}^{M} \rho_j \equiv 1 \quad \text{on } \partial\Omega_1, \tag{2.8}$$

$$\operatorname{supp} \rho_j \cap \partial\Omega_1 \subset \tilde{\Gamma}_j, \tag{2.9}$$

each $\tilde{\Gamma}_j$ has a neighborhood $\mathcal{O}_j$ in $\mathbb{R}^n$ where there exists an orthogonal curvilinear coordinates $(\xi_1, \dots, \xi_n)$ and unit vectors $\vec{e}_1, \dots, \vec{e}_n$ (we temporary drop the dependence on j here) such that, for some $\alpha > 0$

$$\tilde{\Gamma}_j \subset \{\xi \in \mathcal{O}_j,\ \xi_n = 0\}, \tag{2.10a}$$

$$\{\xi \in \mathcal{O}_j,\ \xi_n > 0\} \subset \Omega_1, \tag{2.10b}$$

$$\{\xi = (\xi', \xi_n),\ \xi' \in \tilde{\Gamma}_j,\ 0 < \xi_n < \alpha\} \subset \mathcal{O}_j \cup \Omega_1, \quad j = 1, \dots, M. \tag{2.10c}$$

It is well known (see e.g. [1]) that the divergence operator takes the form

$$\operatorname{div} v = \sum_{i=1}^{n} \left(\frac{\partial(\sqrt{g} v_i)}{\partial \xi_i} \right), \tag{2.11}$$

where

$$v = \sum_{i=1}^{n} v_i \vec{e}_i \tag{2.12}$$

is a given vector field and

$$g = \det(g_{ij}) \tag{2.13}$$

with

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}\, \mathrm{d}\xi_i\, \mathrm{d}\xi_j, \tag{2.14}$$

being the square of the arc length represented in this orthogonal curvilinear coordinate system.

Note that $\Psi = \sum_{j=1}^{M} \rho_j \Psi$ and thus $\rho_j \Psi$ is supported in $\tilde{\Gamma}_j$. Hence we may write

$$\rho_j \Psi = \sum_{i=1}^{n} (\rho_j \Psi \cdot \vec{e}_i)\vec{e}_i \quad \text{in } \tilde{\Gamma}_j. \tag{2.15}$$

Now we take ρ to be such that

$$\rho \in C^{\infty}([0, \infty)), \tag{2.16a}$$
$$\rho'(0) = 1, \tag{2.16b}$$
$$\rho(0) = 0, \tag{2.16c}$$
$$\operatorname{supp} \rho \subset [0, 1]. \tag{2.16d}$$

For $0 < \beta < \alpha$, we define Ψ_j^β as

$$\begin{aligned}\Psi_j^\beta(\xi) = {} & \sum_{i=1}^{n-1} \frac{\sqrt{g(b(\xi))}}{\sqrt{g(\xi)}} (\rho_j(b(\xi))\Psi(b(\xi)) \cdot \vec{e}_i(b(\xi)))\rho'\left(\frac{\xi_n}{\beta}\right) \vec{e}_i(\xi) \\ & - \frac{1}{\sqrt{g(\xi)}} \sum_{j=1}^{n-1} \left(\frac{\partial(\sqrt{g(b(\xi))}\Psi(b(\xi)) \cdot \vec{e}_i(b(\xi)))}{\partial \xi_i} \right) \beta\rho\left(\frac{\xi_n}{\beta}\right) \vec{e}_n(\xi), \\ & \text{for } \xi \in \mathcal{O}_j, \quad 0 \le \xi_n < \alpha, \quad \xi' \in \tilde{\Gamma}_j, \end{aligned} \tag{2.17a}$$

and

$$\Psi_j^\beta(\xi) = 0 \quad \text{elsewhere}, \tag{2.17b}$$

where b is the projection from $\mathcal{O}_j$ to $\partial\Omega_1$ defined by

$$b(\xi) = \xi' \quad \text{for } \xi = (\xi', \xi_n) \in \mathcal{O}_j. \tag{2.18}$$

Thanks to (2.11), (2.16) and (2.17), and our choice of α we see that

$$\Psi_j^\beta \in H \cap \mathbb{W}^{1,\infty}(\Omega_1), \tag{2.19a}$$
$$\Psi_j^\beta = \rho_j \Psi \quad \text{on } \tilde{\Gamma}_j. \tag{2.19b}$$

Let $\Psi^\beta = \sum_{j=1}^{M} \Psi_j^\beta$. We have

$$\Psi^\beta \in H \cap \mathbb{W}^{1,\infty}(\Omega_1), \tag{2.20a}$$
$$\Psi^\beta = \Psi \quad \text{on } \partial\Omega_1. \tag{2.20b}$$

Moreover, it is easy to check, using the explicit expression (2.17), that the inequalities (2.2)–(2.5) hold with φ^β replaced by Ψ^β and L replaced by 1.

Finally for φ given as in the statement of Lemma 1, we set

$$\Psi(x) = \varphi(Lx) \quad \text{for } x \in \Omega_1, \tag{2.21a}$$
$$\varphi^\beta(x) = \Psi^\beta(x/L) \quad \text{for } x \in \Omega. \tag{2.21b}$$

We may then verify that (2.1)–(2.5) are valid.

This completes the proof of the lemma. □

Remark. The construction presented here is in the same spirit as that of Hopf [7]. However, we would like to point out that Hopf's original construction would not give us the desired result since the bound on the diffusive term would be exponential in $1/\nu$ using Hopf's construction while the bound is polynomial in $1/\nu$ in our case (see Section 3).

Thus the construction above can be considered an improvement of Hopf's original construction in this case. There is also the construction of Doering and Constantin [5] which works very well for the channel case but seems hard to generalize to other geometries due to the global nature of their construction.

3. Upper bound on the energy dissipation rate

In this section we shall derive a rigorous upper bound on the time averaged energy dissipation rate. We utilize the one parameter family of background flows constructed in Section 2 and the energy inequality (1.4b) to obtain an upper bound independent of the viscosity.

Unless otherwise specified, κ will represent a generic dimensionless constant in the sequel.

We estimate each term on the right-hand side of (1.4b):

$$\left|\int_0^t \left(u, \frac{\partial \varphi^\beta}{\partial t}\right)(s)\,\mathrm{d}s\right| \le \int_0^t |\nabla u| \left\|\frac{\partial \varphi^\beta}{\partial t}\right\|_{H^{-1}} \mathrm{d}s$$

$$\le \frac{\nu}{4}\int_0^t |\nabla u|^2\,\mathrm{d}s + \frac{1}{\nu}\int_0^t \left\|\frac{\partial \varphi^\beta}{\partial t}\right\|^2_{H^{-1}} \mathrm{d}s \quad \text{(thanks to (2.4))}$$

$$\le \frac{\nu}{4}\int_0^t |\nabla u|^2\,\mathrm{d}s + \frac{\kappa}{\nu}\beta^3 L^{n+2}\int_0^t \left\|\frac{\partial \varphi}{\partial t}\right\|^2_{\mathbb{L}^\infty(\partial\Omega)} \mathrm{d}s, \tag{3.1}$$

$$|(u(t), \varphi^\beta(t))| \le \tfrac{1}{4}|u|^2(t) + |\varphi^\beta|^2(t) \quad \text{(thanks to (2.2))}$$

$$\le \tfrac{1}{4}|u|^2(t) + \kappa\{(\beta+\beta^3)L^n \,\|\varphi\|^2_{\mathbb{L}^\infty(\partial\Omega)} + \beta^3 L^{n+2}\,\|\nabla^T\varphi\|^2_{\mathbb{L}^\infty(\partial\Omega)}\}, \tag{3.2}$$

$$|(u_0, \varphi^\beta(0))| \le |u_0\,\|\,\varphi^\beta|(0), \tag{3.3}$$

$$\left|\nu\int_0^t (\nabla u, \nabla\varphi^\beta)(s)\,\mathrm{d}s\right|$$

$$\le \frac{\nu}{4}\int_0^t |\nabla u|^2\,\mathrm{d}s + \nu\int_0^t |\nabla\varphi^\beta|^2\,\mathrm{d}s$$

$$\le \frac{\nu}{4}\int_0^t |\nabla u|^2\,\mathrm{d}s + \kappa\nu\int_0^t \{(\beta^{-1}+\beta+\beta^3)L^{n-2}\,\|\varphi\|^2_{\mathbb{L}^\infty(\partial\Omega)}$$

$$+ (\beta+\beta^3)L^n\,\|\nabla^T\varphi\|^2_{\mathbb{L}^\infty(\partial\Omega)} + \beta^3 L^{n+2}\,\|\nabla^T\nabla^T\varphi\|^2_{\mathbb{L}^\infty(\partial\Omega)}\}\,\mathrm{d}s, \tag{3.4}$$

$$\left|\int_0^t b(u, u, \varphi^\beta)\,\mathrm{d}s\right|$$

$$\le \int_0^t\int_\Omega \{|u-\varphi^\beta\,\|\,\nabla u\,\|\,\varphi^\beta| + |\varphi^\beta\,\|\,\nabla u\,\|\,\varphi^\beta|\}\,\mathrm{d}x\,\mathrm{d}s$$

(φ^β is supported in a neighborhood of $\partial\Omega$ with thickness $L\beta$)

$$\leq \int_0^t \left|\frac{u(\,,s)-\varphi^\beta(\,,s)}{\mathrm{dist}(x,\partial\Omega)}\right|_{L^2(\mathcal{O}_{L\beta}(\partial\Omega)\cap\Omega)} |\nabla u||\mathrm{dist}(x,\partial\Omega)\varphi^\beta(\,,s)|_{L^\infty}\,\mathrm{d}s + \int_0^t |\varphi^\beta \parallel \nabla u \parallel \varphi^\beta|_{L^\infty}\,\mathrm{d}s$$

(thanks to Hardy's inequality)

$$\leq \kappa L\beta \int_0^t |\nabla(u-\varphi^\beta)|^2 \parallel \varphi^\beta \parallel_{L^\infty(\Omega)}\,\mathrm{d}s + \int_0^t |\varphi^\beta \parallel \nabla u \parallel \varphi^\beta|_{L^\infty}\,\mathrm{d}s$$

$$\leq \kappa L\beta \int_0^t |\nabla u|^2 \parallel \varphi^\beta \parallel_{L^\infty(\Omega)}\,\mathrm{d}s + \kappa L\beta \int_0^t |\nabla\varphi^\beta|^2|\varphi^\beta|_{L^\infty} + L^{-1}\beta^{-1}\int_0^t |\varphi^\beta|^2|\varphi^\beta|_{L^\infty}\,\mathrm{d}s$$

$$\leq \kappa L\beta \int_0^t |\nabla u|^2\{\parallel \varphi \parallel_{\mathbb{L}^\infty(\partial\Omega)} + \beta L \parallel \nabla^T\varphi \parallel_{\mathbb{L}^\infty(\partial\Omega)}\,\mathrm{d}s$$

$$+\ \kappa L^{n-1}|\varphi|^3_{L^\infty} + \kappa\beta L^n|\varphi|^2_{L^\infty}|\nabla^T\varphi|_{L^\infty} + \kappa\beta^2L^{n+1}|\varphi|_{L^\infty}|\nabla^T\varphi|^2_{L^\infty}$$

$$+\ \kappa\beta^3L^{n+2}|\nabla^T\varphi|^3_{L^\infty} + \kappa\beta^4L^{n+3}|\varphi|_{L^\infty}|\nabla^T\nabla^T\varphi|^2_{L^\infty} + \kappa\beta^5L^{n+4}|\nabla^T\varphi|_{L^\infty}|\nabla^T\nabla^T\varphi|^2_{L^\infty}, \tag{3.5}$$

$$\left|\int_0^t (f,u)(s)\,\mathrm{d}s\right| \leq \frac{\nu}{4}\int_0^t |\nabla u|^2\,\mathrm{d}s + \frac{1}{\nu\lambda_1}\int_0^t |f|^2\,\mathrm{d}s, \tag{3.6}$$

$$\left|\int_0^t (f,\varphi^\beta)\,\mathrm{d}s\right| \leq \int_0^t |f \parallel \varphi^\beta|\,\mathrm{d}s$$

$$\leq \kappa\int_0^t |f|\{\beta^{1/2}L^{n/2} \parallel \varphi \parallel_{L^\infty(\partial\Omega)} + \beta^{3/2}L^{n/2+1} \parallel \nabla^T\varphi \parallel_{\mathbb{L}^\infty(\partial\Omega)}\}\,\mathrm{d}s, \tag{3.7}$$

where λ_1 is the first eigenvalue of the Stokes operator on Ω.

Thus, after collecting like terms, denoting

$$U_1 = \limsup_{t\to\infty} |\varphi(t)|_{L^\infty},$$
$$U_2 = L\limsup_{t\to\infty} |\nabla^T\varphi(t)|_{L^\infty},$$
$$U_3 = L^2\limsup_{t\to\infty} |\nabla^T\nabla^T\varphi(t)|_{L^\infty},$$

dividing by t and letting t approach infinity in (1.4b) we have

$$|\Omega|\left\{1 - \kappa\frac{\beta}{\nu}LU_1 - \kappa\frac{\beta^2L}{\nu}\right\}\epsilon$$

$$\leq \frac{2}{\nu\lambda_1}\limsup_{t\to\infty}|f(t)|^2 + \kappa\frac{\beta^3}{\nu}L^{n+2}\limsup_{t\to\infty}\left\|\frac{\partial\varphi}{\partial t}(t)\right\|^2_{\mathbb{L}^\infty(\partial\Omega)}$$

$$+\ \kappa\nu\beta^{-1}L^{n-2}U_1^2 + \kappa\nu\beta L^{n-2}U_2^2 + \kappa\nu\beta^3L^{n-2}U_3^2$$

$$+\ \kappa L^{n-1}U_1^3 + \kappa\beta L^{n-1}U_1^2U_2 + \kappa\beta^2L^{n-1}U_1U_2^2$$

$$+\ \kappa\beta^3L^{n-1}U_2^3 + \kappa\beta^4L^{n-1}U_1U_3^2 + \kappa\beta^5L^{n-1}U_2U_3^2. \tag{3.8}$$

Denoting

$$Re_1 = LU_1/\nu, \qquad Re_2 = LU_2/\nu, \qquad Re_3 = LU_3/\nu$$

and letting

$$\beta = \min\left\{\frac{1}{3\kappa Re_1}, \frac{1}{\sqrt{3\kappa Re_2}}, \alpha\right\}, \tag{3.9}$$

and noticing the fact that $\beta < \alpha \le 1$, we conclude from (3.8)

$$\begin{aligned}\epsilon \le{}& \frac{\kappa}{\nu\lambda_1 L^n}\limsup_{t\to\infty}|f(t)|^2 + \kappa\frac{\beta^3 L^2}{\nu}\limsup_{t\to\infty}\left\|\frac{\partial\varphi}{\partial t}(t)\right\|^2_{L^\infty}\\ &+\kappa\frac{\nu}{\beta L^2}U_1^2 + \kappa\frac{\nu\beta}{L^2}U_2^2 + \kappa\frac{\nu\beta^3}{L^2}U_3^2\\ &+\kappa\frac{U_1^3}{L} + \kappa\beta\frac{U_1^2U_2}{L} + \kappa\beta^2\frac{U_1U_2^2}{L}\\ &+\kappa\beta^3\frac{U_2^3}{L} + \kappa\beta^4\frac{U_1U_3^2}{L} + \kappa\beta^5\frac{U_2U_3^2}{L}.\end{aligned} \tag{3.10}$$

In the case of large Reynolds number

$$Re_1 > 1, \tag{3.11}$$

and eventual absence of body force

$$\limsup_{t\to\infty}|f(t)|^2 = 0, \tag{3.12}$$

we have

$$\begin{aligned}\epsilon \le{}& \kappa\frac{L}{U_1 Re_1^2}\limsup_{t\to\infty}\left\|\frac{\partial\varphi}{\partial t}(t)\right\|^2_{L^\infty(\partial\Omega)} + \kappa\frac{U_1^3}{L} + \kappa\frac{U_1U_2^2}{LRe_1^2} + \kappa\frac{U_1U_3^2}{LRe_1^4}\\ &+\kappa\frac{U_1^2U_2}{LRe_1} + \kappa\frac{U_2^3}{LRe_1^3} + \kappa\frac{U_2U_3^2}{LRe_1^5}.\end{aligned} \tag{3.13}$$

Replacing Re_1 by 1 in (3.13) we obtain upper bounds for ϵ independent of ν.

If one prefers, one can always define a uniform typical velocity and a single Reynolds number. For example let

$$U = \max\{U_1, U_2, U_3\}, \tag{3.14}$$

$$Re = \frac{LU}{\nu}, \quad (Re > 1), \tag{3.15}$$

$$\beta = \kappa\frac{1}{Re}, \quad (\beta < 1). \tag{3.16}$$

Then

$$\epsilon \le \kappa\frac{U^3}{L}. \tag{3.17}$$

4. Conclusion

Eqs. (3.13) and (3.17) are in accordance with Kolmogorov's scaling law (see e.g. [9]). This generalizes the result of Doering and Constantin [5]. However we do not have an explicit estimate on the dimensionless constant κ. In principle we may apply the variational principle developed by Constantin and Doering (see [3,6]) to obtain an "optimal" bound on κ. However, we cannot go into such details unless we confine ourselves to more regular domains like channels, balls, cylinders, etc; otherwise the variational equation derived from the principle would be itself hard to solve.

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