01 Sep 1997

Attractors for Nonautonomous Nonhomogeneous Navier-Stokes Equations

A. Miranville

X. Wang

Attractors for nonautonomous nonhomogeneous Navier - Stokes equations

To cite this article: A Miranville and X Wang 1997 *Nonlinearity* **10** 1047

Nonlinearity **10** (1997) 1047–1061. Printed in the UK PII: S0951-7715(97)78581-5

Attractors for nonautonomous nonhomogeneous Navier–Stokes equations

A Miranville† and X Wang‡§

† Université de Poitiers, Département de Mathématiques, 40 avenue du Recteur Pineau, 86022 Poitiers Cedex, France

‡ Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA

Received 8 October 1996, in final form 3 April 1997
Recommended by S Childress

Abstract. In this paper our aim is to derive an upper bound on the dimension of the attractor of the family of processes associated to the Navier–Stokes equations with nonhomogeneous boundary conditions depending on time. We consider two-dimensional flows with prescribed quasiperiodic (in time) tangential velocity at the boundary, and obtain an upper bound which is polynomial with respect to the viscosity.

AMS classification scheme numbers: 35, 76

0. Introduction

In this paper, we continue the study initiated in [8] of the global attractor associated to the two-dimensional Navier–Stokes equations with prescribed tangential velocity at the boundary. The readers are referred to [11] for a comprehensive review on the subject of attractors and the first work on the attractor for boundary driven flows. In [8], we proved that the fractal and Hausdorff dimensions of the global attractor are bounded by $cRe^{3/2}$, where Re is the Reynolds number and c a nondimensional constant independent of Re in the autonomous case. This is a significant improvement on previous bounds which were exponential with respect to the Reynolds number (see [11]).

Recently, Chepyzhov and Vishik [1] presented a simple approach for the investigation of nonautonomous infinite-dimensional dynamical systems that was well suited for the study of equations arising in mathematical physics (see also Haraux [5] and Smiley [9]). In this approach, to an equation of the type

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A_{\sigma_0(t)}(u) \qquad (*)$$

where $u \in E$ and $\sigma_0(t)$ is called the time symbol, the authors associated a two-parametric family of operators $\{U_{\sigma_0(t)}(t,\tau),\ t \geqslant \tau \in \mathbb{R}\}$ defined by $U_{\sigma_0(t)}(t,\tau)u_\tau = u(t)$, where u is the solution of $(*)$ with initial data u_τ, and called the process associated to $(*)$. To construct the attractors, they considered, together with $(*)$, a family of equations $(*)$ with the symbol $\sigma(t)$ belonging to a space Σ, called the symbol space.

§ On leave from Iowa State University.

When E is a Banach space, Σ a complete metric space, then if some invariant semigroup $\{T(s), s \geqslant 0\}$ (for instance but not necessarily the translation semigroup) acts on Σ (i.e. $T(s)\Sigma = \Sigma$, $s \geqslant 0$) and the translation identity $U_{T(s)\sigma}(t, \tau) = U_\sigma(t + s, \tau + s)$, $\sigma \in \Sigma$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, $s \geqslant 0$ is valid, then the problem can be reduced to an autonomous system on the extended phase space $E \times \Sigma$. The uniform (with respect to σ) attractor of the family of processes will then be the projection on E of the global attractor of this autonomous system, if it exists. Furthermore, if the time symbol is quasiperiodic, then it is possible to obtain an upper bound on the dimension of the attractor.

In this paper, we consider two-dimensional flows with prescribed, quasiperiodic in time, tangential velocity at the boundary. In section 1 we present the equations and obtain *a priori* estimates. Then, in section 2 we construct the family of processes associated to these equations and prove the existence of the uniform attractor. Finally, in section 3 we obtain an upper bound on the Hausdorff dimension of the attractor. This bound is of the same order as the one obtained in the autonomous case.

1. Setting of the problem

Let Ω be a smooth (at least $\mathcal{C}^3$) bounded domain in $\mathbb{R}^2$. We consider the Navier–Stokes equations on Ω:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \tag{1}$$

$$\operatorname{div} u = 0 \tag{2}$$

$$u = \varphi \qquad \text{on } \partial\Omega \tag{3}$$

where $f = f(x, t)$ and $\varphi = \varphi(x, t)$ are quasiperiodic in t.

Unless otherwise stated throughout this paper, c, c' and c'' will denote various generic nondimensional constants (which may depend on the shape of the domain).

Moreover, we make the following assumptions on f and φ:

$$f(\cdot, t) = f(\cdot, \alpha_1 t, \dots, \alpha_k t) \tag{4}$$

$$\varphi(\cdot, t) = \varphi(\cdot, \alpha_1 t, \dots, \alpha_k t) \tag{5}$$

where $f(\cdot, \omega_1, \dots, \omega_k)$ and $\varphi(\cdot, \omega_1, \dots, \omega_k)$ are 2π-periodic in each argument ω_i, the $\{\alpha_i\}$ being rationally independent;

$$f \in \mathcal{C}_b^1(\mathbb{R}; L^2(\Omega)^2) \tag{6}$$

$$\varphi \in \mathcal{C}_b^2(\mathbb{R}; \mathcal{C}^3(\partial\Omega)^2) \tag{7}$$

where b means that we consider bounded functions;

$$\varphi \cdot n = 0 \tag{8}$$

where n denotes the unit outer normal on $\partial\Omega$.

For the mathematical setting of (1)–(3) we consider the spaces

$$H = \{u \in L^2(\Omega)^2, \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega\}$$

$$V = \{u \in H^1(\Omega)^2 \cap H, u = 0 \text{ on } \partial\Omega\}$$

which are endowed with their usual scalar products and norms which we denote $(\cdot, \cdot)$ and $|\cdot|$ for H and $((\cdot, \cdot))$ and $\|\cdot\|$ for V.

Based on a construction by Temam and Wang [12] which improves Hopf's original construction of divergence-free functions with a given velocity field at the boundary we have the following result.

Lemma 1. *For every $\varepsilon > 0$, there exists a smooth function $\phi = \phi_\varepsilon(x, t)$ satisfying:*

(i) $\operatorname{div}\phi = 0$;

(ii) $\phi = \varphi$ *on* $\partial\Omega$;

(iii) $|\int_\Omega (u \cdot \nabla)\phi \cdot u \,\mathrm{d}x| \leqslant c\varepsilon |\nabla\varphi|_{L^\infty(\mathbb{R}\times\partial\Omega)} |\Omega|^{1/2} \|u\|^2, \forall u \in V$;

(iv) $\|\phi\| \leqslant \frac{c'}{\varepsilon^{1/2}} |\nabla\varphi|_{L^\infty(\mathbb{R}\times\partial\Omega)}$;

(v) $|\frac{\partial\phi}{\partial t}| \leqslant c'' |\Omega|^{1/2} \varepsilon^{1/2} \max(|\frac{\partial\varphi}{\partial t}|_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla \frac{\partial\varphi}{\partial t}|_{L^\infty(\mathbb{R}\times\partial\Omega)} \operatorname{diam}\Omega)$

where c, c' and c'' are nondimensional constants, independent of ε and $\operatorname{diam}\Omega$ *denotes the diameter of Ω.*

Proof. The proof here is similar to the one performed in [8], lemma 2.1. Since Ω is bounded and regular there exists $\delta_0 > 0$ such that all normals from $\partial\Omega$ do not intersect in a neighbourhood of width $2\delta_0$ (which we denote $\mathcal{O}_{2\delta_0}(\partial\Omega)$) (see for instance [6, page 354]). Moreover, for every $(x, y) \in \mathcal{O}_{2\delta_0}(\partial\Omega)$, there exists a unique point $b(x, y) \in \partial\Omega$ such that

$$\operatorname{dist}((x, y), \partial\Omega) = \operatorname{dist}((x, y), b(x, y)).$$

Let $T_{b(x,y)}$ denote the clockwise tangent vector to $\partial\Omega$ at the point $b(x, y)$. We consider a function $\rho \in \mathcal{C}^\infty([0, +\infty))$ such that

$$\operatorname{supp}\rho \subset [0, 1] \tag{9}$$

$$\rho(0) = 1 \tag{10}$$

$$|\rho(s)| \leqslant 1 \qquad \forall s \in [0, 1] \tag{11}$$

$$\int_0^1 \rho(s)\,\mathrm{d}s = 0 \tag{12}$$

and we set

$$\psi = \psi_\varepsilon = \varphi(b(x, y), t) \cdot T_{b(x,y)} \int_0^{\operatorname{dist}((x,y),b(x,y))} \rho\left(\frac{s}{|\Omega|^{1/2}\varepsilon}\right) \mathrm{d}s \text{ if } (x, y) \in \mathcal{O}_{2\delta_0}(\partial\Omega)$$

$$\psi = \psi_\varepsilon = 0 \qquad \text{elsewhere.}$$

We finally set

$$\phi = \phi_\varepsilon(x, y, t) = \operatorname{curl}(\psi) = \begin{pmatrix} -\frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial x} \end{pmatrix}.$$

We then prove, exactly as in [8], that (i)–(iv) are true. To obtain (v), we note that if $\phi = (\phi_1, \phi_2)$, then

$$\begin{aligned}\frac{\partial\phi_1}{\partial t} &= -\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}(b(x, y), t) \cdot T_{b(x,y)}\right) \int_0^{\operatorname{dist}((x,y),\partial\Omega)} \rho\left(\frac{s}{|\Omega|^{1/2}\varepsilon}\right) \mathrm{d}s \\ &\quad - \frac{\partial\varphi}{\partial t}(b(x, y), t) \cdot T_{b(x,y)} \rho\left(\frac{\operatorname{dist}((x, y), \partial\Omega)}{|\Omega|^{1/2}\varepsilon}\right) \frac{\partial}{\partial y} \operatorname{dist}((x, y), \partial\Omega) \\ \frac{\partial\phi_2}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t}(b(x, y), t) \cdot T_{b(x,y)}\right) \int_0^{\operatorname{dist}((x,y),\partial\Omega)} \rho\left(\frac{s}{|\Omega|^{1/2}\varepsilon}\right) \mathrm{d}s \\ &\quad + \frac{\partial\varphi}{\partial t}(b(x, y), t) \cdot T_{b(x,y)} \rho\left(\frac{\operatorname{dist}((x, y), \partial\Omega)}{|\Omega|^{1/2}\varepsilon}\right) \frac{\partial}{\partial x} \operatorname{dist}((x, y), \partial\Omega)\end{aligned}$$

and if $(x, y) \notin \mathcal{O}_{|\Omega|^{1/2}\varepsilon}(\partial\Omega)$ (i.e. $\operatorname{dist}((x, y), \partial\Omega) > |\Omega|^{1/2}\varepsilon$) then

$$\rho\left(\frac{\operatorname{dist}((x, y), \partial\Omega)}{|\Omega|^{1/2}\varepsilon}\right) = 0$$

$$\int_0^{\operatorname{dist}((x,y),\partial\Omega)} \rho\left(\frac{s}{|\Omega|^{1/2}\varepsilon}\right) \mathrm{d}s = 0.$$

We then proceed as in [8] lemma 2.1 to verify that (i)–(v) are satisfied. This is straightforward. We omit the details. □

We now set $u = v + \phi$, and we obtain the following equation for v:

$$\frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + (\phi\cdot\nabla)v + (v\cdot\nabla)\phi + \nabla p = f + \nu\Delta\phi - (\phi\cdot\nabla)\phi - \frac{\partial\phi}{\partial t} \tag{13}$$

$$\operatorname{div} v = 0 \tag{14}$$

$$v = 0 \qquad \text{on } \partial\Omega \tag{15}$$

which can be written in functional form (see for instance [11])

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v, v) + Rv = \bar{f} \tag{16}$$

where $\bar{f} = Pf - \nu A\phi - B(\phi,\phi) - P\frac{\partial\phi}{\partial t}$, P is the orthogonal projector from L^2 into H, $A = -P\Delta$, $B(u, v) = P((u\cdot\nabla)v)$ and $Rv = B(v,\phi) + B(\phi, v)$.

As in [8], we prove that for every $\tau \in \mathbb{R}$ and for every $v_\tau \in H$, there exists a unique solution v of (16) with initial data v_τ (i.e. $v(\tau) = v_\tau$) such that:

$$v \in L^2(\tau, +\infty; V) \cap L^\infty(\tau, +\infty; H) \cap C(\tau, +\infty; H).$$

We take the scalar product in L^2 of (13) by v and obtain, by integration by parts and recalling that $(B(u, v), v) = 0$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 + \int_\Omega (v\cdot\nabla)\phi\cdot v\,\mathrm{d}x = (f, v) - \nu((\phi, v)) - \int_\Omega (\phi\cdot\nabla)\phi\cdot v\,\mathrm{d}x - (\phi_t, v). \tag{17}$$

Therefore, taking $\varepsilon = \frac{\nu}{2c|\Omega|^{\frac{1}{2}}|\varphi|_{L^\infty}}$, where c is the constant in lemma 1, (iii),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \frac{\nu}{2}\|v\|^2 \leqslant \frac{\nu}{16}\|v\|^2 + c\nu\|\phi\|^2 + \frac{\nu}{16}\|v\|^2 + \frac{c}{\nu\lambda_1}|f|^2 + \frac{\nu}{8}\|v\|^2$$
$$+ \frac{c}{\nu\lambda_1}|\phi_t|^2 + \left|\int_\Omega (\phi\cdot\nabla)\phi\cdot v\,\mathrm{d}x\right|$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator A on Ω with zero Dirichlet boundary condition (see for instance [11]).

As in [8] we have

$$\left|\int_\Omega (\phi\cdot\nabla)\phi\cdot v\,\mathrm{d}x\right| \leqslant c|\phi|_{L^\infty}|\phi|\|v\|$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \frac{\nu}{2}\|v\|^2 \leqslant \frac{c}{\nu\lambda_1}|f|^2 + \frac{c'}{\nu\lambda_1}|\phi_t|^2 + \frac{c''}{\nu}|\phi|^2|\phi|^2_{L^\infty} + c'''\nu\|\phi\|^2. \tag{18}$$

Integrating (18) between τ and T, we obtain

$$|v(T)|^2 + \frac{\nu}{2}\int_\tau^T \|v\|^2\,\mathrm{d}t \leqslant |v(\tau)|^2 + \frac{c}{\nu\lambda_1}\int_\tau^T |f|^2\,\mathrm{d}t + \frac{c}{\nu\lambda_1}\int_\tau^T |\phi_t|^2\,\mathrm{d}t$$
$$+ \frac{c''}{\nu}\int_\tau^T |\phi|^2_{L^\infty}|\phi|^2\,\mathrm{d}t + c'''\nu\int_\tau^T \|\phi\|^2\,\mathrm{d}t. \tag{19}$$

Moreover, using the Gronwall lemma and Poincaré inequality, we find

$$|v(t)|^2 \leqslant |v(\tau)|^2 \mathrm{e}^{-\frac{\nu\lambda_1}{2}(t-\tau)} + \frac{c}{\nu^2\lambda_1^2}|f|^2_{L^\infty(\mathbb{R},H)} + \frac{c'}{\nu^2\lambda_1^2}|\phi_t|^2_{L^\infty(\mathbb{R},L^2)}$$
$$+\frac{c''}{\nu^2\lambda_1}|\phi|^2_{L^\infty(\mathbb{R},L^2)} \times |\phi|^2_{L^\infty(\mathbb{R},L^\infty)} + \frac{c'''}{\lambda_1}|\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}. \tag{20}$$

Finally, taking the scalar product of (16) with $(t-\tau)Av$, we find

$$(t-\tau)\|v\|^2 \leqslant c_1(\phi, f, T, \tau, \nu)|v(\tau)|^2 + (t-\tau)c_2(\phi, f, T, \tau, \nu). \tag{21}$$

2. Existence of the global attractor

2.1. Preliminary results

In this section we consider the framework of Chepyzhov and Vishik [1]. We recall here the results that will be used in the sequel.

We consider a Banach space E and a two-parametric family of mappings acting on E:

$$U(t,\tau) : E \to E$$

$t \geqslant \tau, \tau \in \mathbb{R}$. The mapping U is called a process if:

(i) $U(\tau,\tau) = I$ (identity operator);

(ii) $U(t,s) \circ U(s,\tau) = U(t,\tau), \forall t \geqslant s \geqslant \tau, \tau \in \mathbb{R}$.

We denote by $\mathcal{B}(E)$ the set of all bounded sets in E. The process is bounded if for any set $B \in \mathcal{B}(E)$, the set $\cup_{\tau\in\mathbb{R}} \cup_{t\geqslant\tau} U(t,\tau)B \in \mathcal{B}(E)$. A set $B_0 \in \mathcal{B}(E)$ is absorbing if for any set $B \in \mathcal{B}(E)$, there exists $T = T(\tau, B)$ such that $U(t,\tau)B \subset B_0$ for $t \geqslant T$. Finally, a set $\mathcal{A}$ is attracting for the process if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(E)$, $\operatorname{dist}(U(t,\tau)B, \mathcal{A}) \to 0$ as $t \to +\infty$.

We now consider a family of processes $\{U_\sigma(t,\tau)\}$ depending on a functional parameter $\sigma \in \Sigma$. The parameter σ is called the symbol of the process, and Σ is called the symbol space, which is here assumed to be a complete metric space. The family of processes is said to be uniformly (with respect to $\sigma \in \Sigma$) bounded if for any $B \in \mathcal{B}(E)$, the set $\cup_{\sigma\in\Sigma} \cup_{\tau\in\mathbb{R}} \cup_{t\geqslant\tau} U_\sigma(t,\tau)B \in \mathcal{B}(E)$. A set $B_0 \in E$ is said to be uniformly absorbing for the family of processes if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(E)$, there exists $T = T(\tau, B)$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$, $\forall t \geqslant T$. Finally, a set $\mathcal{A}$ is said to be uniformly attracting for the family of processes if

$$\lim_{t\to+\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}(U_\sigma(t,\tau)B, \mathcal{A}) = 0 \qquad \forall \tau \in \mathbb{R}, \forall B \in \mathcal{B}(E).$$

A family of processes possessing a compact uniformly absorbing set is said to be uniformly compact and one possessing a compact uniformly attracting set, uniformly asymptotically compact.

A closed set $\mathcal{A}_\Sigma \subset E$ is said to be the uniform (with respect to $\sigma \in \Sigma$) attractor of the family of processes if it is uniformly attracting and it is contained in any closed uniformly attracting set $\mathcal{A}'$ of the family of processes (minimality). We have the following result which is proved in [1].

Proposition 1. *If a family of processes is uniformly asymptotically compact then it possesses a uniform attractor* $\mathcal{A}_\Sigma$.

We now assume that an invariant semigroup $\{T(t)\}_{t\geqslant 0}$ acts on Σ: $T(t)\Sigma = \Sigma$, $\forall t \geqslant 0$, and that:

(iii) $U_\sigma(t+s, \tau+s) = U_{T(s)\sigma}(t,\tau)$, $\forall \sigma \in \Sigma$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$, $s \geqslant 0$.

We then set

$$S(t) : E \times \Sigma \to E \times \Sigma$$
$$(u, \sigma) \mapsto= (U_\sigma(t, 0)u, T(t)\sigma) \qquad t \geqslant 0.$$

The family of mappings $\{S(t)\}$ acting on $E \times \Sigma$ forms a semigroup on $E \times \Sigma$ (see [1]).

Definitions.

(a) A family of processes is said to be $(E \times \Sigma, E)$-continuous if for fixed t and τ, $t \geqslant \tau$, $\tau \in \mathbb{R}$, the mapping $(u, \sigma) \to U_\sigma(t, \tau)u$ is continuous from $E \times \Sigma$ into E.

(b) A curve $u(s)$, $s \in \mathbb{R}$ is a complete trajectory of the process $\{U(t, \tau)\}$ if

$$U(t, \tau)u(\tau) = u(t) \qquad \forall t \geqslant \tau \qquad \tau \in \mathbb{R}.$$

(c) The kernel K of the process $\{U(t, \tau)\}$ consists of all bounded trajectories of the process. The set $K(s) = \{u(s), u \in K\}$ is called the kernel section at time $t = s$.

We end this section with the following result, which is proved in [1].

Proposition 2. *We consider a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, where Σ is a compact metric space. Let $\{T(t)\}$ be a continuous-invariant semigroup on Σ satisfying (iii). We assume that the family of processes is uniformly asymptotically compact and $(E \times \Sigma, E)$-continuous. Then the semigroup $\{S(t)\}$ associated with the family of processes possesses a compact, invariant attractor $\mathcal{A}$. Furthermore:*

(a) $\mathcal{A}_\Sigma = \Pi_1\mathcal{A}$ is the uniform attractor of the family of processes;

(b) $\Pi_2\mathcal{A} = \Sigma$;

(c) $\mathcal{A} = \cup_{\sigma\in\Sigma} K_\sigma(0) \times \{\sigma\}$;

(d) $\mathcal{A}_\Sigma = \cup_{\sigma\in\Sigma} K_\sigma(0)$,

where Π_1 and Π_2 denote the projectors from $E \times \Sigma$ into E and Σ respectively.

2.2. Construction of the global attractor

We write (16) in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{G}_{\sigma_0(t)}(v) \tag{22}$$

where $\sigma_0(t) = (f(t), \phi(t), \phi_t(t))$. Using the *a priori* estimates derived in section 1, we easily prove that (22) possesses a unique solution with initial data $v_\tau \in H$ satisfying

$$v \in L^\infty(\tau, +\infty; H) \cap L^2(\tau, T; V) \cap C(\tau, +\infty; H) \qquad \forall T \geqslant \tau.$$

We now consider the family of problems

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{G}_{\sigma(t)}(v) \tag{23}$$
$$v(\tau) = v_\tau \tag{24}$$

$\sigma(t) \in \Sigma$, where $\Sigma = \mathcal{H}(\sigma_0)$ is the hull of σ_0. We have here

$$\mathcal{H} = \{\sigma_0(\alpha_1 t + \omega_{01}, \ldots, \alpha_k t + \omega_{0k}), (\omega_{01}, \ldots, \omega_{0k}) = \omega_0 \in \mathbb{T}^k\}$$

where $\mathbb{T}^k$ denotes the k-dimensional torus. Therefore it is convenient to consider $\mathbb{T}^k$ as the symbol space of our problem. We also introduce the translation group $\{T(h), h \in \mathbb{R}\}$ which acts on $\mathbb{T}^k$ by the formula

$$T(h)\omega = (\alpha h + \omega) \qquad (\text{mod } \mathbb{T}^k) \qquad \omega \in \mathbb{T}^k.$$

Now, it is easy to check that for every $\sigma \in \mathcal{H}(\sigma_0)$, (23)–(24) possess a unique solution $v(t)$. Moreover, estimates (19)–(21) hold with the same constants, since, for instance,

$|f(\alpha_1 t + \omega_{01}, \ldots, \alpha_k t + \omega_{0k})|_{L^\infty(\mathbb{R}, L^2)} = |f(t)|_{L^\infty(\mathbb{R}, L^2)}$. Therefore, we can consider the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ defined by

$$v(t) = U_\sigma(t, \tau) v_\tau \tag{25}$$

where v is the solution of (23), (24). We have the following result.

Proposition 3. *The family of processes defined by (15) is uniformly bounded, uniformly compact and* $(H \times \mathcal{H}(\sigma_0), H)$*-continuous.*

Proof. To prove that the family of processes is uniformly bounded in H, we use (20). This inequality also enables us to prove the existence of a uniformly absorbing set B_0 in H.

Now the set $\cup_{\sigma \in \Sigma} \cup_{\tau \in \mathbb{R}} U_\sigma(\tau + 1, \tau) B_0$ is also uniformly absorbing in H. Using (21), we prove that this set is also bounded in H^1 and hence precompact in H. Therefore, the family of processes is uniformly compact.

It remains to check that the family of processes is $(H \times \mathcal{H}(\sigma_0), H)$-continuous. We consider two symbols σ_1 and σ_2 and their corresponding solutions v_1 and v_2. Setting $v = v_1 - v_2$, $f = f_1 - f_2$ and $\phi = \phi_1 - \phi_2$, we obtain the following equation for v:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v, v_1) + B(v_2, v) + B(\phi_1, v) + B(\phi, v_2) + B(v, \phi_1) + B(v_2, \phi)$$
$$= f - \nu A\phi - B(\phi, \phi_1) - B(\phi_2, \phi) - P\frac{\partial \phi}{\partial t} \tag{26}$$

$$v|_{t=\tau} = v_{1\tau} - v_{2\tau}. \tag{27}$$

Taking the scalar product of (26) with v we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 = (f, v) - (B(v, v_1), v) - (B(\phi, v_2), v) - (B(v, \phi_1), v)$$
$$-(B(\phi, \phi_1), v) - (B(\phi_2, \phi), v) - (B(v_2, \phi), v) - \nu((\phi, v)) - (\phi_t, v).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 \leqslant c|f|^2 + c'\|\phi\|^2 + c''|\phi_t|^2 + \|v_1\||v|_{L^4}^2 + \|\phi_1\||v|_{L^4}^2 + |\phi|_{L^4}\|v_2\||v|_{L^4}$$
$$+|\phi|_{L^4}\|\phi_1\||v|_{L^4} + |\phi_2|_{L^4}\|\phi\||v|_{L^4} + |v_2|_{L^4}\|\phi\||v|_{L^4}.$$

We then deduce, using Ladyzhenskaya's inequality ($|v|_{L^4}^2 \leqslant c|v|\|v\|$)

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 \leqslant c|f|^2 + c'(1 + \|\phi_1\|^2 + \|\phi_2\|^2 + \|v_2\|^2)\|\phi\|^2 + c''|\phi_t|^2$$
$$+c'''(\|v_1\|^2 + \|\phi_1\|^2 + \|v_2\|^2)|v|^2$$

which yields:

$$|v(t)|^2 \leqslant c\left(|v(\tau)|^2 + \int_\tau^t (c'|f|^2 + c''(1 + \|\phi_1\|^2 + \|\phi_2\|^2 + \|v_2\|^2)\|\phi\|^2 + c'''\|\phi_t\|^2)\,\mathrm{d}\theta\right)$$
$$\times \exp\left(c_0 \int_\tau^t (\|v_1\|^2 + \|\phi_1\|^2 + \|v_2\|^2)\,\mathrm{d}\theta\right)$$

and using (21) (the bound on $\|v\|$), we finally find

$$|v(t)|^2 \leqslant c(|v_{1\tau} - v_{2\tau}|^2 + c'(t - \tau)\max(|f|_{\mathcal{C}(\mathbb{R};H)}, |\phi_t|_{\mathcal{C}(\mathbb{R};L^2)}, |\phi|_{\mathcal{C}(\mathbb{R};H^1)})^2 \mathrm{e}^{c''(t-\tau)})$$

hence the result. □

Now let $\{S(t)\}$ be the semigroup associated with the family of processes defined above. The family of processes satisfies all the conditions of proposition 2. Therefore the following theorem follows.

Theorem 1. *The semigroup $\{S(t)\}$ possesses a global attractor $\mathcal{A}$. Moreover $\mathcal{A}_\Sigma = \Pi_1\mathcal{A}$ is the uniform attractor for the family of processes defined by (25) and $\mathcal{A}_\Sigma = \cup_{\sigma\in\Sigma} K_\sigma(0)$, where $K(0)$ is the kernel of the process $\{U_\sigma(t,\tau)\}$.*

Remark 1. We can also consider almost periodic and asymptotically almost periodic symbols (see [1]). In these two cases, we also obtain a result similar to theorem 1. However, we would not be able to obtain estimates on the dimension of the attractor in these two cases.

3. Upper bound on the dimension of the attractor

3.1. Preliminary results

We give here a general result for the estimation of the dimension of the attractor associated to a nonautonomous system with quasiperiodic symbol. We saw that in that case the symbol space can be identified with the k-dimensional torus $\mathbb{T}^k$. We consider the following system:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u,\omega(t)) \qquad \omega(t) = [\alpha t + \omega_0] \qquad \omega_0 \in \mathbb{T}^k \qquad \alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{R}^k \tag{28}$$

$$u|_{t=\tau} = u_\tau \qquad u_\tau \in H \qquad t \geqslant \tau \qquad \tau \in \mathbb{R} \tag{29}$$

where $\mathcal{G}(u,\omega)$ is a family of nonlinear operators depending on $\omega \in \mathbb{T}^k$ with domain H_1 and with values in H_0, $H_1 \hookrightarrow H \hookrightarrow H_0$ being Hilbert spaces.

We assume that (28), (29) is well posed. It thus generates a family of processes $\{U_{\omega_0}(t,\tau)\}$, $\omega_0 \in \mathbb{T}^k$. We also consider the semigroup

$$S(t) : H \times \mathbb{T}^k \to H \times \mathbb{T}^k$$
$$(u_0,\omega_0) \mapsto (U_{\omega_0}(t,0)u_0, \alpha t + \omega_0)$$

where $t \geqslant 0$. This semigroup can be constructed by considering the following autonomous system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{G}(u,\omega) \tag{30}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \alpha \tag{31}$$

$$u|_{t=0} = u_0 \tag{32}$$

$$\omega|_{t=0} = \omega_0 \tag{33}$$

where $u_0 \in H$, $\omega_0 \in \mathbb{T}^k$, which can be rewritten in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = M(y) \tag{34}$$

$$y|_{t=0} = y_0 \tag{35}$$

where $y = (u,\omega)$ and $M(y) = (\mathcal{G}(u,\omega),\alpha)$. We assume that the family of processes is uniformly asymptotically compact and $(H \times \mathbb{T}^k, H)$-continuous. Therefore the semigroup $\{S(t)\}$ possesses a compact attractor $\mathcal{A}$ in $H \times \mathbb{T}^k$. Moreover, $\mathcal{A}_{\mathbb{T}^k} = \Pi_1\mathcal{A}$ is the uniform attractor of the family of processes. Since

$$\dim \mathcal{A}_{\mathbb{T}^k} \leqslant \dim \mathcal{A}$$

where dim denotes the Hausdorff dimension, it suffices to find an upper bound for $\dim \mathcal{A}$ in order to obtain an upper bound on the dimension of $\mathcal{A}_{\mathbb{T}^k}$. We finally make the following assumptions ((i)–(iii)).

(i) The semigroup is uniformly quasidifferentiable in $H \times \mathbb{T}^k$ on $\mathcal{A}$ (see [11]).

(ii) The quasidifferential $S'(t, y_0)z_0 = z(t)$, $t \geqslant 0$, satisfies the first variation equation of (34) and (35):

$$\frac{\mathrm{d}z}{\mathrm{d}t} = M'(y(t))z \tag{36}$$

$$z|_{t=0} = z_0 \tag{37}$$

where $z = (v, \mu)$, which can be written in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{G}'_u(u(t), \omega(t))v + \mathcal{G}'_\omega(u(t), \omega(t))\mu \tag{38}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0 \tag{39}$$

$$v|_{t=0} = v_0 \tag{40}$$

$$\mu|_{t=0} = \mu_0 \tag{41}$$

where $u(t) = U_{\omega_0}(t, \tau)u_0$, $\omega(t) = [\alpha t + \omega_0]$.

(iii) We have

$$\begin{aligned}(M'(y(t))z, z) &= (\mathcal{G}'_u(u(t), \omega(t))v, v) + (\mathcal{G}'_\omega(u(t), \omega(t))\mu, \mu) \\ &\leqslant (L_1(t, y_0)v, v) + (L_2(t, y_0)\mu, \mu) \equiv (M_1(t, y_0)z, z)\end{aligned} \tag{42}$$

for every $y_0 \in \mathcal{A}$, $t \geqslant 0$ and $z = (v, \mu) \in H_1 \times \mathbb{R}^k$, where L_1 is (for fixed $(t, y_0) \in \mathbb{R} \times \mathcal{A}$) a selfadjoint operator in H, L_1: $H_2 \to H$, $H_2 \subset\subset H$, with a discrete spectrum

$$\lambda_1(t, y_0) \geqslant \lambda_2(t, y_0) \geqslant \cdots \geqslant \lambda_i(t, y_0) \geqslant \dots \qquad \lambda_i \to -\infty \qquad \text{as } i \to \infty$$

and with orthonormal (in H) eigenfunctions $w_i(t, y_0)$, and $L_2 : \mathbb{R}^k \to \mathbb{R}^k$ is selfadjoint, with eigenvalues $\eta_i = \eta_i(t, y_0)$ and corresponding orthonormal in $\mathbb{R}^k$ eigenfunctions $p_i(t, y_0)$, $i = 1, \dots, k$. Moreover, we assume that L_1 and L_2 are uniformly (with respect to $(t, y_0) \in \mathbb{R}^+ \times \mathcal{A}$) semibounded from above.

We have $M_1 = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, $M_1 z = L_1 v + L_2 \mu$, and the eigenvectors of M_1 are $\varphi_j^{(1)} = (w_j, 0)$, $M_1\varphi_j^{(1)} = \lambda_j \varphi_j^{(1)}$, $j \in \mathbb{N}$, and $\rho_i^{(2)} = (0, \zeta_i)$, $M_1\rho_i^{(2)} = \eta_i \rho_i^{(2)}$, $i = 1, \dots, k$.

We have the following result which is proved in [1].

Proposition 4. *We assume that (i)–(iii) are satisfied. Then if*

$$q_d = \liminf_{t \to +\infty} \sup_{y_0 \in \mathcal{A}} \left(\frac{1}{t} \int_0^t \sum_{j=1}^d (M_1 \zeta_j, \zeta_j)\, \mathrm{d}\tau \right) < 0$$

where ζ_j are the eigenfunctions corresponding to the d greatest eigenvalues of M_1, we have

$$\dim \mathcal{A} \leqslant d$$

where dim *denotes the Hausdorff dimension.*

3.2. Application to the nonautonomous nonhomogeneous Navier–Stokes equations

We write (23), (24) as an autonomous system (i.e. in the form (34),(35)):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = M(y) \tag{43}$$

$$y|_{t=0} = y_0 \tag{44}$$

where $y = (v, \omega)$ and $M(y) = (-\nu A v - B(v, v) - B(v, \phi) - B(\phi, v) + f - \nu A\phi - B(\phi, \phi) - \frac{\partial \phi}{\partial t}, \alpha)$.

The proof of (i) and (ii) is classical (see for instance [11]). It remains to check that (iii) is satisfied. In order to do so, we need to estimate $(M'(y)z, z)$, where

$$z = \left(w, \frac{1}{UL}\mu\right)$$

and

$$\begin{aligned} M'(y)z = (&-\nu A w - B(v, w) - B(w, v) + f'_\omega(x, \omega(t))\frac{1}{UL}\mu - B(w, \phi) \\ &-B\left(v, \phi'_\omega(x, \omega(t))\frac{1}{UL}\mu\right) - B(\phi, w) - B\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, v\right) \\ &-\nu A\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right) - B\left(\phi, \phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right) \\ &-B\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, \phi\right) - \frac{\partial}{\partial t}\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, 0\right). \end{aligned}$$

Here U is a velocity and L a length that will be fixed later and are considered for dimensional reasons. Therefore

$$\begin{aligned} (M'(y)z, z) = &-\nu\|w\|^2 - (B(w, v), w) + \left(f'_\omega(x, \omega(t))\frac{\mu}{UL}, v\right) - (B(w, \phi), w) \\ &-\left(B\left(v, \phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right), w\right) - \left(B\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, v\right), w\right) \\ &-\nu\left(\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, w\right)\right) - \left(B\left(\phi, \phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right), w\right) \\ &-\left(B\left(\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, \phi\right), w\right) - \left(\frac{\partial}{\partial t}\phi'_\omega(x, \omega(t))\frac{\mu}{UL}, w\right). \end{aligned}$$

We then obtain, since $|(B(w, \phi), w)| \leqslant \frac{\nu}{2}\|w\|^2$ (by lemma 1 (iii) and our choice of ε at the end of section 1)

$$\begin{aligned} (M'(y)z, z) \leqslant &-\frac{\nu}{2}\|w\|^2 + \int_\Omega |\nabla v||w|^2\,\mathrm{d}x + \int_\Omega \left|f'_\omega\frac{\mu}{LU}\right||v|\,\mathrm{d}x \\ &+\int_\Omega \left|\frac{\partial}{\partial t}\phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right||w|\,\mathrm{d}x + \nu\left\|\phi'_\omega(x, \omega(t))\frac{\mu}{UL}\right\|\|w\| \\ &+\int_\Omega |\nabla\phi||w|^2\,\mathrm{d}x + |\phi|_{L^\infty}\left\|\phi'_\omega\frac{\mu}{UL}\right\||w| + \left|\phi'_\omega\frac{\mu}{UL}\right|_{L^\infty}\|\phi\||w| \\ &+\left|\left(B\left(\phi'_\omega\frac{\mu}{UL}, v\right), w\right)\right| + \left|\left(B\left(v, \phi'_\omega\frac{\mu}{UL}\right), w\right)\right|. \end{aligned}$$

We have

$$\left|\left(B\left(v, \phi'_\omega\frac{\mu}{UL}\right), w\right)\right| \leqslant \left|\nabla\phi'_\omega\frac{\mu}{LU}\right|_{L^\infty}|v||w|.$$

Moreover $v \in \mathcal{A}_{\mathbb{T}^k} \subset B_0$, where B_0 is the absorbing set in H, thanks to the minimality property. Thanks to (20), $B_0 = B_H(0, 2R_0)$, where

$$R_0^2 = \frac{c}{\nu^2\lambda_1^2}|f|^2_{L^\infty(\mathbb{R},H)} + \frac{c'}{\nu^2\lambda_1^2}|\phi_t|^2_{L^\infty(\mathbb{R},L^2)} + \frac{c''}{\nu^2\lambda_1}|\phi|^2_{L^\infty(\mathbb{R},L^\infty)}|\phi|^2_{L^\infty(\mathbb{R},L^2)} + \frac{c'''}{\lambda_1}|\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}$$

is an absorbing set in H. Therefore

$$\left|\left(B\left(v, \phi'_\omega \frac{\mu}{LU}\right), w\right)\right| \leqslant 2R_0 \left|\nabla\phi'_\omega \frac{\mu}{LU}\right|_{L^\infty} |w|.$$

Similarly we obtain, by integration by parts

$$\left|\left(B\left(\phi'_\omega \frac{\mu}{LU}, v\right), w\right)\right| \leqslant 2R_0 \left|\nabla\phi'_\omega \frac{\mu}{LU}\right|_{L^\infty} |w| + 2R_0 \left|\phi'_\omega \frac{\mu}{LU}\right|_{L^\infty} \|w\|.$$

Therefore

$$\begin{aligned}(M'(y)z, z) \leqslant{}& -\frac{\nu}{4}\|w\|^2 + \int_\Omega |\nabla v||w|^2\,\mathrm{d}x + \int_\Omega |\nabla\phi||w|^2\,\mathrm{d}x \\ &+ \frac{c}{L^2U^2}\left(\frac{1}{\nu\lambda_1}|f'_\omega|^2_{L^\infty(\mathbb{R},L^2)} + \frac{1}{\nu\lambda_1}\left|\frac{\partial}{\partial t}\phi'_\omega\right|^2_{L^\infty(\mathbb{R},L^2)} + \nu|\nabla\phi'_\omega|^2_{L^\infty(\mathbb{R},L^2)}\right. \\ &+ \frac{1}{\nu\lambda_1}|\phi|^2_{L^\infty(\mathbb{R},L^\infty)}|\nabla\phi'_\omega|^2_{L^\infty(\mathbb{R},L^2)} + \frac{1}{\nu\lambda_1}|\phi'_\omega|^2_{L^\infty(\mathbb{R},L^\infty)}|\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)} \\ &\left.+ \frac{R_0^2}{\nu\lambda_1}|\nabla\phi'_\omega|^2_{L^\infty(\mathbb{R},L^\infty)} + \frac{R_0^2}{\nu}|\phi'_\omega|^2_{L^\infty(\mathbb{R},L^\infty)}\right)\mu^2\end{aligned}$$

which can be rewritten as

$$(M'(y)z, z) \leqslant -\frac{\nu}{4}\|w\|^2 + \int_\Omega |\nabla v||w|^2\,\mathrm{d}x + \int_\Omega |\nabla\phi||w|^2\,\mathrm{d}x + K\mu^2. \tag{45}$$

We then find

$$(M'(y)z, z) \leqslant (M_1 z, z) = (L_1 w, w) + (L_2\mu, \mu)$$

where

$$L_1 w = -\frac{\nu}{4}Aw + P(|\nabla v| + |\nabla\phi|)w$$

and

$$L_2\mu = K I_k \mu$$

where I_k is the identity operator in $\mathbb{R}^k$.

Since $y_0 \in \mathcal{A}$, the function $|\nabla v(x,t)|$ is smooth (see for instance [11, ch 4, section 6]) and consequently L_1 is selfadjoint with a discrete spectrum, each eigenvalue having a finite multiplicity. We then have $M_1 = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, and the eigenfunctions are $\varphi_j^{(1)} = (w_j, 0)$ and $\varphi_i^{(2)} = (0, \zeta_i)$, where $\{w_j\}$ is orthonormal in H, $L_1 w_j = \lambda_j w_j$, and $\{\zeta_i\}$ is an arbitrary basis of $\mathbb{R}^k$, $i = 1, \ldots, k$, $L_2\zeta_i = K\zeta_i$. We note that the d greatest eigenvalues of M_1 can be written in the form

$$\lambda_{d-k} \leqslant \cdots \leqslant \lambda_p \leqslant K \leqslant \cdots \leqslant K \leqslant \lambda_{p-1} \leqslant \cdots \leqslant \lambda_1$$

for d large enough. Moreover, we have

$$\sum_{j=1}^{d}(M_1\varphi_j,\varphi_j) = -\frac{\nu}{4}\sum_{j=1}^{d-k}\|w_j\|^2 + \int_\Omega |\nabla v|\sum_{j=1}^{d-k}|w_j|^2\,\mathrm{d}x + \int_\Omega |\nabla\phi|\sum_{j=1}^{d-k}|w_j|^2\,\mathrm{d}x + K\sum_{j=1}^{k}|\zeta_j|^2$$

$$= -\frac{\nu}{4}\sum_{j=1}^{d-k}\|w_j\|^2 + \int_\Omega |\nabla v|\sum_{j=1}^{d-k}|w_j|^2\,\mathrm{d}x + \int_\Omega |\nabla\phi|\sum_{j=1}^{d-k}|w_j|^2\,\mathrm{d}x + Kk.$$

Using the Lieb–Thirring inequality (see [11], appendix, theorem 4.1) we find

$$\sum_{j=1}^{d}(M_1\varphi_j,\varphi_j) \leqslant -\frac{\nu}{8}\sum_{j=1}^{d-k}\|w_j\|^2 + \frac{c}{\nu}(\|v\|^2 + |\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}) + Kk.$$

Moreover

$$d-k = \int_\Omega \rho\,\mathrm{d}x \leqslant |\Omega|^{\frac12}|\rho|$$

where $\rho(x) = \sum_{j=1}^{d-k}|w_j|^2$. Therefore, we obtain

$$q_d \leqslant -\frac{c\nu}{|\Omega|}(d-k)^2 + \frac{c}{\nu}(\gamma + |\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}) + Kk$$

where

$$\gamma = \liminf_{t\to+\infty}\sup_{y_0\in\mathcal{A}}\left\{\frac{1}{t}\int_0^t \|v\|^2\,\mathrm{d}\tau\right\}.$$

Therefore $q_d < 0$ if

$$d-k > c\left(\frac{|\Omega|}{\nu^2}(\gamma + |\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}) + \frac{K|\Omega|}{\nu}k\right)^{1/2}.$$

That is to say

$$d > k + c\left(\frac{|\Omega|}{\nu^2}(\gamma + |\nabla\phi|^2_{L^\infty(\mathbb{R},L^2)}) + \frac{K|\Omega|}{\nu}k\right)^{1/2}.$$

Using (19) we find

$$\gamma \leqslant \frac{c}{\nu^2\lambda_1}|f|^2_{L^\infty(\mathbb{R},L^2)} + \frac{c'}{\nu^2\lambda_1}|\phi_t|^2_{L^\infty(\mathbb{R},L^2)} + \frac{c''}{\nu^2}|\phi|^2_{L^\infty(\mathbb{R},L^\infty)}|\phi|^2_{L^\infty(\mathbb{R},L^2)} + c'''|\nabla\phi|^2_{L^\infty(0,T;L^2)}.$$

Therefore

$$\frac{|\Omega|}{\nu^2}\gamma \leqslant c\Bigg(\frac{|\Omega|}{\nu^4\lambda_1}|f|^2_{L^\infty(\mathbb{R},L^2)} + \frac{|\Omega|^2}{\nu^4\lambda_1}\max\left(|\varphi_t|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, \left|\nabla\frac{\partial\varphi}{\partial t}\right|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega\right)\varepsilon$$
$$+\frac{c''|\Omega|}{\nu^4\lambda_1}(|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\cdot|\nabla\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega|\Omega|\varepsilon) + \frac{c'''}{\varepsilon}\frac{|\Omega|}{\nu^2}|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\Bigg).$$

Similarly, we have

$$R_0^2 \leqslant \frac{c}{\nu^2\lambda_1^2}|f|^2_{L^\infty(\mathbb{R},L^2)} + \frac{c'|\Omega|}{\nu^2\lambda_1^2}\max(|\varphi_t|^2_{L^\infty(\mathbb{R},\partial\Omega)}, |\nabla\varphi_t|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega)\varepsilon$$
$$+\frac{c''|\Omega|}{\nu^2\lambda_1}\max(|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega)^2\varepsilon + \frac{c'''}{\varepsilon}|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}$$

and we then find (the computations are similar to those performed in lemma 1) (note that K is as defined by (45))

$$
\begin{aligned}
\frac{|\Omega|}{\nu}K \leqslant \frac{c}{L^2U^2}\Bigg(& \frac{|\Omega|}{\nu^2\lambda_1}|f'_\omega|^2_{L^\infty(\mathbb{R},L^2)} \\
& + \frac{|\Omega|^2}{\nu^2\lambda_1}\max\left(\left|\frac{\partial}{\partial t}\varphi'_\omega\right|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, \left|\nabla\frac{\partial}{\partial t}\varphi'_\omega\right|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega\right)\varepsilon \\
& + \frac{1}{\varepsilon}|\Omega||\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)} + \frac{|\Omega|}{\nu^2\lambda_1}\max(|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)} \\
& \times \operatorname{diam}^2|\Omega|)|\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\frac{1}{\varepsilon} + \frac{|\Omega|}{\nu^2\lambda_1}\max(|\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)} \\
& \times \operatorname{diam}^2|\Omega|)|\varphi|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\frac{1}{\varepsilon} + \frac{R_0^2}{\nu^2\lambda_1}|\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\frac{1}{\varepsilon^2} + \frac{R_0^2|\Omega|}{\nu^2} \\
& \times \max(|\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi'_\omega|^2_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}^2\Omega)\Bigg).
\end{aligned}
$$

We now set

$$
\begin{aligned}
& U = \max(|\varphi|_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi|_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}\Omega, |\varphi'_\omega|_{L^\infty(\mathbb{R}\times\partial\Omega)}, |\nabla\varphi'_\omega|_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}\Omega) \\
& L = \max\left(|\Omega|^{1/2}, \left(\frac{1}{\lambda_1}\right)^{1/2}\right) \\
& G_1 = \frac{L^2}{\nu^2}|f|_{L^\infty(\mathbb{R},L^2)} \\
& G_2 = \frac{L}{\nu U}|f'_\omega|_{L^\infty(\mathbb{R},L^2)} \\
& Re_1 = \frac{UL}{\nu} \\
& Re_2 = \max(|\varphi_t|_{L^\infty(\mathbb{R}\times\partial\Omega)}|\nabla\varphi_t|_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}\Omega)^{1/2}\frac{L^{3/2}}{\nu} \\
& Re_3 = \max\left(\left|\frac{\partial}{\partial t}\varphi'_\omega\right|_{L^\infty(\mathbb{R}\times\partial\Omega)}, \left|\nabla\frac{\partial}{\partial t}\varphi'_\omega\right|_{L^\infty(\mathbb{R}\times\partial\Omega)}\operatorname{diam}\Omega\right)\frac{L^2}{U\nu} \\
& Re = \max(Re_1, Re_2, Re_3).
\end{aligned}
$$

We then find, taking $\varepsilon = cRe^{-1}$, c being a nondimensional constant, independent of Re,

$$
\begin{aligned}
& \frac{|\Omega|}{\nu^2}\gamma \leqslant (G_1^2 + Re^3) \\
& \frac{|\Omega|}{\nu^2}|\nabla\varphi|^2_{L^\infty(\mathbb{R},L^2)} \leqslant c' Re^3 \\
& R_0^2 \leqslant \frac{c\nu^2}{L^4}(G_1^2 + Re^3) \\
& \frac{|\Omega|K}{\nu} \leqslant c(G_2^2 + Re + G_1 + Re^3).
\end{aligned}
$$

we finally deduce that the Hausdorff dimension of $\mathcal{A}_{\mathbb{T}^k}$ is bounded, as follows

$$
\dim\mathcal{A}_{\mathbb{T}^k} \leqslant k + c(G_1 + Re^{3/2}) + c'(G_1 + G_2 + Re^{3/2})k^{1/2} \tag{46}
$$

where c and c' are nondimensional constants, independent of G_1, G_2, and Re.

Roughly speaking, the Reynolds numbers are nondimensional numbers illustrating the ratio of (in this case) the strength of external forcing via boundary velocity and the viscous force strength while the generalized Grashoff numbers are nondimensional numbers illustrating the ratio of the strength of body force and the the strength of the viscous force. Since both the boundary velocity and the body force are time and space dependent in general, we may define different Reynolds numbers and Grashoff numbers to emphasize different effects of different scales of the external forcing. Of course, we may always define one single Reynolds number, Re, as above and one single Grashoff number as the sum of G_1 and G_2.

Remark 2. If $k = 0$ (autonomous case), then (46) becomes the estimate derived in [8]. Moreover, it was proved in [1] that the term k is optimal if we consider the dependence in k. Now if $\varphi = 0$, (46) reduces to the estimates derived in [1]

Thanks to (46), we find

$$\dim \mathcal{A}_{\mathbb{T}^k} \leqslant \frac{c}{\nu^2} \tag{47}$$

if $f \neq 0$ and,

$$\dim \mathcal{A}_{\mathbb{T}^k} \leqslant \frac{c'}{\nu^{3/2}} \tag{48}$$

if $f = 0$, where c and c' are independent of ν. These estimates are of the same order as those derived in [8] for the autonomous case. Moreover, if $f = 0$, (48) agrees with Kolmogorov's heuristical estimate of the number of degrees of freedom of a two-dimensional turbulent flow (see [11]), when the dependence on ν is considered.

Remark 3. We can also obtain similar bounds for the fractal dimension of $\mathcal{A}_{\mathbb{T}^k}$ just as in the autonomous case.

Remark 4. For optimal bounds on the dimension of attractors for two-dimensional periodic flows, the readers are referred to [3]. For more general boundary conditions, see [2].

Remark 5. In the case of shear driven channel flow we may sort out the dependence of the dimension of the attractor on the shape of the domain (aspect ratio = length/width) in the same way as was done by Doering and Wang [4] where the authors proved that the dimension of the attractor for two-dimensional autonomous shear driven channel flows has an upper bound of the form an absolute constant times the aspect ratio times $Re^{3/2}$.

Acknowledgments

This work was initiated during Professor M I Vishik's visit to the Institute for Scientific Computing and Applied Mathematics at Indiana University, Bloomington, where Professor Vishik presented a series of lectures on the theory of attractors for nonautonomous dynamical systems. The authors wish to thank him for very interesting discussions and his interest in their work. They also wish to thank Professor Temam for several helpful comments. We also wish to thank an anonymous referee for several useful comments.

This work was partially supported by the National Science Foundation under grant NSF-DMS-9400615 and the Research Fund of Indiana University.

References

[1] Chepyzhov V V and Vishik M I 1994 Attractors of non-autonomous dynamical systems and their dimension *J. Math. Pures Appl.* **73** 279–333

[2] Constantin P, Foias C, and Temam R 1985 Attractors representing turbulent flows *Mem. AMS* **53**

[3] Constantin P, Foias C, and Temam R 1988 On the dimension of the attractors in two-dimensional turbulence *Physica* D **30** 284–96

[4] Doering C and Wang X Degrees of freedom for two dimensional shear driven channel flow *Preprint*

[5] Haraux A 1991 *Systèmes Dynamiques Dissipatifs* (Paris: Masson)

[6] Gilbarg D and Trudinger N 1983 *Elliptic Partial Differential Equations of Second Order* (Berlin: Springer)

[7] Miranville A 1993 Shear layer flow in a channel: Estimate on the dimension of the attractor *Physica* D **65** 135–53

[8] Miranville A and Wang X 1996 Upper bound on the dimension of the attractor for the nonhomogeneous Navier–Stokes equations *Disc. Contin. Dynam. Syst.* **2** 95–110

[9] Smiley M 1993 Global attractors and approximate inertial manifolds for nonautonomous dissipative equations *Appl. Anal.* **50** 217–41

[10] Temam R 1984 *Navier–Stokes Equations, Theory and Numerical Analysis* 3rd edition (Amsterdam: North-Holland)

[11] Temam R 1988 *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (Berlin: Springer)

[12] Temam R and Wang X 1995 Asymptotic analysis of the linearized Navier–Stokes equations in a 2-D channel *J. Diff. Int. Eqns* **8** 1591–1618