12 Sep 1997

The Convergence of the Solutions of the Navier-Stokes Equations to that of the Euler Equations

R. Temam

X. Wang

Pergamon

Appl. Math. Lett. Vol. 10, No. 5, pp. 29–33, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00079-7

The Convergence of the Solutions of the Navier-Stokes Equations to that of the Euler Equations

R. TEMAM
Laboratoire d'Analyse Numérique
Université Paris-Sud, 91405 Orsay, France
and
The Institute for Scientific Computing and Applied Mathematics
Indiana University, Bloomington, IN 47405, U.S.A.

X. WANG
Courant Institute, New York University,
New York, NY 10012, U.S.A.
and
Department of Mathematics
Iowa State University, Ames, IA 50011, U.S.A.

(Received January 1997; accepted February 1997)

Abstract—In this article, we establish partial results concerning the convergence of the solutions of the Navier-Stokes equations to that of the Euler equations. Convergence is proved in space dimension two under a physically reasonable assumption, namely that the gradient of the pressure remains bounded at the boundary as the Reynolds number converges to infinity.

Keywords—Boundary layer, Small viscosity, Navier-Stokes equation, Euler equation.

1. INTRODUCTION

The understanding of turbulent boundary layers and the behavior of the solutions of the Navier-Stokes equations at large Reynolds numbers are outstanding problems in mathematics and physics. An abundance of literature is available in the fluid mechanic and mathematics literatures; on the mathematical side, see, e.g., [1–6], and the references therein. The most recent results include a new approach to the law of the wall for turbulent boundary layers [7,8], a proof of the convergence of the solutions of the Navier-Stokes equations to that of the Euler equations, for a small interval of time in the context of analytic solutions [9], and the study of the inviscid limit of vortex patches [10].

In earlier works we have studied the convergence, for large Reynolds numbers, of the solutions of linearized Navier-Stokes of the Oseen type, see [11–13]; see also [14] for related but distinct

This work was supported in part by the National Science Foundation under Grant NSF-DMS-9400615, by ONR under Grant NAVY-N00014-91-J-1140 and by the Research Fund of Indiana University. The second author was supported by an NSF post-doctoral position at the Courant Institute.

situations. In this article, we apply some of the technics developed in [11–13], to the full (nonlinear) Navier-Stokes equations in space dimension two. Under physically reasonable assumptions we prove that the solutions to the Navier-Stokes equations converge to the solutions of the Euler equations on any finite interval of time: the assumptions that we make on the solutions are either the boundedness at the wall of the gradient of the pressure, an assumption which is confirmed by physical experiments and well accepted in turbulence theory, or we assume a moderate growth condition for the tangential derivative of the tangential flow near the wall, an assumption whose physical relevance is discussed in the text.

For the sake of simplicity, we restrict ourselves to the case of rectangular geometry, considering the flow in a channel; however the flow in a general domain can be handled in a similar manner [12].

This article is organized as follows. In Section 2, we set the notations and state the main results; then in Sections 3 and 4, we give some indications on the proofs. The details of the proofs will be given elsewhere [15].

2. THE MAIN RESULTS

We consider the Navier-Stokes in space dimension two in an infinite channel $\Omega_\infty = \mathbb{R} \times (0,1)$:

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = f, \qquad \text{in } \Omega_\infty \times \mathbb{R}_+, \tag{2.1}$$

$$\nabla \cdot u^\varepsilon = 0, \qquad \text{in } \Omega_\infty \times \mathbb{R}_+. \tag{2.2}$$

The velocity u^ε vanishes at the boundary $\partial\Omega_\infty$ of the channel (i.e., at $y = 0, 1$),

$$u^\varepsilon = 0 \text{ on } \partial\Omega_\infty \times \mathbb{R}_+, \tag{2.3}$$

and periodicity (period 2Π) is assumed in the horizontal (x) direction. We set $\Omega = (0, 2\pi) \times (0,1)$, $\Gamma = (0, 2\pi) \times \{0, 1\}$, and introduce the natural function spaces

$$V = \left\{ v \in (H^1_{\text{loc}}(\Omega_\infty))^2,\ \text{div } v = 0, v|_{\partial\Omega_\infty} = 0,\ v \text{ is } 2\pi\text{-periodic in } x \right\},$$
$$H = \left\{ v = (v_1, v_2) \in (L^2_{\text{loc}}(\Omega_\infty))^2,\ \text{div } v = 0,\ v_2|_{\partial\Omega_\infty = 0},\ v_1 \text{ is } 2\pi\text{-periodic in } x \right\}.$$

Finally, equations (2.1)–(2.3) are supplemented with the initial condition

$$u^\varepsilon = u_0, \qquad \text{(given in V) at } t = 0. \tag{2.4}$$

We intend to compare the solutions of (2.1)–(2.4) to the solutions (u^0, p^0) of the corresponding Euler equations, i.e.,

$$\frac{\partial u^0}{\partial t} + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f, \qquad \text{in } \Omega_\infty \times \mathbb{R}_+, \tag{2.5}$$

$$\nabla \cdot u^0 = 0, \qquad \text{in } \Omega_\infty \times \mathbb{R}_+, \tag{2.6}$$

$$u^0 \cdot n = 0, \qquad \text{on } \partial\Omega_\infty \times \mathbb{R}_+, \tag{2.7}$$

$$u^0 = u_0, \qquad \text{at } t = 0. \tag{2.8}$$

In (2.7), n is the unit outward normal, so that for $u^0 = (u_1^0, u_2^0)$, (2.7) is equivalent to $u_2^0 = 0$ at $y = 0, 1$. Furthermore, space periodicity in x is understood.

We are not interested in issues related to the possible loss of regularity for u^ε and u^o; hence we assume hereafter that u^ε, p^ε, u^0, p^0 are as smooth as needed. In fact, the classical results on regularity of the solutions of the Navier-Stokes equations provide all the desired regularity for u^ε, p^ε, provided f and u_0 are sufficiently smooth; for the Euler equations (2.5)–(2.8) existence and regularity are shown in [16] by a proper generalization of the corresponding results for the Euler equations with the nonpenetration boundary condition [3,17–20], (see also [21,22]).

Our main result asserts that if the tangential derivative of p^ε (x-derivative on Γ) does not grow too fast, then u^ε converges to u^0 as $\varepsilon \to 0$; as demonstrated in Theorem 1.

THEOREM 1. *Let u^ε and u^0 be the solutions of the Navier-Stokes and Euler equations above.*

We assume that $T > 0$ is fixed and that there exist two constants κ_1, δ independent of ε, $0 \leq \delta < 1/2$, such that either

$$\|p^\varepsilon\|_{L^2(0,T;H^{1/2}(\Gamma))} \leq \kappa_1 \varepsilon^{-\delta}, \tag{2.9}$$

or

$$\|p^\varepsilon_{,x}\|_{L^2(0,T;L^2(\Gamma))} = \|p^\varepsilon\|_{L^2(0,T;H^1(\Gamma))} \leq \kappa_1 \varepsilon^{-\delta-1/4}. \tag{2.10}$$

Then, there exists κ_2 independent of ε such that

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T;H)} \leq \kappa_2 \varepsilon^{(1-2\delta)/5}. \tag{2.11}$$

The principle of the proof of Theorem 1 is given in Section 4; all the details will appear in [15].

REMARK 1. The convergence in (2.11) is in the strong (norm) topology of $L^\infty(0,T;L^2(\Omega)^2)$; as usual in boundary layer phenomena, convergence in the $L^\infty(\Omega)$ or $H^1(\Omega)$-norm is not expected (is not true in general), because of the discrepancy in the boundary values of u^ε and u^0.

REMARK 2. As mentioned in the Introduction, it is expected on physical grounds that p^ε and $p^\varepsilon{}_{,x} = \frac{\partial p^\varepsilon}{\partial x}$ remain bounded on and near Γ; therefore (2.9) or (2.10) are physically very realistic hypotheses, since they even allow some growth of $p^\varepsilon, p^\varepsilon{}_{,x}$.

Of course, a complete mathematical proof of the convergence of u^ε to u^0 would necessitate proving these hypothesis.

3. A RELATED RESULT

The proof of Theorem 1 is based on a related result which has some interest on its own. We state this result.

THEOREM 2. *Let u^ε and u^0 be the solutions of the Navier-Stokes and Euler equations above and let $T > 0$ be fixed.*

We assume that there exist two constants α and $\kappa_3, 3/4 \leq \alpha < 1$, such that

$$\int_0^T \left|\frac{\partial u_2^\varepsilon}{\partial x}\right|^2_{L^2(\Omega\cap\Gamma_{\varepsilon^\alpha})} dt \leq \kappa_3 \varepsilon^{3-4\alpha}, \tag{3.1}$$

where Γ_r is the r-neighborhood of Γ.

Then, there exists a constant κ_4 independent of ε such that

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T;H)} \leq \kappa_4 \varepsilon^{2(1-\alpha)/5}. \tag{3.2}$$

REMARK 3. On physical grounds, we expect that the normal component u_2^ε does not display a boundary layer phenomenon since $u_2^\varepsilon = u_2^0 = 0$ on Γ, although its normal derivative $\frac{\partial u_x^\varepsilon}{\partial y}$ might display such a boundary layer. Hence, the assumption (3.1) is physically reasonable as well, although this is less transparent than for hypotheses (2.9),(2.10) in Theorem 1.

REMARK 4. Condition (3.1) is also close to be mathematically necessary for (3.2) to be true. Indeed by the energy equations for u^ε and u^0,

$$|u^\varepsilon(T)|_H^2 + 2\varepsilon \int_0^T |u^\varepsilon|_V^2 \, dt = |u_0|_H^2 + 2\int_0^T (f, u^\varepsilon)_H \, dt,$$

$$|u^0(T)|_H^2 = |u^0|_H^2 + 2\int_0^T (f, u^0)_H \, dt.$$

Hence, if (3.2) occurs then, as $\varepsilon \to 0$,

$$\varepsilon \int_0^T |u^\varepsilon|_V^2 \, dt = \varepsilon \int_0^T |\nabla u^\varepsilon|_{L^2(\Omega)}^2 \, dt \to 0,$$

which implies (3.1) with $\alpha = 1$. ∎

The details of the proof of Theorem 2 appear in [15]; the principle of the proof is as follows.

We consider as in [11] a divergence free function φ^ε which agrees with $-u^0$ on the wall (i.e., on $\partial\Omega_\infty$), so that $w^\varepsilon = u^\varepsilon - u^0 - \varphi^\varepsilon$ is divergence free and vanishes on $\partial\Omega_\infty$; see [11] for the construction and properties of φ^ε.

Then we consider the equations, boundary conditions and initial conditions satisfied by w^ε. Finally we write the energy equation satisfied by w^ε and we properly estimate from above all the nonpositive terms.

4. PROOF OF THEOREM 1

We multiply the Navier-Stokes equation (2.1) by $-\varepsilon\Delta u^\varepsilon$ and integrate over Ω. For the nonlinear term we use the property, valid in space dimension two, for the present boundary conditions that

$$\int_\Omega [(u^\varepsilon \cdot \nabla) u^\varepsilon] \cdot \Delta u^\varepsilon \, dx\, dy = \sum_{i,j=1}^{2} \int_\Omega u_i^\varepsilon \frac{\partial u_j^\varepsilon}{\partial x_i} \Delta u_j^\varepsilon \, dx\, dy = 0.$$

Hence, we find

$$\frac{\varepsilon}{2}\frac{d}{dt}|\nabla u^\varepsilon|_{L^2}^2 + \varepsilon^2 |\Delta u^\varepsilon|_{L^2}^2 = -\varepsilon (f, \Delta u^\varepsilon)_{L^2} + \varepsilon \int_\Omega \nabla p^\varepsilon \Delta u^\varepsilon \, dx\, dy;$$

we then conclude by integration by parts on the term involving p^ε.

REFERENCES

1. W. Eckhaus, *Asymptotic Analysis of Singular Perturbations*, North-Holland, (1979).
2. P. Fife, Considerations regarding the mathematical basis for Prandtl's boundary layer theory, *Arch. Rational Mech. Anal.*, **38**, 184–216, (1967).
3. T. Kato, On the classical solutions of two dimensional nonstationary Euler equations, *Arch. Rational Mech. Anal.*, **25**, 188–200, (1967).
4. J.L. Lions, Perturbations singuliéres dans les problèmes aux limites et en contrôle optimal, In *Lecture Notes in Math 323*, Springer-Verlag, New York, (1973).
5. O. Oleinik, The Prandtl system of equations in boundary layer theory, *Dokl. Akad. Nauk S.S.S.R. 150* **4** (3), 583–586, (1963).
6. M.I. Vishik and L.A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with small parameter, *Uspekki Mat. Nauk* **12**, 3–122, (1957).
7. G.I. Barenblatt and A.J. Chorin, Small viscosity asymptotics for the inertial range of local structure and for the wall region of wall-bounded turbulent shear flow, *Proc. Natl. Acad. Sci., Applied Mathematics*, **93**, (1996).
8. G.I. Barenblatt and A.J. Chorin, Scaling laws and zero viscosity limits for wall-bounded shear flows and for local structure in developed turbulence (preprint), Center for Pure and Applied Mathematics, University of California at Berkeley, n° PAM–678, (1996).
9. M. Sammartino and R.E. Caflisch Zero viscosity limit for analytic solutions of the Navier-Stokes equations on a half-space, I and II, Preprint, (1996).
10. P. Constantin and J. Wu, Inviscid limit for vortex patches, *Nonlinearity* **8**, 735–742, (1995).
11. R. Temam and X. Wang, Asymptotic analysis of the linearized Navier-Stokes equations in a channel, *Differential and Intergral Equations* **8** (7), 1591–1618, (1995).
12. R. Temam and X. Wang, Asymptotic analysis of the linearized Navier-Stokes equations in a general 2D domain, *Asymptotic Analysis* **9**, 1–30, (1996).
13. R. Temam and X. Wang, Asymptotic analysis of Oseen type equations in a channel at small viscosity, *Indiana Univ. Math. J.*, **45** (3), 863–916, (1996).
14. S.N. Alekseenko, Existence and asymptotic representation of weak solutions to the flowing problem under the condition of regular slippage on solid walls, *Siberian Math. J.*, **35** (2), 209–229, (1994).

15. R. Temam and X. Wang, On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity, *Annali della Scuola Normale Superiore di Pisa* (to appear).
16. X. Wang, On the Euler equation in a channel, (in preparation).
17. J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, (1969).
18. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, (1984).
19. R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.*, **20**, 32–43, (1975).
20. F. Browder, Editor, Remarks on the Euler equations, In *Proc. of Symposia in Pure Mathematics*, Volume 45, pp. 429–430, AMS, Providence, RI, (1986).
21. L. Lichtenstein, *Grundlagen der Hydromechanik*, Springer-Verlag, (1923).
22. W. Wolibner, Un théorème sur l'existence du mouvement plan d'un fluide parfait homogène incompressible, pendant un temps infiniment long, *Math. Z.*, **39**, 698–726, (1933).
23. J.T. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equations, *Comm. Math. Phys.*, **94**, 61–66, (1984).
24. A.J. Chorin, Turbulence as a near-equilibrium process, in *Lecture in Applied Mathematics,* **31**, 235-249, (1996).
25. D.G. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Arch. Rational Mech. Anal.*, **46**, 241–279, (1972).