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Attractor dimension estimates for two-dimensional shear flows

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Abstract

We study the large time behavior of boundary and pressure-gradient driven incompressible fluid flows in elongated two-dimensional channels with emphasis on estimates for their degrees of freedom, i.e., the dimension of the attractor for the solutions of the Navier–Stokes equations. For boundary driven shear flows and flux driven channel flows we present upper bounds for the degrees of freedom of the form $c\alpha\, Re^{3/2}$ where c is a universal constant, α denotes the aspect ratio of the channel (length/width), and Re is the Reynolds number based on the channel width and the imposed "outer" velocity scale. For fixed pressure gradient driven channel flows we obtain an upper bound of the form $c'\alpha\, Re^2$, where c' is another universal positive constant and the Reynolds number is based on a velocity defined by the infimum, over all possible trajectories, of the time averaged mass flux per unit channel width. We discuss these results in terms of physical arguments based on small length scales in turbulent flows. Copyright © 1998 Elsevier Science B.V.

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1. Introduction

The connection between finite-dimensional dynamical systems theory and the long-time behavior of solutions of a priori infinite-dimensional continuum systems described by partial differential equations is of great interest and importance. Indeed, applications of many results and ideas from dynamical systems theories to continuum problems such as turbulence in fluid dynamics depend on this relationship. In recent years, techniques have been developed to establish this connection in a rigorous and quantitative way by showing how the dimension of the global attractor may be estimated for some dissipative partial differential equations (see e.g. [1,4,22]). Of primary interest are fluid dynamical systems where the study of the behavior of solutions on the attractor for strongly driven systems is the essence of turbulence theory. The sharpest results to date in these studies regarding the two-dimensional Navier–Stokes equations have been for the case of a fixed time-independent force $f(x)$ driving a (possibly turbulent) flow

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in a periodic box of side length L. In that case it has been proven that the global attractor dimension has an upper bound of the form [5,8].

$$d_{\text{att}} \le cG^{2/3}(1+\log G)^{1/3},$$

where c is an absolute constant and the Grashof number G is

$$G = \frac{L^2\|f\|_2}{\nu^2} = \frac{L^3 f_{\text{rms}}}{\nu^2}.$$

See [18] for a rigorous lower bound which agrees with this upper bound within logarithmic terms. Furthermore, disregarding the logarithmic term, the $G^{2/3}$ scaling is consistent with the number of degrees of freedom in a flow whose smallest scale is the Kraichnan length determined by the viscosity ν and the entropy dissipation rate χ according to $l_{\text{Kraichnan}} \sim (\nu^3/\chi)^{1/6}$. Hence, one way to interpret the attractor dimension estimate is in terms of the implied small scale in the flow. That is, we may identify a small scale l in (presumably isotropic) two-dimensional turbulent flows via the relation $d_{\text{att}} \sim L^2/l^2$. For the case of forced flow in a periodic box the rigorous result is not completely satisfactory from a heuristic point of view because the ostensibly "intensive" small scale thereby estimated gets smaller and smaller for larger and larger systems at fixed driving intensity due to the logarithmic term:

$$l \sim \frac{L}{d_{\text{att}}^{1/2}} \ge c' \frac{\nu^{2/3}}{f_{\text{rms}}^{1/3}} \frac{1}{(1+\log(L^3 f_{\text{rms}}/\nu^2))^{1/6}} \to 0$$

as $L \to \infty$ with fixed aspect ratio of the box (c' is another absolute constant). The requirement on the attractor dimension estimate, in order that this interpretation in terms of a small scale makes sense, is that it be strictly proportional to the system size (area, in two dimensions) with the remaining factors defining a length scale that only depends on the "intensity" of the driving that maintains the dynamics (in this externally forced case, f_{rms}) and the viscosity in such a way that the scale does not vanish for large systems. This is similar in spirit to the notion of a thermodynamic limit for the small length scale defined by the attractor dimension, i.e., a dimension density. In different situations we may interpret the "intensity" in different ways, but in each case predictions may in principle be tested.

In this paper we study the attractor dimension and the associated small length scales for two-dimensional flows confined between rigid no-slip boundaries and driven by (i) a boundary-imposed shear gradient, (ii) a fixed-flux constraint, or (iii) an applied pressure gradient. We consider only simple rectangular channel geometries for the domain of length L in the x (x_1)-direction and height h in the y (x_2)-direction. The rigid boundaries are at $y = 0$ and $y = h$, and in each case periodic conditions are imposed in the x-direction. The aspect ratio of the domain is

$$\alpha = \frac{L}{h}.$$

For a fluid of kinematic viscosity ν and flows with an imposed velocity scale U, there is a natural dimensionless measure of the "intensity" of the driving, namely the Reynolds number

$$Re = \frac{Uh}{\nu}.$$

The situation we are interested in here is high intensity driving ($Re \gg 1$) in long channels ($\alpha \gg 1$).

The results of our study for the shear- and flux-driven flows is compactly summarized in the inequality

$$d_{\text{att}} \le c\alpha Re^{3/2}, \tag{1.1}$$

where c is an absolute constant. This result may be interpreted in terms of the small length scale estimate

$$l \sim \frac{L}{d_{\mathrm{att}}^{1/2}} \geq c' h\, Re^{-3/4} = c' \frac{h^{1/4} \nu^{3/4}}{U^{3/4}}. \tag{1.2}$$

This lower bound is independent of the system length L and so makes sense for long channels. The intensity of the driving necessarily depends on one of the boundary length scales (here, h) which sets the natural "outer" length scale for these flows. We observe that the estimate displays all the correct qualitative behavior with regard to the remaining variables: this turbulent length scale increases as the viscosity increases and smaller eddies are dissipated more rapidly, and it decreases as the velocity scale increases driving the flow harder.

Moreover, the length scale in (1.2) is precisely the familiar Kolmogorov length scale from three-dimensional turbulence theory. There, the key scaling hypothesis is that the rate of energy dissipation becomes independent of the Reynolds number in the limit $Re \to \infty$. For long channels the length L should not matter with regard to "intensive" quantities, so given this assumption the dissipation per unit mass ϵ must be only a function of U and h. By dimensional analysis this implies

$$\epsilon \sim \frac{U^3}{h}.$$

The Kolmogorov scale is then presumed to be the smallest effective eddy size in the flow, i.e., the length scale below which viscosity dominates the local dynamics. This means that it is the only length scale which can be constructed solely from the viscosity ν and the energy dissipation rate ϵ:

$$l_{\mathrm{Kolmogorov}} \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim h\, Re^{-3/4}.$$

The appearance of this Kolmogorov length scale for two-dimensional flows is somewhat surprising because it is usually associated with three-dimensional turbulence. The Kraichnan scale might be more natural for these two-dimensional flows, and mimicking the scaling argument given above we can predict what an attractor dimension estimate would be, were this the case. That is, if the entropy dissipation rate (per unit mass) χ was independent of viscosity (and the channel length) then by dimensional analysis we must have

$$\chi \sim \frac{U^3}{h^3},$$

so the Kraichnan length would be

$$l_{\mathrm{Kraichnan}} \sim \left(\frac{\nu^3}{\chi}\right)^{1/6} = h\, Re^{-1/2}.$$

One would then expect an attractor dimension estimate of the form

$$Lh / l_{\mathrm{Kraichnan}}^2 \sim \alpha\, Re,$$

lower than our result by a factor of $Re^{1/2}$ but with precisely the same dependence on aspect ratio α. (We remark that these identifications in terms of Kolmogorov or Kraichnan lengths gloss over the fact that we are not considering homogeneous isotropic turbulence. However, at high Reynolds numbers it is useful to consider the boundary conditions to be forcing the bulk in which the inhomogeneities and anisotropies may not be so dominant. There is also a no-slip boundary layer which would presumably vary between $h\, Re^{-1/2}$ and $h\, Re^{-1}$ in thickness. But if the degrees of freedom at those scales are localized to thin layers of length L along the boundaries, then we expect them to contribute terms $\mathrm{O}(\alpha\, Re^{1/2})$ to $\mathrm{O}(\alpha\, Re)$ to the attractor dimension. In all these cases this will be no more than the "bulk" contribution to the total count of the degrees of freedom.)

In any case this is perhaps the most fundamental correspondence: with all other parameters and constraints on the channel flow held fixed, the attractor dimension for turbulent flows ought to be no more than directly proportional to the length of the channel. And unless the flows generate ever-longer correlations along ever-longer channels, then we expect precisely a linear dependence of the attractor dimension on the channel length. Indeed, if an associated time-independent laminar solution – a fixed point on the attractor for the system – shows a linear instability at some value Re_c of the Reynolds number, as is the case with plane Poiseuille flow for example, then for $Re > Re_c$ there will be a band of unstable horizontal wavenumbers. See, for instance, [10] for a curve of marginal stability of plane Poiseuille flow in an infinite channel, and note that the number of unstable horizontal wave numbers with horizontal periodicity L depends linearly on the aspect ratio for fixed Reynolds number. The number of unstable directions about such a fixed point bounds the attractor dimension from below [1], so we conclude for such problems that the linear dependence of the attractor dimension on α is sharp.

For the pressure gradient driven channel flow we obtain an upper bound for the degrees of freedom of the form $c'\alpha\, Re^2$, where c' is another universal positive constant and the typical velocity is defined as the infimum over all possible trajectories of the time averaged mass flux per unit length. In this configuration we must estimate a velocity scale associated with the applied pressure gradient in order to identify a Reynolds number. This is not the expected result in the light of our estimates for the fixed shear and fixed flux flows: we anticipated a similar $Re^{3/2}$ dependence of the bound, but some technical difficulties concerning the extent of the global attractor for the problem as we define it, prevent our obtaining that result (see calculation in the next section). If a functional invariant set were considered instead of the entire global attractor in the pressure gradient driven case, then we may derive the same kind of estimates as in the shear driven and fixed-flux channel flow case where an "alternative" typical velocity on this functional invariant set is defined as the maximum over all possible trajectories of the time averaged horizontal mass flux per unit length.

Regardless of the accuracy of the Reynolds number dependence of the estimate in these results, however, the linear dependence of the attractor dimension estimate on L in (1.1) represents some progress beyond the best previous results – the Grashof number bound for periodic boundary conditions [5,8] – as far as the system size dependence is concerned (but see also the very recent work of Zaine [26] for further developments for the fully periodic case). Here we have established that the associated small length scale remains an effective and physically meaningful bound in the limit of infinite domain area, in this case meaning in the limit of an infinitely long channel.

There are two key technical ingredients that go into the new results for these two-dimensional boundary condition driven flows. The first is the use of rigorous bounds on the time averaged energy dissipation rate for solutions on the attractor using the method of Hopf [14] as developed in [3,6,7]. These give Kolmogorov-type bounds on the energy dissipation rate, independent of viscosity at large Reynolds numbers. The second is a re-working of the Lieb–Thirring inequality for this geometry to sharpen the dependence of the prefactor on the aspect ratio of the domain. Indeed, as discussed further in the following sections, a naive application of the Lieb–Thirring inequality results in an unphysical dependence of the attractor dimension bound on the channel length.

The rest of this paper is organized as follows. In Section 2 we use the Constantin–Foias–Temam technology to derive upper bounds on the dimension of the attractor for three specific flows, boundary-driven shear flow (the laminar form of which is planar Couette flow), a fixed-flux channel flow, and a pressure-gradient induced channel flow (the latter two laminar forms of which are Poiseuille flow). The proof of the new version of the Lieb–Thirring inequality constitutes the content of Section 3.

2. The dimension of attractors

We consider the Navier–Stokes equations for incompressible Newtonian fluid in an elongated channel $\Omega = (0, L) \times (0, h)$, $\Omega_\infty = \mathbb{R}^1 \times (0, h)$ with $L \geq h$:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

where $u = (u_1, u_2)$ is the velocity vector field, p is the pressure field determined by the divergence-free condition on u and f is a body force. The boundary conditions on u are taken to be no-slip in the vertical (y) direction, and periodic in the horizontal (x) direction with period L. The initial velocity vector field $u|_{t=0} = u_0$ is square integrable.

The first case here is boundary driven shear flow (the geometry for planar Couette flow) where the upper boundary is held at rest while the lower boundary moves with velocity $\mathbf{i}U$ ($\mathbf{i}$ is the unit vector in the horizontal direction) and there is no body force.

For pressure-gradient driven flow (the usual set-up for plane Poiseuille flow) both the upper and lower boundaries are at rest and the body force takes the form

$$f = \mathbf{i}\frac{P}{L}, \tag{2.3}$$

where P is a positive constant, the applied pressure drop.

For the fixed-flux problem (an alternative set-up for plane Poiseuille flow) we take

$$f(t) = \mathbf{i}\frac{\nu}{Lh}\int_0^L \left[\frac{\partial u_1}{\partial y}(x, 0, t) - \frac{\partial u_1}{\partial y}(x, h, t)\right] \mathrm{d}x$$

and we require that the initial velocity field satisfies

$$\int_0^h \mathbf{i} \cdot u_0 \,\mathrm{d}y = Uh.$$

The Navier–Stokes evolution with body force above, combined with the incompressibility condition, ensures the constant flux constraint

$$\int_0^h u_1(x, y, t)\,\mathrm{d}y = Uh.$$

(In the vorticity-stream function formulation this constant flux constraint may be simply implemented with inhomogeneous Dirichlet conditions on the stream function, but we choose to set it up this way to enable a more unified analysis for the attractor dimension estimate.)

We observe that, after homogenizing the boundary conditions if necessary, the Navier–Stokes equations can be viewed as dynamical systems on Hilbert space H specified below (see e.g. [1,4,9,12,13,22,27] among many other references):

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v, v) + R(v) = \tilde{f}, \tag{2.4}$$

where $A = -\mathbf{P}\Delta$ is the Stokes operator with homogeneous no-slip boundary condition in the vertical direction and periodic boundary condition in the horizontal direction, $\mathbf{P}$ is the Leray–Hopf projector from $(L^2(\Omega))^2$ onto H, where

$$H = \{v \in (L^2_{\mathrm{loc}}(\Omega_\infty))^2,\ \mathrm{div}\ v = 0,\ v_2 = 0 \text{ on } \partial\Omega_\infty\}, \tag{2.5}$$

$$V = \{v \in (H^1_{\mathrm{loc}}(\Omega_\infty))^2,\ \mathrm{div}\ v = 0,\ v = 0 \text{ on } \partial\Omega_\infty\}, \tag{2.6}$$

with periodicity in x understood. The bilinear operator is defined as

$$\langle B(u, v)w\rangle = ((u \cdot \nabla)v, w) \quad \text{for } u, v, w \in V. \tag{2.7}$$

The linear operator R and the adjusted body force $\tilde{f}$ take different forms for the different cases. In the boundary driven shear flow case $v = u - \phi$ where

$$\phi(x, y) = (\varphi(y), 0), \tag{2.8}$$

and φ may be chosen as (a mollified version, if you wish, of) the piecewise linear velocity profile (see [6,7])

$$\varphi(y) = \begin{cases} (U/2\delta)(2\delta - y), & 0 \le y \le \delta, \\ U/2, & \delta \le y \le h - \delta, \\ (U/2\delta)(h - y), & h - \delta \le y \le h, \end{cases} \tag{2.9}$$

where δ, referred to as the boundary layer thickness, is a parameter to be specified later. Then R is defined by

$$\langle Rv, w\rangle = ((v \cdot \nabla)\phi, w) + ((\phi \cdot \nabla)v, w) \quad \text{for } v, w \in V, \tag{2.10}$$

and

$$\tilde{f} = \nu\mathbf{P}\Delta\phi - B(\phi, \phi) = \mathbf{i}\nu\varphi''(y). \tag{2.11}$$

The vector field ϕ is the *background flow* and the construction in [6,7] is an extension of the original construction of Hopf [14]; see also [23] for similar constructions for general two-dimensional smooth domain and [25] for general three-dimensional smooth domain).

In the case of pressure-gradient driven flow we have

$$v = u,\ \phi = 0,\ \tilde{f} = f = \mathbf{i}\frac{P}{L}. \tag{2.12}$$

In the case of the fixed-flux problem we take $v = u - \phi$ using $\phi(x, y) = (\varphi(y), 0)$ where the background profile function, for a specified mean velocity U and boundary layer thickness $\delta \le h/2$ is (a mollified version, if you wish, of)

$$\varphi(y) = \begin{cases} \frac{y}{\delta}\frac{U}{1-\delta/h}, & 0 \le y \le \delta, \\ \frac{U}{1-\delta/h}, & \delta \le y \le h - \delta, \\ \frac{h-y}{\delta}\frac{U}{1-\delta/h}, & h - \delta \le y \le h. \end{cases}$$

Then we have R and $\tilde{f}$ defined by

$$\langle Rv, w\rangle = ((v \cdot \nabla)\phi, w) + ((\phi \cdot \nabla)v, w) + \frac{\nu}{Lh}\left(\int_0^L \mathrm{d}x \int_0^h \mathrm{d}y\, w_1\right)\left(\int_0^L \left[\frac{\partial v_1}{\partial y}(x, h, t) - \frac{\partial v_1}{\partial y}(x, 0, t)\right] \mathrm{d}x\right)$$

and

$$\tilde{f} = \mathbf{i}\left(\nu\varphi''(y) + \frac{\nu}{h}(\varphi'(0) - \varphi'(h))\right).$$

Note in this case that, by construction,

$$\int_0^h v_1(x, y, t)\,\mathrm{d}y = 0$$

for each value of x.

Existence and uniqueness of strong solutions of these problems are classical results; see e.g. [16,18,21,27]. Furthermore, there exists a finite-dimensional global attractor $\mathcal{A}$ (see e.g. [4,22, Chapter 3]. Miranville and Wang [20] derived an upper bound of the form $c(\Omega)Re^{3/2}$ on the dimension of attractor for two-dimensional boundary driven shear flows, but those estimates do not have an explicit dependence on the aspect ratio α. In order to sort out the dependence on α, and hence on L, we apply the Constantin–Foias–Temam [4] approach with special care taken whenever an expression appears which might depend on the aspect ratio.

We observe that (2.4) can be rewritten as

$$\dot{v} = F(v), \tag{2.13}$$

with

$$F(v) = \tilde{f} - \nu A v - B(v, v) - R(v). \tag{2.14}$$

We take the first variation of (2.13) to find

$$\dot{W} = F'(v)W, \quad W(0) = \xi, \tag{2.15}$$

where $F'(v)$ acts according to

$$F'(v)W = -(\nu AW + B(v, W) + B(W, v) + R(W)). \tag{2.16}$$

Now for a particular time τ, let $\varphi_j(\tau)$, $j \in \mathbb{N}$, be an orthonormal basis of H, with $\varphi_1(\tau), \ldots, \varphi_m(\tau)$ spanning $Q_m H(\tau) = \mathrm{Span}[W_1(\tau), \ldots, W_m(\tau)]$, where $W_1, \ldots, W_m$ are m solutions of (2.30) corresponding to $\xi = \xi_1, \ldots, \xi_m$. We then have $\varphi_j(\tau) \in V$, $j = 1, \ldots, m$ for a.e. $\tau \in \mathbb{R}_+$. The trace of $F'(v(\tau)) \circ Q_m(\tau)$ is given by

$$\begin{aligned} &Tr(F'(v(\tau)) \circ Q_m(\tau)) \\ &\quad = \sum_{j=1}^{\infty}((F'(v(\tau)) \circ Q_m(\tau))\varphi_j(\tau), \varphi_j(\tau)) = \sum_{j=1}^{m}(F'(v(\tau))\varphi_j(\tau), \varphi_j(\tau)). \end{aligned}$$

We then obtain, in light of (2.7), (2.10) and (2.16),

$$Tr(F'(v(\tau)) \circ Q_m(\tau)) \le -\nu \sum_{j=1}^{m} |\nabla\varphi_j(\tau)|_2^2 + |\nabla(v(\tau) + \phi)|_2 |\rho(\tau)|_2,$$

where

$$\rho(\tau) = \sum_{j=1}^{m} |\varphi_j(\tau)|^2.$$

Note that in order to derive this last estimate in the fixed flux case we explicitly used

$$\int_0^h \mathbf{i} \cdot \varphi_j(x, y, t)\, \mathrm{d}y = 0.$$

Then, thanks to the anisotropic version of the classical Lieb–Thirring inequality (3.8) (Lemma 1, Section 3), we have

$$Tr(F'(v(\tau)) \circ Q_m(\tau)) \leq -\frac{\nu}{2} \sum_{j=1}^{m} |\nabla \varphi_j(\tau)|_2^2 + \frac{\kappa}{2\nu} |\nabla(v(\tau) + \phi)|_2^2 + \frac{\tilde{\kappa} m \nu}{2\kappa L^2}, \tag{2.17}$$

where κ and $\tilde{\kappa}$ are universal positive constants.

Now

$$m = \int_\Omega \rho(x, y)\, \mathrm{d}x\, \mathrm{d}y \leq |\Omega|^{1/2} \left(\int_\Omega \rho^2(x, y)\, \mathrm{d}x\, \mathrm{d}y \right)^{1/2}.$$

This implies, when combined with (3.8) (Section 3, Lemma 1),

$$\frac{m^2}{Lh} \leq \int_\Omega \rho^2(x, y)\, \mathrm{d}x\, \mathrm{d}y \leq \kappa \sum_{j=1}^{m} |\nabla \varphi_j(\tau)|_2^2 + \frac{\tilde{\kappa} m}{L^2}. \tag{2.18}$$

Thus we obtain

$$q_m \leq -\frac{\nu m^2}{2\kappa L h} + \frac{\tilde{\kappa} m \nu}{2\kappa L^2} + \frac{\kappa L h}{2\nu^2} \bar{\epsilon}, \tag{2.19}$$

where

$$q_m = \limsup_{t \to +\infty} \sup_{v_0 \in \mathcal{A}} \sup_{\substack{\xi_i \in H \\ |\xi_i| \leq 1 \\ i=1,\ldots,m}} \left(\frac{1}{t} \int_0^t \mathrm{Tr}(F'(v(\tau)) \circ Q_m(\tau))\, \mathrm{d}\tau \right),$$

and, recalling $u = v + \phi$,

$$\bar{\epsilon} = \frac{\nu}{Lh} \limsup_{t \to +\infty} \sup_{v_0 \in \mathcal{A}} \left(\frac{1}{t} \int_0^t |\nabla u(\tau)|_2^2\, \mathrm{d}\tau \right). \tag{2.20}$$

Let

$$m = \left[\max \left\{ \frac{2\tilde{\kappa}}{\alpha}, \sqrt{2}\kappa \alpha h^2 \bar{\epsilon}^{1/2} \nu^{-3/2} \right\} \right] + 1. \tag{2.21}$$

Then we have, denoting the dimension of the attractor by $\mathrm{d}(\bar{\epsilon})$,

$$\mathrm{d}(\bar{\epsilon}) \leq 2m. \tag{2.22}$$

Now the game is to estimate the time averaged energy dissipation rate per unit mass, i.e. $\bar{\epsilon}$.

In the case of boundary driven shear flow, we define the natural Reynolds number as

$$Re = \frac{hU}{\nu}, \tag{2.23}$$

and set

$$\delta = 4\sqrt{2}\frac{\nu}{U} = 4\sqrt{2}h\, Re^{-1}.$$

We then deduce, in exactly the same fashion as in [6,7] that

$$\bar{\epsilon} \le \kappa_0 \frac{U^3}{h}, \tag{2.24}$$

where κ_0 is an absolute constant. This implies, thanks to (2.21) and (2.22), that

$$\dim(\mathcal{A}) \le 2\kappa {\kappa_0}^{1/2} \alpha\, Re^{3/2} \tag{2.25}$$

for large Reynolds number. This completes the argument for the boundary driven shear flow case.

For the fixed flux case, a similar estimate to (2.24) may be derived using the same methods. Indeed, given the Reynolds number based on the mean flow, $Re = hU/\nu > \sqrt{2}$, and making the choice

$$\delta = \frac{h}{1 + Re/\sqrt{2}}$$

we conclude

$$\bar{\epsilon} \le \kappa_1 \left(1 + \frac{\sqrt{2}}{Re}\right)^3 \frac{U^3}{h},$$

where κ_1 is another absolute constant. Hence, for large Reynolds number,

$$\dim(\mathcal{A}) \le 2\kappa {\kappa_1}^{1/2} \alpha\, Re^{3/2}.$$

For the pressure gradient driven flow, we define the instantaneous mass flux in the horizontal direction (x) as

$$\Phi(t) = \int_0^h u_1(t; x, y)\, \mathrm{d}y. \tag{2.26}$$

In light of incompressibility and the boundary condition, this flux is independent of x. Take the inner product of (2.1) with u and integrate over Ω to deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \nu|\nabla u|_2^2 = P\Phi. \tag{2.27}$$

We observe that

$$\Phi \le \alpha^{-1/2}|u|_2 \le \alpha^{-1/2}h|\nabla u|_2 \le \frac{\nu}{2P}|\nabla u|_2^2 + \frac{Ph^2}{2\alpha\nu}. \tag{2.28}$$

When this combined with (2.27), we deduce that

$$\bar{\epsilon} \le \frac{P^2}{\alpha^2\nu}. \tag{2.29}$$

Indeed this bound is optimal for flows characterized only by the applied pressure drop because Poiseuille flow is an exact solution which satisfies the estimate up to an absolute multiplicative constant. This, together with (2.21) and (2.22), implies

$$\dim(\mathcal{A}) \leq 2\kappa \frac{h^2 P}{\nu^2}. \tag{2.30}$$

We would like to interpret the result in terms of the mass flux in the horizontal direction, i.e. Φ, and an associated Reynolds number. For this purpose, we define

$$\underline{U} = \frac{1}{h} \inf_{u_0 \in H} \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \Phi(\tau)\, \mathrm{d}\tau. \tag{2.31}$$

Using the Hopf technique (i.e. using the background flow technique) in exactly the same way as in the work of Constantin and Doering [3], we have

$$h\underline{U} \geq \kappa_1 \frac{h P^{1/2}}{\alpha^{1/2}} - \kappa_2 \nu, \tag{2.32}$$

where κ_1 and κ_2 are two (more) universal positive constants. We then define the new Reynolds number as

$$Re = \frac{h\underline{U}}{\nu}. \tag{2.33}$$

This, when combined with (2.30) and (2.32), yields

$$\dim(\mathcal{A}) \leq \kappa_3 \alpha\, Re^2, \tag{2.34}$$

for large Reynolds number, where κ_3 is another universal positive constant. We can reinterpret this result in the following, although not altogether satisfying, way: if we observe the averaged mass flux averaged velocity to be $\underline{U}$, then the number of degrees of freedom in the flow (in the long time limit) is bounded above by the right-hand side of (2.34). It is the fact that the estimate in (2.29) is saturated by the *laminar* flow, rather than by a turbulent state, that leads to the higher than expected Reynolds number dependence in this bound.

3. The proof of the anisotropic Lieb–Thirring inequality

Let Ω and Ω_∞ be as in the previous section.

It is known that the following type of Lieb–Thirring inequality is true (cf. [11]):

$$\int_\Omega \left(\sum_{j=1}^N \varphi_j(x)^2 \right)^2 \mathrm{d}x \leq c(\Omega) \sum_{j=1}^N \int_\Omega |\nabla \varphi_j(x)|^2\, \mathrm{d}x, \tag{3.1}$$

where $\{\varphi_j, j = 1, \ldots, N\}$ is an orthonormal family in $L^2(\Omega)$ and $\varphi_j \in V$ where

$$V = \{v \in H^1_{\mathrm{loc}}(\Omega_\infty),\ v = 0 \text{ on } \partial\Omega_\infty\} \tag{3.2}$$

and periodicity in the horizontal direction (period L) understood.

Our goal here is to sort out the dependence on the aspect ratio $\alpha = L/h$ in the Lieb–Thirring type inequality (3.1).

A naive approach is to scale the domain in the horizontal direction.

Let

$$\Omega_1 = (0, h) \times (0, h), \tag{3.3}$$

and we define $\psi_j(y)$ on Ω_1 in the following way:

$$\psi_j(y_1, y_2) = \sqrt{\alpha}\varphi_j(\alpha y_1, y_2). \tag{3.4}$$

It is easily checked that $\{\psi_j, j = 1, \dots, N\}$ form an orthonormal family in $L^2(\Omega_1)$ and $\psi_j \in V(\Omega_1)$. Thus the Lieb–Thirring inequality applies to ψ_j on Ω_1 with a constant independent of α. More precisely we have

$$\int_{\Omega_1} \left(\sum_{j=1}^{N} \psi_j(y)^2 \right)^2 \mathrm{d}y \le c_1 \sum_{j=1}^{N} \int_{\Omega_1} |\nabla \psi_j(y)|^2 \, \mathrm{d}y. \tag{3.5}$$

This is equivalent to, thanks to (3.4),

$$\int_{\Omega} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le c_1 \sum_{j=1}^{N} \int_{\Omega} \left\{ \alpha \left| \frac{\partial \varphi_j(x)}{\partial x_1} \right|^2 + \frac{1}{\alpha} \left| \frac{\partial \varphi_j(x)}{\partial x_2} \right|^2 \right\} \mathrm{d}x, \tag{3.6}$$

and it immediately leads to

$$\int_{\Omega} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le c_1 \alpha \sum_{j=1}^{N} \int_{\Omega} |\nabla \varphi_j(x)|^2 \, \mathrm{d}x, \tag{3.7}$$

where c_1 is a non-dimensional constant independent of α.

However, we suspect that (3.7) is not sharp based on two reasons. Firstly, if we use (3.7) to estimate the dimension of the attractor for two-dimensional shear driven flows, we obtain an upper bound which does not agree with physical arguments for the linear dependence on α. Secondly, if we have the pure homogeneous Dirichlet boundary condition, the dependence on α in (3.7) should not be there due to the classical Lieb–Thirring inequality. Thus we suspect that the dependence on α in (3.7) is an artifact of our estimates.

A key observation is the following. If we could write the right-hand side of (3.6) in product form instead of the summation form in terms of the partial derivatives, the α and $1/\alpha$ will cancel and we obtain the desired result.

This inspires us to use a classical approach to Agmon's inequality (see e.g. [2, Chapter 3]). The idea was to consider the whole space case first which does not feel the scaling, write the inequality in product form and then use extension and truncation to extend the result to bounded domain.

This motivates the following new version of the Lieb–Thirring inequality:

Lemma 1. Let $\{\varphi_j, j = 1, \dots, N\}$ be an orthonormal family in $L^2(\Omega)$ and $\varphi_j \in V$. Then there exist absolute constants κ and $\tilde{\kappa}$, such that

$$\int_{\Omega} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le \kappa \sum_{j=1}^{N} \int_{\Omega} |\nabla \varphi_j(x)|^2 \, \mathrm{d}x + \frac{\tilde{\kappa} N}{L^2}. \tag{3.8}$$

Before we prove the lemma, we first derive the product version of the classical Lieb–Thirring inequality in the whole space.

Lemma 2. Let $\{\varphi_j, j = 1, \ldots, N\}$ be an orthonormal family in $L^2(\mathbb{R}^2)$ and $\varphi_j \in H^1(\mathbb{R}^2)$. Then there exists an absolute constant κ_0, such that

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le 2\kappa_0 \prod_{i=1}^{2} \left(\sum_{j=1}^{N} \int_{\mathbb{R}^2} \left(\frac{\partial \varphi_j}{\partial x_i} \right)^2 \mathrm{d}x \right)^{1/2}. \tag{3.9}$$

Proof. Let us recall the classical Lieb–Thirring inequality which indicates that there exists an absolute constant κ_0, such that

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le \kappa_0 \sum_{j=1}^{N} \int_{\mathbb{R}^2} |\nabla \varphi_j|^2 \,\mathrm{d}x. \tag{3.10}$$

Let λ_1, λ_2 be two arbitrary positive real numbers. Let

$$\psi_j(x) = \sqrt{\lambda_1 \lambda_2}\varphi_j(\lambda_1 x_1, \lambda_2 x_2). \tag{3.11}$$

Then

$$\int_{\mathbb{R}^2} \psi_i(x)\psi_j(x)\,\mathrm{d}x = \delta_{ij}, \tag{3.12}$$

where δ_{ij} is the classical Kronecker symbol.

Thus the classical Lieb–Thirring inequality (3.10) applies to ψ_j. In particular, we set

$$\lambda_i = \frac{\prod_{k=1}^{2} (\sum_{j=1}^{N} \int_{\mathbb{R}^2} (\partial \varphi_j / \partial x_k)^2 \,\mathrm{d}x)^{1/4}}{(\sum_{j=1}^{N} \int_{\mathbb{R}^2} (\partial \varphi_j / \partial x_i)^2 \,\mathrm{d}x)^{1/2}}, \tag{3.13}$$

and we observe

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^{N} \psi_j(x)^2 \right)^2 \mathrm{d}x = \lambda_1 \lambda_2 \int_{\mathbb{R}^2} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x, \tag{3.14}$$

and

$$\sum_{j=1}^{N} \int_{\mathbb{R}^2} |\nabla \psi_j(x)|^2 \,\mathrm{d}x = \sum_{j=1}^{N} \int_{\mathbb{R}^2} \lambda_1^2 \left| \frac{\partial \psi_j(x)}{\partial x_1} \right|^2 \mathrm{d}x + \sum_{j=1}^{N} \int_{\mathbb{R}^2} \lambda_2^2 \left| \frac{\partial \psi_j(x)}{\partial x_2} \right|^2 \mathrm{d}x. \tag{3.15}$$

Then (3.9) follows immediately.

This completes the proof of Lemma 2. □

Our next task is to prove a parallel version of Lemma 2 for the bounded domain Ω_1.

Lemma 3. Let $\{\varphi_j, j = 1, \ldots, N\}$ be an orthonormal family in $L^2(\Omega_1)$ and $\varphi_j \in V(\Omega_1)$. Then there exist absolute constants κ_1 and κ_2, such that

$$\int_{\Omega_1} \left(\sum_{j=1}^{N} \varphi_j(x)^2 \right)^2 \mathrm{d}x \le \kappa_1 \left(\sum_{j=1}^{N} \int_{\Omega_1} \left(\frac{\partial \varphi_j}{\partial x_1} \right)^2 \mathrm{d}x + \frac{\kappa_2 N}{h^2} \right)^{1/2} \left(\sum_{j=1}^{N} \int_{\Omega_1} \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \mathrm{d}x \right)^{1/2}. \tag{3.16}$$

Proof. We follow the approach of Ghidaglia et al. [11], namely we try to extend φ_j to the whole space, truncate, normalize and then apply Lemma 2.

Let ξ be a smooth cut-off function satisfying:

$$\xi \in C_0^\infty(\mathbb{R}^1), \xi = 1 \text{on } (-\tfrac{1}{2}, \tfrac{1}{2}), \text{ supp } \xi \in [-1, 1]. \tag{3.17}$$

We define a linear extension operator $\mathcal{E}$ from $V(\Omega_1)$ to $H^1(\mathbb{R}^2)$ in the following way:

$$(\mathcal{E}\varphi)(x) = \begin{cases} \varphi(x), & \text{for } x \in \Omega_1, \\ \xi(x_1/h - 1)\varphi(x), & \text{if } x \in [h, 2h] \times [0, h], \\ \xi(x_1/h)\varphi(x), & \text{if } x \in [-h, 0] \times [0, h], \\ 0, & \text{else.} \end{cases} \tag{3.18}$$

It is easily verified that $\mathcal{E}\varphi \in H^1(\mathbb{R}^2)$ and

$$|\mathcal{E}\varphi|^2_{L^2(\mathbb{R}^2)} \le 3|\xi|^2_\infty |\varphi|^2_{L^2(\Omega_1)}, \tag{3.19}$$

$$\left|\frac{\partial \mathcal{E}\varphi}{\partial x_1}\right|^2_{L^2(\mathbb{R}^2)} \le 3|\xi|^2_\infty \left|\frac{\partial \varphi}{\partial x_1}\right|^2_{L^2(\Omega_1)} + \frac{2}{h^2}|\xi'|^2_\infty |\varphi|^2_{L^2(\Omega_1)}, \tag{3.20}$$

and

$$\left|\frac{\partial \mathcal{E}\varphi}{\partial x_2}\right|^2_{L^2(\mathbb{R}^2)} \le 3|\xi|^2_\infty \left|\frac{\partial \varphi}{\partial x_2}\right|^2_{L^2(\Omega_1)}. \tag{3.21}$$

Let $\eta > 0$ and we define

$$w_j(x) = \begin{cases} \frac{2}{\eta}\sin(j\pi x_1/\eta)\sin(\pi x_2/\eta), & \text{for } x \in (0, \eta) \times (-2\eta, -\eta), \\ 0, & \text{else.} \end{cases} \tag{3.22}$$

Then $\{w_j, j = 1, \dots, N\}$ form an orthonormal family in $L^2(\mathbb{R}^2)$ and $w_j \in H^1(\mathbb{R}^2)$.

Next we try to normalize $\mathcal{E}\varphi$ using w_j. More specifically, we look for $\{\alpha_{ij}, 1 \le i, j \le N\}$, such that

$$\begin{aligned} &\alpha_{ij} = 0 \quad \text{for } i < j, \\ &\psi_1 = \mathcal{E}\varphi_1 + \alpha_{11}w_1, \\ &\psi_2 = \mathcal{E}\varphi_2 + \alpha_{21}w_1 + \alpha_{22}w_2, \\ &\quad\vdots \\ &\psi_N = \mathcal{E}\varphi_N + \alpha_{N1}w_1 + \dots + \alpha_{NN}w_N, \end{aligned} \tag{3.23}$$

satisfying the property

$$\int_{\mathbb{R}^2} \psi_i\psi_j \,\mathrm{d}x = 6|\xi|^2_\infty \delta_{ij}. \tag{3.24}$$

We observe that the support of $\mathcal{E}\varphi_j$ and w_s are disjoint and hence

$$\int_{\mathbb{R}^2} \psi_i\psi_j \,\mathrm{d}x = \int_{\mathbb{R}^2} \mathcal{E}\varphi_i\mathcal{E}\varphi_j \,\mathrm{d}x + \sum_{s=1}^N \alpha_{is}\alpha_{js}. \tag{3.25}$$

Let

$$b_{ij} = 6|\xi|_\infty^2 \delta_{ij} - \int_{\mathbb{R}^2} \mathcal{E}\varphi_i \mathcal{E}\varphi_j \,\mathrm{d}x. \tag{3.26}$$

Then (3.24) is equivalent to solving the system of equations

$$b = \alpha \cdot \alpha^{\mathrm{T}}, \tag{3.27}$$

with $\alpha = (\alpha_{ij})$ being a lower triangular matrix with positive diagonal elements.

Such a system is solvable if and only if b is a symmetric non-negative definite matrix. It is obvious that b is symmetric. For the non-negative definite part we observe for $(\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N$,

$$\begin{aligned} \sum_{i,j=1}^N \zeta_i \zeta_j \int_{\mathbb{R}^2} \mathcal{E}\varphi_i \mathcal{E}\varphi_j \,\mathrm{d}x &= \left| \mathcal{E}\left(\sum_{i=1}^N \zeta_i \varphi_i \right) \right|^2_{L^2(\mathbb{R}^2)}, \\ &\le 3|\xi|_\infty^2 \left| \sum_{i=1}^N \zeta_i \varphi_i \right|^2_{L^2(\Omega_1)} \quad \text{(Thanks to (3.19))}, \\ &\le 3|\xi|_\infty^2 \sum_{i=1}^N \zeta_i^2 \quad (\text{since } \varphi_i \text{ are orthonormal in } L^{2(\Omega_1)}). \end{aligned} \tag{3.28}$$

Thus

$$\sum_{i,j=1}^N \zeta_i \zeta_j b_{ij} = 6|\xi|_\infty^2 \sum_{i=1}^N {\zeta_i}^2 - \sum_{i,j=1}^N \zeta_i \zeta_j \int_{\mathbb{R}^2} \mathcal{E}\varphi_i \mathcal{E}\varphi_j \,\mathrm{d}x \ge 3|\xi|_\infty^2 \sum_{i=1}^N {\zeta_i}^2, \tag{3.29}$$

and hence b is non-negative definite.

Now we observe that the functions

$$\left\{ \frac{\psi_j}{\sqrt{6}|\xi|_\infty}, j = 1, \ldots, N \right\} \tag{3.30}$$

form an orthonormal family in $L^2(\mathbb{R}^2)$ and belong to $H^1(\mathbb{R}^2)$. Thus Lemma 2 applies and we have

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\sum_{j=1}^N \frac{\psi_j^2}{6|\xi|_\infty^2} \right)^2 \mathrm{d}x &\le 2\kappa_0 \prod_{i=1}^2 \left(\sum_{j=1}^N \int_{\mathbb{R}^2} \frac{(\partial\psi_j/\partial x_i)^2}{6|\xi|_\infty^2} \,\mathrm{d}x \right)^{1/2} \\ &= \frac{2\kappa_0}{6|\xi|_\infty^2} \prod_{i=1}^2 \left(\sum_{j=1}^N \int_{\mathbb{R}^2} \left(\frac{\partial\psi_j}{\partial x_i} \right)^2 \mathrm{d}x \right)^{1/2}, \end{aligned} \tag{3.31}$$

or equivalently,

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^N \psi_j^2 \right)^2 \mathrm{d}x \le 12\kappa_0 |\xi|_\infty^2 \prod_{i=1}^2 \left(\sum_{j=1}^N \int_{\mathbb{R}^2} \left(\frac{\partial\psi_j}{\partial x_i} \right)^2 \mathrm{d}x \right)^{1/2}. \tag{3.32}$$

We note that $\psi_j = \varphi_j$ on Ω_1 and thus we deduce

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^{N} \psi_j^2 \right)^2 \mathrm{d}x \geq \int_{\Omega_1} \left(\sum_{j=1}^{N} \varphi_j^2 \right)^2 \mathrm{d}x. \tag{3.33}$$

We also observe, thanks to the orthogonality between $\mathcal{E}\varphi_j$ and w_s and the orthogonality between the w_s's,

$$\begin{aligned} {\alpha_{js}}^2 &= \left| \int_{\mathbb{R}^2} \psi_j w_s \, \mathrm{d}x \right|^2 \\ &\leq |\psi_j|^2_{L^2(\mathbb{R}^2)} |w_s|^2_{L^2(\mathbb{R}^2)} \\ &\leq 6|\xi|^2_\infty \quad \text{(Thanks to (3.22) and (3.24))}. \end{aligned} \tag{3.34}$$

Then

$$\begin{aligned} \sum_{j=1}^{N} \int_{\mathbb{R}^2} \left(\frac{\partial \psi_j}{\partial x_1} \right)^2 d &= \sum_{j=1}^{N} \int_{\mathbb{R}^2} \left\{ \left(\frac{\partial (\mathcal{E}\varphi_j)}{\partial x_1} \right)^2 + \sum_{s=1}^{N} \alpha_{js}^2 \left(\frac{\partial w_s}{\partial x_1} \right)^2 \right\} \mathrm{d}x \\ &\leq 3|\xi|^2_\infty \sum_{j=1}^{N} \int_{\Omega_1} \left| \frac{\partial \varphi_j}{\partial x_1} \right|^2 \mathrm{d}x + \frac{2N}{h^2} |\xi'|^2_\infty + 6N|\xi|^2_\infty \sum_{s=1}^{N} \frac{s^2\pi^2}{\eta^2} \\ &\qquad \text{(Thanks to (3.20), (3.22) and (3.34))} \\ &\leq 3|\xi|^2_\infty \sum_{j=1}^{N} \int_{\Omega_1} \left| \frac{\partial \varphi_j}{\partial x_1} \right|^2 \mathrm{d}x + \frac{2N}{h^2} |\xi'|^2_\infty + \frac{6N^4\pi^2|\xi|^2_\infty}{\eta^2}, \end{aligned} \tag{3.35}$$

and similarly

$$\sum_{j=1}^{N} \int_{\mathbb{R}^2} \left(\frac{\partial \psi_j}{\partial x_2} \right)^2 \mathrm{d}x \leq 3|\xi|^2_\infty \sum_{j=1}^{N} \int_{\Omega_1} \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \mathrm{d}x + \frac{6N^2\pi^2|\xi|^2_\infty}{\eta^2}. \tag{3.36}$$

Thus we may deduce, thanks to (3.32), (3.33), (3.35) and (3.36),

$$\begin{aligned} \int_{\Omega_1} \left(\sum_{j=1}^{N} \varphi_j^2 \right)^2 \mathrm{d}x &\leq 12\kappa_0 |\xi|^2_\infty \left(3|\xi|^2_\infty \sum_{j=1}^{N} \int_{\Omega_1} \left| \frac{\partial \varphi_j}{\partial x_1} \right|^2 \mathrm{d}x + \frac{2N}{h^2} |\xi'|^2_\infty + \frac{6N^4\pi^2|\xi|^2_\infty}{\eta^2} \right)^{1/2} \\ &\quad \times \left(3|\xi|^2_\infty \sum_{j=1}^{N} \int_{\Omega_1} \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \mathrm{d}x + \frac{6N^2\pi^2|\xi|^2_\infty}{\eta^2} \right)^{1/2}. \end{aligned} \tag{3.37}$$

Observe that η is an arbitrary parameter in (3.37), thus we may take the limit for $\eta \longrightarrow \infty$, denoting

$$\kappa_1 = 36\kappa_0 |\xi|^2_\infty \quad \text{and} \quad \kappa_2 = \frac{|\xi'|^2_\infty}{|\xi|^2_\infty}, \tag{3.38}$$

we obtain (3.16).

This completes the proof of Lemma 3. □

Now we are in position to prove the new version of the Lieb–Thirring inequality needed below.

Proof of Lemma 1. We scale the domain and apply Lemma 3.

Let

$$\psi_j(x_1, x_2) = \sqrt{\alpha}\varphi_j(\alpha x_1, x_2) \quad \text{for } (x_1, x_2) \in \Omega_1. \tag{3.39}$$

Then

$$\int_{\Omega_1} \psi_j \psi_i \, \mathrm{d}x = \delta_{ij}. \tag{3.40}$$

Thus Lemma 3 applies to $\{\psi_j, j = 1, \ldots, N\}$.

We observe that

$$\int_{\Omega_1} \left(\sum_{j=1}^{N} \psi_j{}^2\right)^2 \mathrm{d}x = \int_{\Omega_1} \left(\sum_{j=1}^{N} \alpha\varphi_j(\alpha x_1, x_2)^2\right)^2 \mathrm{d}x = \alpha \int_{\Omega} \left(\sum_{j=1}^{N} \varphi_j{}^2\right)^2 \mathrm{d}x, \tag{3.41}$$

$$\int_{\Omega_1} \left(\frac{\partial \psi_j}{\partial x_1}\right)^2 \mathrm{d}x = \int_{\Omega_1} \alpha^3 \left(\frac{\partial \varphi_j(\alpha x_1, x_2)}{\partial x_1}\right)^2 \mathrm{d}x = \alpha^2 \int_{\Omega} \left(\frac{\partial \varphi_j}{\partial x_1}\right)^2 \mathrm{d}x, \tag{3.42}$$

$$\int_{\Omega_1} \left(\frac{\partial \psi_j}{\partial x_2}\right)^2 \mathrm{d}x = \int_{\Omega_1} \alpha \left(\frac{\partial \varphi_j(\alpha x_1, x_2)}{\partial x_2}\right)^2 \mathrm{d}x = \int_{\Omega} \left(\frac{\partial \varphi_j}{\partial x_2}\right)^2 \mathrm{d}x. \tag{3.43}$$

Combining (3.16), (3.41), (3.42) and (3.43) we deduce

$$\alpha \int_{\Omega} \left(\sum_{j=1}^{N} \varphi_j{}^2\right)^2 \mathrm{d}x \le \kappa_1 \left(\alpha^2 \sum_{j=1}^{N} \int_{\Omega} \left(\frac{\partial \varphi_j}{\partial x_1}\right)^2 \mathrm{d}x + \frac{\kappa_2 N}{h^2}\right)^{1/2} \left(\int_{\Omega} \left(\frac{\partial \varphi_j}{\partial x_2}\right)^2 \mathrm{d}x\right)^{1/2}. \tag{3.44}$$

Divide by α, apply Cauchy–Scharwz, we obtain Lemma 1 with

$$\kappa = \frac{\kappa_1}{2} \quad \text{and} \quad \tilde{\kappa} = \frac{\kappa_1 \kappa_2}{2}.$$

This completes the proof of Lemma 1. □

Remark 1. Though only the two-dimensional case and the Dirichlet-Periodic boundary conditions are considered here, the technique carries over to higher dimensions and other boundary conditions. Indeed, one could prove the following result:

Let $\{\varphi_j, j = 1, \ldots, N\}$ be an orthonormal family in $L^2(Q)$ and $\varphi_j \in H^1(Q)$, where $Q = \prod_{i=1}^{n}(0, \alpha_i L)$, $\alpha_i \ge 1$. Then there exist constants $\kappa_0, \kappa_1, \ldots, \kappa_n$, depending on p and n only, such that

$$\left[\int_Q \left(\sum_{j=1}^{N} \varphi_j^2\right)^{p/(p-1)} \mathrm{d}x\right]^{2(p-1)/n} \le \left(\kappa_0 \sum_{j=1}^{N} \int_Q |\nabla \varphi_j|^2 \, \mathrm{d}x + \sum_{i=1}^{n} \frac{\kappa_i N}{\alpha_i^2 L^2}\right) \left(\prod_{i=1}^{n} \alpha_i\right)^{1-2p/n+2/n}, \tag{3.45}$$

with

$$\max\left(1, \frac{n}{2}\right) < p \le 1 + \frac{n}{2}.$$

We leave the detail to the interested reader.

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