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Boundary Layers Associated with Incompressible Navier–Stokes Equations: The Noncharacteristic Boundary Case

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The goal of this article is to study the boundary layer of wall bounded flows in a channel at small viscosity when the boundaries are uniformly noncharacteristic, i.e., there is injection and/or suction everywhere at the boundary. Following earlier work on the boundary layer for linearized Navier–Stokes equations in the case where the boundaries are characteristic (no-slip at the boundary and non-permeable), we consider here the case where the boundary is permeable and thus noncharacteristic. The form of the boundary layer and convergence results are derived in two cases: linearized equation and full nonlinear equations. We prove that there exists a boundary layer at the outlet (downwind) of the form $e^{-Uz/\varepsilon}$ where U is the speed of injection/suction at the boundary, z is the distance to the outlet of the channel, and ε is the kinematic viscosity. We improve an earlier result of S. N. Alekseenko (1994, *Siberian Math. J.* **35**, No. 2, 209–230) where the convergence in L^2 of the solutions of the Navier–Stokes equations to that of the Euler equations at vanishing viscosity was established. In the two dimensional case we are able to derive the physically relevant uniform in space (L^∞ norm) estimates of the boundary layer. The uniform in space estimate is derived by properly developing our previous idea of better control on the tangential derivative and the use of an anisotropic Sobolev imbedding. To the best of our knowledge this is the first rigorously proved result concerning boundary layers for the full (nonlinear) Navier–Stokes equations for incompressible fluids.

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Contents

0. INTRODUCTION

Earth is surrounded by fluids such as air and water whose dynamics are characterized by the Navier–Stokes equations for incompressible homogeneous Newtonian fluids to a good approximation (see Section 1 for the equations, see also [7, 9, 18, 19, 21, 28, 30] among many others). One characteristic of fluids like air and water is the smallness of their kinematic viscosity. Hence one may formally drop the viscosity term in the Navier–Stokes equations and arrive at the Euler equations for incompressible inviscid homogeneous fluids (see Section 1 for the equations). This raised the natural question whether the Euler equations are a good approximation of the Navier–Stokes equations at small viscosity. In other words, we would like to know if the solutions to the Navier–Stokes equations converge to the solutions of the Euler equations as the viscosity decreases to zero for suitable fixed data.

In the case of no physical boundary, i.e., the case when the fluid occupies the whole space, or the case with periodic boundary conditions, or the case with free[1] boundary condition, the convergence is true (see for instance the work of T. Kato [16] and Swann [25]). Indeed Yudovitch [38] proved the global (in time) existence and uniqueness of solutions of the two-dimensional Euler equations by obtaining uniform (in terms of viscosity) estimates of solutions to the Navier–Stokes equations with free (free-slip) boundary condition and passed to the limit as the viscosity approaches zero. More recently it was shown that the convergence is true even for certain rough data (vortex patch) in the case of no physical boundary (see for instance Constantin and Wu [8]). Mathematically speaking, due to the absence of a physical boundary, the Navier–Stokes system viewed as a perturbation of the Euler system is a regular perturbation problem. Hence the desired convergence result is relatively easier to derive (in the smooth case) when there is no boundary.

In the physically interesting case where there exist physical boundaries, and hence the fluid velocity is specified at the physical boundary, the problem of convergence is essentially open. It is clear that due to the

[1] Free in the sense of free-slip in contrast to no-slip; not to be confused with the usual free boundary value problems.

disparity of boundary conditions between the one for the Navier–Stokes equations (no-slip) and the one for the Euler equations (free-slip, impermeable), the convergence of derivatives (say H^1) or uniform in space convergence (L^∞) cannot be true. Thus the right question to ask concerns the convergence in the interior and/or convergence in some averaged sense (say L^2) in the presence of a solid wall. This problem has been open for a long time, despite the fact that existence and uniqueness of smooth solutions for both the Navier–Stokes and the Euler equations are known for bounded domains with a wall (for all time in space dimension two, for a small interval of time in space dimension 3; see, e.g., [7, 19, 28] for the Navier–Stokes equations and [2, 26, 27] for the Euler equations). The general goal is to investigate if such a convergence in the interior is true. If it is not true, one would like to derive the effective equation at vanishing viscosity (perhaps a formidable task). If the convergence is true, one would like to study what corrections (*correctors*) are needed to ensure the uniform convergence or convergence of derivative. In other words, one would like to study the boundary layer behavior since it is believed that the difference between the solutions of the Navier–Stokes equations and that of the Euler equations is mainly concentrated in a narrow region called the *boundary layer* near the boundary.

The study of the boundary layer is of great physical and engineering importance. There are quite a few partial results in the case with a solid wall: see for instance, Ladyzhenskaya [18] who established the convergence in L^2 of the linearized problem, Alekseenko [1] who established the convergence in L^2 when the boundary is noncharacteristic, Temam and Wang [31, 32, 34] on boundary layers analysis for linearized Navier–Stokes equations, Asano [3] and Sammartino and Caflisch [4] on analytic solutions of the Navier–Stokes system in half-space, Kato [17] on convergence under the assumption that energy dissipation rate approaches zero at vanishing viscosity, and Temam and Wang [33] on convergence under mild assumptions on the tangential derivative of the velocity field or the behavior of pressure at the physical boundary, among many others. See also the survey paper by Weinan E [11]. Hence, to the best of our knowledge, the articles of Alekseenko [1] and Sammartino and Caflisch [4] are the only fully rigorously proven convergences of the solutions of the incompressible NSE to that of the Euler equations: [4] relates to analytic solutions and [1] studies the case of noncharacteristic boundaries, but the convergence is in L^2 and convergence in H^1 and the question of the boundary layer are not addressed.

The difficulty in proving the convergence and analyzing the near boundary behavior is fourfold. First it is a singular perturbation problem since there is a disparity of boundary conditions between the viscous and inviscid problems. This is in contrast to the case without physical boundary.

Second this is a nonlinear problem due to the nonlinear convection term. Third this is a global (nonlocal) equation due to the presence of the pressure term which ensures the incompressibility condition. Fourth the boundaries are parabolic (characteristic) boundaries in the case of a no-slip boundary condition.

In this article we study the asymptotic behavior of the Navier–Stokes equations, both linear and nonlinear, with small viscosity when the physical boundary is *noncharacteristic* (i.e., when $v \cdot n \neq 0$ at the boundary, where v is the velocity field and n is the unit outward normal). Physically such a situation could happen in fluid control where fluids are injected and/or sucked out of the region occupied by the fluids. This could also arise if we consider a moving domain and apply a Galilean transformation. See also Doering *et al.* [10] about discussion on energy dissipation rate and the physical relevance of a related problem with injection and suction and about results of a different type. Mathematically the problem is relatively simpler due to the noncharacteristic nature of the physical boundary. In our boundary layer analysis, this amounts to the cancellation of the highest order singular terms from the dissipative term with that from the convection term in the noncharacteristic case (see Section 3 for more details). Roughly speaking, the Prandtl type equation in this case is approximately a linear elliptic equation which is similar to a model problem for the boundary layer proposed by Friedrichs [14], and it can be solved explicitly which is not true in the case of characteristic boundary (no-slip boundary). For the sake of simplicity we consider here a channel geometry with periodic boundary conditions in the horizontal directions and Dirichlet boundary conditions in the vertical direction. We also restrict ourselves to the simple case of fluids uniformly pumped into the channel from the top with speed U and sucked out at the bottom of the channel with the same speed U. The noncharacteristic boundary causes some technical difficulties in studying the inviscid problem. This is due to the fact that upwind boundary conditions have to be imposed and hence fluids are coming in and going out of the channel resulting in the failure of adapting Kato's method [15] of proving existence of classical solutions to the Euler system (Kato's method relies heavily on the fact that particle trajectories remain in the same region forever). For the linearized problem we are able to prove the well-posedness of the inviscid-problem using a semi-group method by constructing an appropriate space which incorporates the upwind boundary condition (see Section 1). Thus we are able to conclude, for the linearized case, that the solutions of the Navier–Stokes system converge (in L^2) to that of the Euler system as the viscosity decreases to zero provided that they have the same initial velocity field and body forces which satisfy certain compatibility conditions. We also prove that there exists a boundary layer at the bottom of the channel only, of the form $e^{-Uz/\varepsilon}$ where z is the vertical coordinate

with $z=0$ being the bottom of the channel and ε being the kinematic viscosity. This boundary layer analysis is done in both the H^1 and L^∞ norms. In the fully nonlinear case we have a short time result since we need the relative smallness of the horizontal velocity with respect to the vertical flushing U. This is perhaps anticipated since otherwise the boundary layer profile may become unstable and turbulence might develop. The well-posedness problem of the Euler system in this noncharacteristic case is not trivial, see however the work of Yudovitch [38] where he presented a very sketchy proof for the noncharacteristic case with normal velocity and vorticity (2D case) specified at the upwind (top) boundary and the work of Antontsev *et al.* [2]. Notice that we must have the full velocity specified at the upwind boundary for the convergence to be true.

A very similar problem with noncharacteristic boundary conditions was treated by Alekseenko [1] who proved the convergence in L^2 of the solutions of the Navier–Stokes equations to that of the Euler equations. However, his result has no implication on the boundary layer as no convergence in the interior is proved in either the H^1 or L^∞ norm. See also the numerical work of Fix and Gunzburger [13]. There is a body of work on boundary layers related to the compressible case (or to hyperbolic systems with noncharacteristic boundary condition in half space; see for instance the references list in the survey paper by Weinan E [11]). However, the compressible case is totally different from the incompressible case due to the presence, in the latter case, of the incompressibility condition which makes the pressure a global function of the velocity.

The purpose of this paper is to present explicit boundary layer analysis of the Navier–Stokes equations in the case when the boundary is noncharacteristic. Our boundary layer analysis is performed in both the H^1 space (hence the trace of the function can be defined and thus the term boundary layer makes sense) and the physically more appealing uniform space L^∞. In both cases we conclude that there is a boundary layer of the form $e^{-Uz/\varepsilon}$ existing at the outlet of the boundary only. More precisely we prove that the viscous solution can be approximated by the inviscid solution plus a boundary layer type function of the form $e^{-Uz/\varepsilon}$ and a small term which vanishes at vanishing viscosity, in both the uniform space (L^∞) and the derivative space (H^1). As a consequence we proved that *the viscous solution can be approximated by the inviscid solution uniformly away from the boundary*.

The proof of the space-time uniform estimates is quite interesting since we have to employ the physical idea of better control on the tangential derivative than that of the whole gradient (see for instance [31–34]). The *better control on the tangential derivative* of the velocity field together with an appropriate *anisotropic Sobolev imbedding* (see for instance [32]) gives us the L^∞ estimate of the boundary layer which is more appealing to the

physicists and engineers than the previous H^1 estimates. The space-time uniform estimate is established in space dimension two only. Such a better control of the tangential derivatives in the boundary layer still needs to be verified/proved for the case of classical no-slip boundary condition.

The article is organized as follows. In the first section we present the viscous and the inviscid linearized problems and prove the well-posedness of the linearized Euler system. Then in the second section we present the convergence result together with the boundary layer analysis for the linearized problem, both in H^1 and L^∞. In the third section we present the short time result that we mentioned earlier on the fully nonlinear case. We also assumed the well-posedness of the inviscid problem whose proof we shall address elsewhere. Finally in the fourth section we derive the L^∞ boundary layer analysis to the nonlinear case in space dimension two.

1. THE VISCOUS AND INVISCID PROBLEMS

Consider a channel

$$\Omega = (0, L_1) \times (0, L_2) \times (0, h). \tag{1.1}$$

We are interested in the asymptotic behavior of the Navier–Stokes equations in this channel at small viscosity with fluids pumped into the channel at the top ($z = h$) and fluids sucked out the channel at the bottom ($z = 0$), i.e., the case with non-vanishing normal velocity. Hence in this case the boundary which is permeable is noncharacteristic; i.e., it is not a stream surface. We recall the incompressible Navier–Stokes equations which take the form

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)\, u^\varepsilon + \nabla p^\varepsilon = f, \\ \operatorname{div} u^\varepsilon = 0, \\ u^\varepsilon = (0, 0, -U) \qquad \text{at} \quad z = 0, h. \end{cases} \tag{1.2}$$

Throughout this article we will assume periodicity in x (with period L_1) and in y (with period L_2). To start with, we consider the simpler case where we linearize the Navier–Stokes equations (1.2) around a constant steady state (which is an exact solution in the case without external forcing)

$$(0, 0, -U) \qquad (U = \text{constant} > 0), \tag{1.3}$$

Denoting now v^ε the perturbation, i.e., $v^\varepsilon = u^\varepsilon - (0, 0, -U)$, we find after dropping the nonlinear terms:

$$(1.4)\qquad \begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon - U D_3 v^\varepsilon + \nabla p^\varepsilon = f, \\ \operatorname{div} v^\varepsilon = 0, \\ v^\varepsilon = 0 \qquad \text{at the wall, i.e. at} \quad z = 0, h. \end{cases}$$

We are interested in the asymptotic behavior of the solutions of (1.4) for small kinematic viscosity ε.

We first need to determine the *inviscid* limit v^0 of v^ε, corresponding to the convergence in the interior of Ω (i.e., compact subsets of Ω). If we set formally $\varepsilon = 0$ in (1.4) we find

$$(1.5)\qquad \begin{cases} \dfrac{\partial v^0}{\partial t} - U D_3 v^0 + \nabla p^0 = f, \\ \operatorname{div} v^0 = 0, \\ v^0_3 = 0 \qquad \text{at} \quad z = 0, h. \end{cases}$$

Notice that (1.5) is a coupled hyperbolic-elliptic system and the walls of the channels are noncharacteristic; thus we should not expect well-posedness for (1.5). This can be illustrated via the following example. Consider the special case where f and v_0 are both x, y independent; then (1.5) reduces to a linear convection equation for which an upwind boundary condition has to be specified in order to obtain uniqueness (see for instance Cheng [5] or Cheng *et al.* [6]). This suggests that we should add an "upwind" boundary condition (at $z = h$) to (1.5) to make the problem well-posed. The natural "upwind" boundary condition is $v^0 = 0$ at $z = h$, which we may observe from (1.4) and by working out an asymptotic expansion for a convection-diffusion problem like (1.4) with small diffusivity (see however Yudovitch [37] for an alternate "upwind" boundary condition.)

Hence we propose to replace (1.5) by the following "new" inviscid problem

$$(1.6)\qquad \begin{cases} \dfrac{\partial v^0}{\partial t} - U D_3 v^0 + \nabla p^0 = f, \\ \operatorname{div} v^0 = 0, \\ v^0 = 0 \qquad \text{at} \quad z = h, \\ v^0_3 = 0 \qquad \text{at} \quad z = 0. \end{cases}$$

The first question we need to address is the well-posedness of the problem (1.6).

For this purpose we employ the classical semigroup theory (see, e.g., Lions [19], Yosida [36]). The natural Hilbert space H is the classical solenoidal subspace of $(L^2(\Omega))^3$ corresponding to the current boundary conditions (see, e.g., Lions [19], Temam [28]), namely

$$H = \{v \in (L^2(\Omega))^3,\ \operatorname{div} v = 0,\ v_1|_{x=0} = v_1|_{x=L_1},\\ v_2|_{y=0} = v_2|_{y=L_2},\ v_3 = 0 \text{ at } z = 0, h\}.$$

For the semigroup approach, we need to specify a subspace X of H such that

$$\begin{cases} -UD_3 u + \nabla p + \lambda u = f, \\ \operatorname{div} u = 0, \\ u = 0 \qquad \text{at} \quad z = h, \\ u_3 = 0 \qquad \text{at} \quad z = 0, \end{cases} \tag{1.7}$$

has a unique solution in X for all $f \in H$ and $\lambda > 0$, and the solution satisfies the estimate

$$|u|_H \leqslant \frac{1}{\lambda} |f|_H, \qquad \forall f \in H. \tag{1.8}$$

We also need the solution operator to be a closed operator for some fixed value of $\lambda = \lambda_0$.

The difficulty here is to find the right space X such that the boundary conditions are well defined and (1.7) is solvable. For the equation without the pressure and the $\operatorname{div} u = 0$ condition, the natural space would be the space of functions in $L^2(\Omega)^3$ such that their vertical derivative belongs to $L^2(\Omega)^3$ as well. However, such a space seems not appropriate for our problem due to regularity issues. Instead we propose the following space

$$X = \{v \in H \mid zD_3 v \in (L^2(\Omega))^3,\ v = 0 \text{ at } z = h,\\ \exists p \in \mathscr{D}'(\Omega) \text{ s.t.} -UD_3 v + \nabla p \in H\}. \tag{1.9}$$

This space is equipped with the "natural" norm

$$\|v\|_X = (|v|_H^2 + |zD_3 v|^2_{(L^2(\Omega))^3} + |UD_3 v - \nabla p|_H^2)^{\frac{1}{2}}. \tag{1.10}$$

Note that, in (1.9) and (1.10), p is unique up to an additive constant, due to the definition of H. Indeed if p_1 and p_2 are two admissible p's, then $\nabla(p_1 - p_2) \in H$, so that $p_1 - p_2$ a constant.

The existence and uniqueness of solutions to (1.7) can be derived via Fourier series in the horizontal directions, or by other methods; the details

will be given elsewhere. Next we derive the key a priori estimates which guarantee the well-posedness of (1.7), with solutions in X.

We first take the scalar product of (1.7) with u, integrate over Ω and we deduce easily the inequality (1.8), by integration by parts and utilizing the boundary conditions.

For the closedness of the solution operator for fixed $\lambda_0 > 0$ we observe from Eq. (1.7), that

$$
\begin{aligned}
|\nabla p|_{H^{-1}} &\leqslant |f|_{H^{-1}} + \lambda_0\, |u|_{H^{-1}} + U \left|\frac{\partial u}{\partial z}\right|_{H^{-1}} \\
&\leqslant \text{(thanks to (1.8))} \\
&\leqslant \kappa(\lambda_0)\, |f|_H.
\end{aligned}
\tag{1.11}
$$

Hence, by a theorem of Magenes and Stampacchia [22] (see also, Lions [19], Temam [28]),

$$
|p|_{L^2} \leqslant \kappa(\lambda_0)\, |f|_H, \tag{1.12}
$$

where $\kappa(\lambda_0)$ is a constant depending only on the domain Ω and on λ_0 only.

Next we multiply (1.7) by $-z^2 D_3 u$, integrate over Ω, and integrate by parts. Notice also that

$$
\int_\Omega -\lambda_0 z^2 u D_3 u = \lambda_0 \int_\Omega z\, |u|^2 \geqslant 0,
$$

$$
\begin{aligned}
\left|\int_\Omega \nabla p\, z^2 D_3 u\right| &\leqslant \left|2 \int_\Omega p z D_3 u\right| \\
&\leqslant \frac{U}{4}\, |z D_3 u|_{L^2}^2 + \frac{4}{U}\, |p|_{L^2}^2 \\
&\leqslant \text{(thanks to (1.12), and with another } \kappa(\lambda_0)), \\
&\leqslant \frac{U}{4}\, |z D_3 u|_{L^2}^2 + \frac{\kappa(\lambda_0)}{U}\, |f|_H^2,
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_\Omega f z^2 D_3 u\right| &\leqslant \frac{U}{4}\, |z D_3 u|_{L^2}^2 + \frac{1}{U}\, |z f|_{L^2}^2 \\
&\leqslant \frac{U}{4}\, |z D_3 u|_{L^2}^2 + \frac{h^2}{U}\, |f|_H^2.
\end{aligned}
$$

Therefore we deduce that

$$
|z D_3 u|_{L^2} \leqslant \frac{\kappa}{U}\, |f|_H. \tag{1.13}
$$

This implies

$$\|u\|_X \leqslant \kappa(\lambda_0, \Omega, U)\,|f|_H, \tag{1.14}$$

and hence we deduce that the operator $v \in X \to Av = -UD_3 v + \lambda_0 v + \nabla p$ is a closed operator on X.

We conclude then that the semigroup theory applies and we have the well-posedness for the initial value problem associated with (1.6), X being the domain of A in H: given f and $v_0 \in H$, there exists a unique solution of (1.6), i.e.,

$$\frac{dv^0}{dt}(t) + Av^0(t) = f, \qquad t > 0,$$

such that $v^0(0) = v_0$, and $v^0 \in \mathscr{C}(\mathbb{R}_+; H)$, $(dv^0/dt) \in L^2(0, T; H)$, $\forall T > 0$.

Further regularity results for the solutions of (1.6) can be derived in a standard way using translations parallel to $0x$ or $0y$: if we differentiate (1.6) with respect to x or y, e.g., we apply $D_1^k D_2^l$, then $D_1^k D_2^l v^0$ is the solution of the same problem with f replaced by the corresponding derivative. Hence we obtain tangential regularity provided f and v_0 are sufficiently smooth; for instance if f, $D_1 f$, $D_2 f$ are in $L^\infty(H)$, v^0, $D_1 v^0$, $D_2 v^0$ are in H then $D_i v^0$ are in $L^\infty(H)$, $D_i p$ are in $L^2(L^2)$, for $i = 1, 2$ and the third component of Eq. (1.6) together with the incompressibility of v^0 implies that $D_3 p^0$ is in $L^2(L^2)$ so that $\nabla p^0 \in L^2(L^2)$ and finally

$$p \in L^2(H^1) \qquad \text{and} \qquad v^0 \in L^2((H^1)^3). \tag{1.15}$$

by additional tangential regularity.

For further regularity we differentiate Eq. (1.6) in time and obtain

$$\begin{cases} \dfrac{\partial^2 v^0}{\partial t^2} - UD_3 \dfrac{\partial v^0}{\partial t} + \nabla \dfrac{\partial p^0}{\partial t} = \dfrac{\partial f}{\partial t}, \\ \operatorname{div} \dfrac{\partial v^0}{\partial t} = 0, \\ \dfrac{\partial v^0}{\partial t} = 0 \qquad \text{at} \quad z = h, \\ \dfrac{\partial v_3^0}{\partial t} = 0 \qquad \text{at} \quad z = 0, \\ \dfrac{\partial v^0}{\partial t} = UPD_3 v_0 + f(0) \in H, \end{cases} \tag{1.16}$$

where P is the Leray–Hopf projector, provided

$$v_0 \in H^1 \qquad \text{and} \qquad f \in H^1(H). \tag{1.17}$$

Thus the same regularity applies to $\partial v^0/\partial t$ and hence we deduce

$$\frac{\partial v^0}{\partial t} \in L^\infty(H).$$

This further implies, when combined with the 3rd component of Eq. (1.6), the incompressibility and the tangential regularity

$$\frac{\partial p^0}{\partial z} \in L^\infty(L^2).$$

Thus by free tangential regularity on the pressure we deduce

$$p^0 \in L^\infty(H^1)$$

and hence, when combined with (1.6) and the time derivative estimate

$$v^0 \in L^\infty((H^1)^3).$$

Furthermore, since

$$\begin{aligned} D_3^2 p^0 &= D_3 f_3 + U D_3^2 v_3^0 - D_3 \frac{\partial v_3^0}{\partial t} \\ &= D_3 f_3 - U D_3 (D_1 v_1^0 + D_2 v_2^0) + D_1 \frac{\partial v_1^0}{\partial t} + D_2 \frac{\partial v_2^0}{\partial t} \in L^\infty(L^2) \end{aligned}$$

we have

$$p^0 \in L^\infty(H^2)$$

which further implies, together with tangential regularity

$$v^0 \in L^2((H^2)^3) \tag{1.18}$$

since, thanks to the Eq. (1.6),

$$-U D_3 v^0 = f - \nabla p^0 \in L^\infty(H^1) - \frac{\partial v_2^0}{\partial t} \in L^2((H^1)^3).$$

More regularity results can be derived in a similar fashion. From now on we will assume as much regularity as we need for the solution v^0 of (1.6).

2. THE LINEAR CASE

Now we want to study the convergence, as $\varepsilon \to 0$, of v^ε to v^0. Because the boundary conditions for v^ε and v^0 at the out flow ($z=0$) are different we do not expect v^ε to converge to v^0 on the whole domain $\bar{\Omega}$. However, we can expect and we will prove that convergence holds in $(0, L_1) \times (0, L_2) \times [0, h)$: this will be achieved by constructing a corrector φ^ε, singular in $H^1(\Omega)$ and which absorbs the singularity of $v^\varepsilon - v^0 = \theta^\varepsilon$. The convergence will then follow from the convergence to 0 of $w^\varepsilon = v^\varepsilon - v^0 + \varphi^\varepsilon$.

Utilizing a stretched coordinates as in Prandtl's work [24], we discover that to the leading order, the difference between the viscous solution and the inviscid solution should satisfy the following *Prandtl type equation*[2]

$$
\begin{cases}
-\varepsilon \Delta \theta^\varepsilon - U D_3 \theta^\varepsilon + \nabla q^\varepsilon = 0, \\
\operatorname{div} \theta^\varepsilon = 0, \\
\theta^\varepsilon = 0, & \text{at} \quad z = h, \\
\theta^\varepsilon = -v^0, & \text{at} \quad z = 0.
\end{cases}
\tag{2.1}
$$

This is the incompressible fluid version of Friedrichs [14] classical example on boundary layer.

This inspires us to construct the following background flow φ^ε which solves the Prandtl type Eq. (2.1) to the leading order.

$$
\varphi^\varepsilon = \operatorname{curl} \psi^\varepsilon = \left(-\frac{\partial \psi_2^\varepsilon}{\partial z}, \frac{\partial \psi_1^\varepsilon}{\partial z}, -\frac{\partial \psi_1^\varepsilon}{\partial y} + \frac{\partial \psi_2^\varepsilon}{\partial x} \right),
\tag{2.2}
$$

where the vorticity potential ψ^ε takes the form

$$
\begin{cases}
\psi^\varepsilon = (\psi_1^\varepsilon, \psi_2^\varepsilon, 0), \qquad \text{with} \\
\psi_1^\varepsilon = v_2^0(x, y, 0, t)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U} (1 - e^{-Uz/\varepsilon}), \\
\psi_2^\varepsilon = -v_1^0(x, y, 0, t)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U} (1 - e^{-Uz/\varepsilon}),
\end{cases}
\tag{2.3}
$$

[2] Our approach is different from Prandtl's since we do not use the matched asymptotic method.

and the cut-off function $\rho \in \mathscr{C}^\infty[0, \infty)$ satisfies

$$\text{supp}\,\rho \subset [0, 1), \qquad \rho(0) = 1, \qquad \rho'(0) = \rho''(0) = \rho'''(0) = 0. \tag{2.4}$$

The last component of φ^ε can be rewritten as, after utilizing the incompressibility condition of u^0,

$$\varphi_3^\varepsilon = D_3 v_3^0(x, y, 0, t)\, \rho\left(\frac{z}{h}\right) \frac{\varepsilon}{U} (1 - e^{-Uz/\varepsilon}). \tag{2.5}$$

Remembering that $v_3^0 = 0$ at $z = 0$ and h, we then observe that

$$\varphi^\varepsilon = v^0 \qquad \text{at} \quad z = 0, h. \tag{2.6}$$

Notice that the *background* flow φ^ε has the natural decomposition into a boundary layer type part $\varphi^{\varepsilon, 1}$ and a regular type part $\varphi^{\varepsilon, 2}$ satisfying the properties

$$\varphi^\varepsilon = \varphi^{\varepsilon, 1} + \varphi^{\varepsilon, 2} = \text{curl}\, \psi^{\varepsilon, 1} + \text{curl}\, \psi^{\varepsilon, 2}, \tag{2.7}$$

where the boundary layer part $\psi^{\varepsilon, 1}$ of the vorticity potential and the regular part $\psi^{\varepsilon, 2}$ of the vorticity potential take the form

$$\begin{cases} \psi^{\varepsilon, 1} = (\psi_{11}^\varepsilon, \psi_{21}^\varepsilon, 0), \\ \psi_{11}^\varepsilon = -v_2^0(x, y, 0)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U} e^{-Uz/\varepsilon}, \\ \psi_{21}^\varepsilon = v_1^0(x, y, 0)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U} e^{-Uz/\varepsilon}, \\ \psi^{\varepsilon, 2} = (\psi_{12}^\varepsilon, \psi_{22}^\varepsilon, 0), \\ \psi_{12}^\varepsilon = v_2^0(x, y, 0)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U}, \\ \psi_{22}^\varepsilon = -v_1^0(x, y, 0)\, \rho\left(\dfrac{z}{h}\right) \dfrac{\varepsilon}{U}. \end{cases} \tag{2.8}$$

This is equivalent to

$$
\left\{
\begin{aligned}
&\varphi_1^\varepsilon = \varphi_{11}^\varepsilon + \varphi_{12}^\varepsilon,\\
&\varphi_{11}^\varepsilon = -\frac{\partial \psi_{22}^\varepsilon}{\partial z} = v_1^0(x, y, 0)\, e^{-Uz/\varepsilon}\left(\rho\left(\frac{z}{h}\right) - \frac{\varepsilon}{hU}\,\rho'\left(\frac{z}{h}\right)\right),\\
&\varphi_{12}^\varepsilon = -\frac{\partial \psi_{21}^\varepsilon}{\partial z} = \frac{\varepsilon}{hU}\, v_1^0(x, y, 0)\,\rho'\left(\frac{z}{h}\right),\\
&\varphi_2^\varepsilon = \varphi_{21}^\varepsilon + \varphi_{22}^\varepsilon,\\
&\varphi_{21}^\varepsilon = \frac{\partial \psi_{12}^\varepsilon}{\partial z} = v_2^0(x, y, 0)\, e^{-Uz/\varepsilon}\left(\rho\left(\frac{z}{h}\right) - \frac{\varepsilon}{hU}\,\rho'\left(\frac{z}{h}\right)\right),\\
&\varphi_{22}^\varepsilon = \frac{\partial \psi_{11}^\varepsilon}{\partial z} = \frac{\varepsilon}{hU}\, v_2^0(x, y, 0)\,\rho'\left(\frac{z}{h}\right),\\
&\varphi_3^\varepsilon = \frac{\varepsilon}{U}\,\rho\left(\frac{z}{h}\right) D_3 v_3^0(x, y, 0)(1 - e^{-Uz/\varepsilon}).
\end{aligned}
\right.
\tag{2.9}
$$

It is obvious from our explicit construction that the approximate corrector satisfies the estimates

$$
\left\{
\begin{aligned}
&\|\varphi^{\varepsilon,2}\|_{L^\infty(0,T;H^k)} \leqslant \kappa\varepsilon, \qquad \forall k,\\
&\|\varphi^{\varepsilon,1}\|_{L^\infty(0,T;H^k)} \leqslant \kappa\varepsilon^{1/2-k},\\
&\|\varphi^{\varepsilon,1}\|_{L^\infty(0,T;L^\infty)} \leqslant \kappa,\\
&\|z\varphi^{\varepsilon,1}\|_{L^\infty(0,T;L^\infty)} \leqslant \kappa\varepsilon,\\
&\|z\varphi^{\varepsilon,1}\|_{L^\infty(0,T;L^2)} \leqslant \kappa\varepsilon^{\frac{3}{2}}\\
&\|z^2\nabla\varphi^{\varepsilon,1}\|_{L^\infty(0,T;L^2)} \leqslant \kappa\varepsilon^{\frac{3}{2}},
\end{aligned}
\right.
\tag{2.10}
$$

where κ, is a generic constant independent of the kinematic viscosity ε.

As in the case with impermeable boundary (see for instance Temam and Wang [31]), we consider the adjusted difference between the viscous and inviscid solutions. Namely we set

$$
w^\varepsilon = v^\varepsilon - v^0 + \varphi^\varepsilon = u^\varepsilon - u^0 + \varphi^\varepsilon,
\tag{2.11}
$$

and we deduce that the adjusted difference satisfies the equation

$$
\left\{
\begin{aligned}
&\frac{\partial w^\varepsilon}{\partial t} - \varepsilon\Delta w^\varepsilon - UD_3 w^\varepsilon + \nabla(p^\varepsilon - p^0) = -\frac{\partial \varphi^\varepsilon}{\partial t} + \varepsilon\Delta v^0 - \varepsilon\Delta\varphi^\varepsilon - UD_3\varphi^\varepsilon,\\
&\operatorname{div} w^\varepsilon = 0,\\
&w^\varepsilon = 0 \qquad \text{at} \quad z = 0, h,\\
&w^\varepsilon = 0 \qquad \text{at} \quad t = 0.
\end{aligned}
\right.
\tag{2.12}
$$

Now if we carelessly estimate the right hand side (RHS) of the first equation (2.12), term by term, we obtain an uninteresting estimate because

$$|D_3\varphi^\varepsilon|_{L^q} \sim \varepsilon^{\frac{1}{q}-1}$$

is not small.

The key observation here is that there exists *cancellation* (at the leading order) between the diffusion and convection terms due to our construction. In another word, our ansatz for corrector does solve the Prandtl type equation (2.1) to the leading order. More precisely, we have, for the first component,

$$
\begin{aligned}
(2.13)\qquad \varepsilon\Delta\varphi_1^\varepsilon + UD_3\varphi_1^\varepsilon &= -\varepsilon\Delta D_3\psi_2^\varepsilon - UD_3^2\psi_2^\varepsilon \\
&= \varepsilon\Delta\left\{ v_1^0(x,y,0,t)\,\rho'\left(\frac{z}{h}\right)\frac{\varepsilon}{UH}(1-e^{-Uz/\varepsilon}) \right. \\
&\quad \left. + v_1^0(x,y,0,t)\,\rho\left(\frac{z}{h}\right)e^{-Uz/\varepsilon}\right\} \\
&\quad + UD_3^2\left(v_1^0(x,y,0,t)\,\rho\left(\frac{z}{h}\right)\frac{\varepsilon}{U}(1-e^{-Uz/\varepsilon})\right) \\
&= \frac{\varepsilon^2}{Uh}\Delta v_1^0(x,y,0,t)\,\rho'\left(\frac{z}{h}\right)(1-e^{-Uz/\varepsilon}) \\
&\quad + 2\frac{\varepsilon}{h^2}v_1^0(x,y,0,t)\,\rho''\left(\frac{z}{h}\right)e^{-Uz/\varepsilon} \\
&\quad - v_1^0(x,y,0,t)\,\rho'\left(\frac{z}{h}\right)\frac{U}{h}e^{-Uz/\varepsilon} \\
&\quad + \frac{\varepsilon^2}{Uh^3}v_1^0(x,y,0,t)\,\rho'''\left(\frac{z}{h}\right)(1-e^{-Uz/\varepsilon}) \\
&\quad + \varepsilon\Delta v_1^0(x,y,0,t)\,\rho\left(\frac{z}{h}\right)e^{-Uz/\varepsilon} \\
&\quad + \varepsilon v_1^0(x,y,0,t)\,D_3^2\left(\rho\left(\frac{z}{h}\right)e^{-Uz/\varepsilon}\right) \\
&\quad + \frac{\varepsilon}{h^2}v_1^0(x,y,0,t)\,\rho''\left(\frac{z}{h}\right) \\
&\quad - \varepsilon v_1^0(x,y,0,t)\,D_3^2\left(\rho\left(\frac{z}{h}\right)e^{-Uz/\varepsilon}\right) \\
&= -v_1^0(x,y,0,t)\,\rho'\left(\frac{z}{h}\right)\frac{U}{h}e^{-Uz/\varepsilon} + \mathrm{Rem},
\end{aligned}
$$

where the remainder Rem satisfies

$$
\begin{aligned}
\text{(2.14)}\quad \mathrm{Rem} &= \frac{\varepsilon^2}{Uh} \Delta v_1^0(x, y, 0, t)\, \rho'\left(\frac{z}{h}\right)(1-e^{-Uz/\varepsilon}) \\
&\quad + 2\frac{\varepsilon}{h^2} v_1^0(x, y, 0, t)\, \rho''\left(\frac{z}{h}\right) e^{-Uz/\varepsilon} \\
&\quad + \frac{\varepsilon^2}{Uh^3} v_1^0(x, y, 0, t)\, \rho'''\left(\frac{z}{h}\right)(1-e^{-Uz/\varepsilon}) \\
&\quad + \varepsilon \Delta v_1^0(x, y, 0, t)\, \rho\left(\frac{z}{h}\right) e^{-Uz/\varepsilon} + \frac{\varepsilon}{h^2} v_1^0(x, y, 0, t)\, \rho''\left(\frac{z}{h}\right) \\
&= \mathrm{Rem}_1 + \mathrm{Rem}_2,
\end{aligned}
$$

where Rem_1 is the boundary layer part and Rem_2 is the regular part satisfying the estimates

$$
\text{(2.15)}\qquad \begin{cases} \|\mathrm{Rem}\|_{L^\infty(0,T;L^\infty)} \leqslant \kappa\varepsilon, \\ \|z\, \mathrm{Rem}_1\|_{L^\infty(0,T;L^\infty)} \leqslant \kappa\varepsilon^2, \\ \|\mathrm{Rem}_2\|_{L^\infty(0,T;H^1)} \leqslant \kappa\varepsilon. \end{cases}
$$

Similar estimates hold for the other two components and (2.13) and (2.10) imply that the right hand side (RHS) of (2.12) can be decomposed into a boundary layer part RHS_1 and a regular part RHS_2 satisfying the estimates

$$
\text{(2.16)}\qquad \begin{cases} \|z\, \mathrm{RHS}_1\|_{L^\infty(0,T,L^2)} \leqslant \kappa\varepsilon^{\frac{3}{2}} \\ \|\mathrm{RHS}_2\|_{L^\infty(0,T,L^2)} \leqslant \kappa\varepsilon. \end{cases}
$$

Next we proceed with the energy method and we multiply the equation for the adjusted difference (2.12) by w^ε and integrate over Ω. Utilizing a weighted energy norm on the boundary layer part with weight z on the right hand side (RHS_1) and Hardy's inequality[3] we have

$$
\begin{aligned}
\left|\int_\Omega \mathrm{RHS}_1 \cdot w^\varepsilon\right| &\leqslant |z\, \mathrm{RHS}_1|_{L^2} \left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\leqslant \kappa\varepsilon^{\frac{3}{2}} \|\nabla w^\varepsilon\|_{L^2} \\
&\leqslant \frac{\varepsilon}{2} \|\nabla w^\varepsilon\|_{L^2} + \kappa\varepsilon^2
\end{aligned}
$$

[3] Hardy's inequality says that $\|\frac{w}{z}\|_{L^2} \leqslant 2\, \|\nabla w\|_{L^2}$ provided $w(0)=0$.

and

$$\left|\int_{\Omega} \mathrm{RHS}_2 \cdot w^{\varepsilon}\right| \leqslant \tfrac{1}{2}\,|w^{\varepsilon}|_{L^2}^2 + \kappa\varepsilon^2.$$

Thus we have

$$\frac{d}{dt}\,|w^{\varepsilon}|_{L^2}^2 + \varepsilon\,\|\nabla w^{\varepsilon}\|_{L^2}^2 \leqslant |w^{\varepsilon}|_{L^2}^2 + \kappa\varepsilon^2$$

which further implies, via Gronwall type technique,

$$\begin{cases} \|w^{\varepsilon}\|_{L^{\infty}(0,T;H)} \leqslant \kappa\varepsilon, \\ \|w^{\varepsilon}\|_{L^2(0,T;V)} \leqslant \kappa\varepsilon^{\frac{1}{2}}. \end{cases} \tag{2.17}$$

This already implies our claim on the boundary layer in this linear case in the space H^1.

Combining (2.10) and (2.17) we further deduce that

$$\begin{cases} \|v^{\varepsilon} - v^0\|_{L^{\infty}(0,T;H)} \leqslant \kappa\varepsilon^{\frac{1}{2}}, \\ \|v^{\varepsilon}\|_{L^2(0,T;V)} \leqslant \kappa\varepsilon^{-1/2}. \end{cases} \tag{2.18}$$

To derive L^{∞} bounds on the convergence rate we proceed as in the case of impermeable walls (see for instance Temam and Wang [31]) using an anisotropic Sobolev imbedding, the $L^{\infty}(H)$ and $L^{\infty}(V)$ estimates, and the free regularity in the direction Ox, Oy.

In order to derive the $L^{\infty}(H^1)$ estimate we differentiate the equation for adjusted difference (2.12) in time and we have

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\, w^{\varepsilon} - \varepsilon\Delta\,\dfrac{\partial}{\partial t}\, w^{\varepsilon} - UD_3\,\dfrac{\partial}{\partial t}\, w^{\varepsilon} + \nabla\,\dfrac{\partial}{\partial t}\, p^{\varepsilon} = \dfrac{\partial}{\partial t}\,\mathrm{RHS}, \\ \operatorname{div}\dfrac{\partial}{\partial t}\, w^{\varepsilon} = 0, \\ \dfrac{\partial}{\partial t}\, w^{\varepsilon} = 0, \qquad \text{at} \quad z = 0, h, \\ \dfrac{\partial}{\partial t}\, w^{\varepsilon} = \varepsilon P\Delta v_0, \qquad \text{at} \quad t = 0, \end{cases} \tag{2.19}$$

where P is the Leray–Hopf projector. We multiply both sides of (2.19) by $\partial w^{\varepsilon}/\partial t$ and integrate over Ω, utilizing the same energy estimates as for w^{ε}

and the fact that the time derivative of the right hand side (RHS) satisfies the same kind of estimate as itself, and we have

$$\begin{cases} \left\| \dfrac{\partial}{\partial t} w^{\varepsilon} \right\|_{L^{\infty}(0,T;H)} \leqslant \kappa\varepsilon, \\ \left\| \dfrac{\partial}{\partial t} w^{\varepsilon} \right\|_{L^{2}(0,T;V)} \leqslant \kappa\varepsilon^{\frac{1}{2}}. \end{cases} \tag{2.20}$$

Integrating in time we deduce

$$\|w^{\varepsilon}\|_{L^{\infty}(0,T;V)} \leqslant \kappa\varepsilon^{\frac{1}{2}}. \tag{2.21}$$

Applying an anisotropic Sobolev imbedding (see Temam and Wang [31 Remark 4.2]) and utilizing free tangential estimates, we obtain

$$\|w^{\varepsilon}\|_{L^{\infty}((0,T)\times\Omega)} \leqslant \kappa\varepsilon^{\frac{3}{4}}. \tag{2.22}$$

In summary, we have proved the following

THEOREM 2.1. *For f an v_0 given, let v^{ε} and v^0 be the solutions of* (1.4) *and* (1.6) *and assume that v^0 is sufficiently regular.*
Then, as $\varepsilon \to 0$, $v^{\varepsilon} - v^0$ is estimated by (2.12) *and*

$$\begin{cases} \|w^{\varepsilon}\|_{L^{\infty}([0,T]\times\Omega)} = \|u^{\varepsilon} - u^0 - \varphi^{\varepsilon}\|_{L^{\infty}([0,T]\times\Omega)} \leqslant \kappa\varepsilon^{3/4}, \\ \|w^{\varepsilon}\|_{L^{\infty}(0,T;H)} \leqslant \kappa\varepsilon, \\ \|w^{\varepsilon}\|_{L^{2}(0,T;(H^1)^3)} \leqslant \kappa\varepsilon^{\frac{1}{2}}. \end{cases} \tag{2.23}$$

The difference $w^{\varepsilon} = v^{\varepsilon} - v^0 - \varphi^{\varepsilon}$, the corrector φ^{ε} being given by (2.2), *is estimated by* (2.10).

Remark 2.1. Theorem 2.1 confirms the following.

(a) There is no boundary layer at the top (upwind direction $z = h$).

(b) There is a boundary layer of thickness ε/U at the bottom (downwind direction $z = 0$),

Remark 2.2. The estimates in the spaces H and V are optimal thanks to a concrete example (a special case of the NSE) considered in [6]. The space time uniform estimate here is not optimal. The optimal rate of ε can be derived via higher order asymptotic expansion.

3. A SHORT TIME RESULT IN THE NONLINEAR CASE

Here we would like to test the technique developed in the previous section on the nonlinear (full) Navier–Stokes equations. The technique

produces indeed a similar result, i.e., the convergence of the solutions of the NSE to the solutions of the corresponding equations at least for a short period of time which ensures some smallness condition (to be specified in the subsequent context). This restriction is expected in the nonlinear case as, otherwise, turbulence appears. Our result relies also on the well-posedness of the corresponding Euler equation that we will develop in a separate work.

We recall the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)\, u^\varepsilon + \nabla p^\varepsilon = f, & \text{in } \Omega, \\ \operatorname{div} u^\varepsilon = 0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

supplemented with the boundary condition

$$\begin{aligned} &u^\varepsilon = (0, 0, -U) \qquad \text{at} \quad z = 0, h, \qquad (U > 0), \\ &u^\varepsilon,\, p^\varepsilon \text{ periodic in the } x \text{ and } y \text{ directions.} \end{aligned} \tag{3.2}$$

Here

$$\Omega = (0, L_1) \times (0, L_2) \times (0, h), \tag{3.3}$$

is a channel (with periodicity in x and y understood). The boundary condition (3.2) amounts to saying that fluids flow in the channel at the top ($z = h$) and flows out of the channel at the bottom ($z = 0$).

The corresponding "inviscid" Euler equation ($\varepsilon = 0$) is

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla)\, u^0 + \nabla p^0 = f & \text{in } \Omega, \\ \operatorname{div} u^0 = 0, & \text{in } \Omega. \end{cases} \tag{3.4}$$

We propose, based on our result on the linearized problem, the following boundary conditions:

$$\begin{cases} u^0 = (0, 0, -U) & \text{at} \quad z = h, \\ u_3^0 = -U & \text{at} \quad z = 0. \end{cases} \tag{3.5}$$

The boundary condition amount to specifying the full velocity at the top or upwind condition, and the normal velocity at the bottom.

As before we would like to consider a translated problem (the reason behind is that, in general, the trilinear form $\int_\Omega (u \cdot \nabla)\, v \cdot w$ is skew symmetric in the last two variables only for velocity field with zero normal velocity at the boundary).

Let

$$
(3.6) \qquad \begin{cases} u^\varepsilon = v^\varepsilon + (0, 0, -U), \\ u^0 = v^0 + (0, 0, -U). \end{cases}
$$

We then deduce from (3.1), (3.2), (3.4), and (3.5) that

$$
(3.7) \qquad \begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon + (v^\varepsilon \cdot \nabla)\, v^\varepsilon - U D_3 v^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega, \\ \operatorname{div} v^\varepsilon = 0 & \text{in } \Omega, \\ v^\varepsilon = 0 & \text{at} \quad z = 0, h, \end{cases}
$$

and

$$
(3.8) \qquad \begin{cases} \dfrac{\partial v^0}{\partial t} + (v^0 \cdot \nabla)\, v^0 - U D_3 v^0 + \nabla p^0 = f & \text{in } \Omega, \\ \operatorname{div} v^0 = 0 & \text{in } \Omega, \\ v^0 = 0 & \text{at} \quad z = h, \\ v_3^0 = 0 & \text{at} \quad z = 0, \end{cases}
$$

where $D_3 = \partial / \partial z$.

The short time existence, uniqueness, smoothness and continuous dependence or parameters for the viscous problems (3.7) can be derived by adapting classical methods (see for instance [7, 18, 19, 28] among many others). However, we will only need a Leray–Hopf weak solution for the viscous problem whose global existence can be derived by following classical literatures [7, 18, 19, 28]. The well posedness of the *inviscid* Euler problem is much more involved (see for instance [2, Chap. 4]). A similar problem was roughly treated in the early work of Yudovitch [37]. We will assume in this article and will prove elsewhere that the inviscid problem (3.8) is well-posed, that its solution is as smooth as needed below, and that it depends continuously on the data in appropriate norms.

First we would like to prove the convergence of v^ε solution of (3.7) to v^0 solution of (3.8). The convergence in L^2 is established by Alekseenko [1]. However, we shall derive optimal rate of convergence, together with convergence in H^1 with the aid of a corrector (boundary layer function). For that purpose we would like to consider, just as in the linearized case, a zero order correction function φ^ε which represents the singular part of $v^0 - v^\varepsilon$.

A stretched coordinate argument together with the incompressibility yields that the Prandtl type equation for the nonlinear case is the same as the linear case (2.1) to the leading order. Thus it make sense to keep (1.2) as our ansatz for corrector.

Next we consider the adjusted difference, i.e.,

$$w^\varepsilon = v^\varepsilon - v^0 + \varphi^\varepsilon = u^\varepsilon - u^0 + \varphi^\varepsilon. \tag{3.9}$$

Notice that we may rewrite the nonlinear term in the form

$$\begin{aligned}
&(v^\varepsilon \cdot \nabla)\, v^\varepsilon - (v^0 \cdot \nabla)\, v^0 - (\varphi^\varepsilon \cdot \nabla)\, \varphi^\varepsilon \\
&\quad = (v^\varepsilon \cdot \nabla)(v^\varepsilon - v^0 + \varphi^\varepsilon) + ((v^\varepsilon - v^0) \cdot \nabla)\, v^0 - ((v^\varepsilon + \varphi^\varepsilon) \cdot \nabla)\, \varphi^\varepsilon \\
&\quad = (v^\varepsilon \cdot \nabla)(v^\varepsilon - v^0 + \varphi^\varepsilon) + ((v^\varepsilon - v^0 + \varphi^\varepsilon) \cdot \nabla)\, v^0 \\
&\qquad - ((v^\varepsilon - v^0 + \varphi^\varepsilon) \cdot \nabla)\, \varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla)\, v^0 - (v^0 \cdot \nabla)\, \varphi^\varepsilon \\
&\quad = (v^\varepsilon \cdot \nabla)\, w^\varepsilon + (w^\varepsilon \cdot \nabla)\, v^0 - (w^\varepsilon \cdot \nabla)\, \varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla)\, v^0 - (v^0 \cdot \nabla)\, \varphi^\varepsilon.
\end{aligned} \tag{3.10}$$

Hence the adjusted difference w^ε satisfies

$$\begin{cases}
\dfrac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + (v^\varepsilon \cdot \nabla)\, w^\varepsilon + (w^\varepsilon \cdot \nabla)\, v^0 - (w^\varepsilon \cdot \nabla)\, \varphi^\varepsilon - U D_3 w^\varepsilon + \nabla(p^\varepsilon - p^0) \\
\quad = \dfrac{\partial \varphi^\varepsilon}{\partial t} - \varepsilon \Delta v^0 - \varepsilon \Delta \varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla)\, \varphi^\varepsilon - U D_3 \varphi^\varepsilon + (\varphi^\varepsilon \cdot \nabla)\, v^0 + (v^0 \cdot \nabla)\, \varphi^\varepsilon \\
\quad = \text{R.H.S.} \\
\operatorname{div} w^\varepsilon = 0, \\
w^\varepsilon = 0 \qquad \text{at} \quad z = 0, h
\end{cases} \tag{3.11}$$

We proceed with energy estimates, i.e., multiply (3.15) by w^ε and integrate over Ω. We have, with $|\cdot|$ denoting the norm in $L^2(\Omega)^3$:

$$\int_\Omega \frac{\partial w^\varepsilon}{\partial t}\, w^\varepsilon = \frac{1}{2} \frac{d}{dt} |w^\varepsilon|^2, \tag{3.12}$$

$$\int_\Omega -\varepsilon \Delta w^\varepsilon \cdot w^\varepsilon = \varepsilon\, |\nabla w^\varepsilon|^2, \tag{3.13}$$

$$\int_\Omega (v^\varepsilon \cdot \nabla)\, w^\varepsilon \cdot w^\varepsilon = 0, \tag{3.14}$$

$$\left| \int_\Omega (w^\varepsilon \cdot \nabla)\, v^0 \cdot w^\varepsilon \right| \leqslant |\nabla v^0|_{L^\infty}\, |w^\varepsilon|^2, \tag{3.15}$$

$$U \int_\Omega D_3 w^\varepsilon \cdot w^\varepsilon = 0, \tag{3.16}$$

$$\int_\Omega \nabla(p^\varepsilon - p^0) \cdot w^\varepsilon = 0. \tag{3.17}$$

We notice that the right hand side is the same as in the linear case except for the quadratic terms. We utilize the decomposition of the ansatz (into boundary layer part and regular part) to estimate the quadratic terms,

(3.18)

$$
\begin{aligned}
\left|\int_\Omega (\varphi^\varepsilon\cdot\nabla)\,\varphi^\varepsilon\cdot w^\varepsilon\right| &= \left|\int_\Omega (\varphi_1^\varepsilon D_1\varphi^\varepsilon\cdot w^\varepsilon+\varphi_2^\varepsilon D_2\varphi^\varepsilon\cdot w^\varepsilon+\varphi_3^\varepsilon D_3\varphi^\varepsilon\cdot w^\varepsilon)\right| \\
&\leqslant \left|\int_\Omega \varphi_1^\varepsilon D_1\varphi^{\varepsilon,1}\cdot w^\varepsilon\right|+\left|\int_\Omega \varphi_1^\varepsilon D_1\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\quad +\left|\int_\Omega \varphi_2^\varepsilon D_2\varphi^{\varepsilon,1}\cdot w^\varepsilon\right|+\left|\int_\Omega \varphi_2^\varepsilon D_2\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\quad +\left|\int_\Omega \varphi_3^\varepsilon D_3\varphi^{\varepsilon,1}\cdot w^\varepsilon\right|+\left|\int_\Omega \varphi_3^\varepsilon D_3\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\leqslant |\varphi_1^\varepsilon|_{L^\infty}\,|D_1\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2}+|\varphi_1^\varepsilon|_{L^\infty}\,|zD_1\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\quad +|\varphi_2^\varepsilon|_{L^\infty}\,|D_2\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2}+|\varphi_2^\varepsilon|_{L^\infty}\,|zD_2\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\quad +|\varphi_3^\varepsilon|_{L^\infty}\,|D_3\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2}+|\varphi_3^\varepsilon|_{L^\infty}\,|zD_3\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\leqslant \text{(thanks to (2.10) and Hardy's inequality)} \\
&\leqslant \kappa\varepsilon\,|w^\varepsilon|_{L^2}+\kappa\varepsilon^{\frac{3}{2}}\,|\nabla w^\varepsilon|_{L^2} \\
&\leqslant \frac{\varepsilon}{8}\,|\nabla w^\varepsilon|_{L^2}^2+|w^\varepsilon|_{L^2}^2+\kappa\varepsilon^2
\end{aligned}
$$

(3.19)

$$
\begin{aligned}
\left|\int_\Omega (\varphi^\varepsilon\cdot\nabla)\,v^0\cdot w^\varepsilon\right| &= \left|\int_\Omega (\varphi^{\varepsilon,1}\cdot\nabla)\,v^0\cdot w^\varepsilon+(\varphi^{\varepsilon,2}\cdot\nabla)\,v^0\cdot w^\varepsilon\right| \\
&\leqslant |\varphi^{\varepsilon,2}|_{L^2}\,|\nabla v^0|_{L^\infty}\,|w^\varepsilon|_{L^2}+|\nabla v^0|_{L^\infty}\,|z\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\leqslant \text{(thanks to (2.10) and Hardy's inequality)} \\
&\leqslant \kappa\varepsilon\,|w^\varepsilon|_{L^2}+\kappa\varepsilon^{3/2}\,|\nabla w^\varepsilon|_{L^2} \\
&\leqslant \frac{\varepsilon}{8}\,|\nabla w^\varepsilon|_{L^2}^2+|w^\varepsilon|_{L^2}^2+\kappa\varepsilon^2.
\end{aligned}
$$

(3.20)

$$
\begin{aligned}
\left|\int_\Omega (v^0\cdot\nabla)\,\varphi^\varepsilon\cdot w^\varepsilon\right| &= \left|\int_\Omega v_1^0 D_1\varphi^\varepsilon\cdot w^\varepsilon + v_2^0 D_2\varphi^\varepsilon\cdot w^\varepsilon + v_3^0 D_3\varphi^\varepsilon\cdot w^\varepsilon\right| \\
&\leqslant \left|\int_\Omega v_1^0 D_1\varphi^{\varepsilon,1}\cdot w^\varepsilon + v_1^0 D_1\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\quad + \left|\int_\Omega v_2^0 D_2\varphi^{\varepsilon,1}\cdot w^\varepsilon + v_2^0 D_2\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\quad + \left|\int_\Omega v_3^0 D_3\varphi^{\varepsilon,1}\cdot w^\varepsilon + v_3^0 D_3\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\leqslant |v_1^0|_{L^\infty}\,|D_1\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2} + |v_1^0|_{L^\infty}\,|zD_1\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\quad + |v_2^0|_{L^\infty}\,|D_2\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2} + |v_2^0|_{L^\infty}\,|zD_2\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\quad + |v_3^0|_{L^\infty}\,|D_3\varphi^{\varepsilon,2}|_{L^2}\,|w^\varepsilon|_{L^2} + \left|\frac{v_3^0}{z}\right|_{L^\infty}|z^2D_3\varphi^{\varepsilon,1}|_{L^2}\left|\frac{w^\varepsilon}{z}\right|_{L^2} \\
&\leqslant \text{(thanks to (2.10), Hardy's inequality,} \\
&\qquad \text{and the fact that } v_3^0|_{z=0}=0 \text{ and } v^0 \text{ is smooth)} \\
&\leqslant \kappa\varepsilon\,|w^\varepsilon|_{L^2} + \kappa\varepsilon^{3/2}\,|\nabla w^\varepsilon|_{L^2} \\
&\leqslant \frac{\varepsilon}{8}\,|\nabla w^\varepsilon|_{L^2}^2 + |w^\varepsilon|_{L^2}^2 + \kappa\varepsilon^2.
\end{aligned}
$$

Now we estimate the most difficult nonlinear term on the left hand side.

(3.21)

$$
\begin{aligned}
\left|\int_\Omega (w^\varepsilon\cdot\nabla)\,\varphi^\varepsilon\cdot w^\varepsilon\right| &= \left|\int_\Omega w_1^\varepsilon D_1\varphi^\varepsilon\cdot w^\varepsilon + w_2^\varepsilon D_2\varphi^\varepsilon\cdot w^\varepsilon + w_3^\varepsilon D_3\varphi^\varepsilon\cdot w^\varepsilon\right| \\
&\leqslant (|D_1\varphi^\varepsilon|_{L^\infty} + |D_2\varphi^\varepsilon|_{L^\infty})\,|w^\varepsilon|_{L^2}^2 \\
&\quad + \left|\int_\Omega w_3^\varepsilon D_3\varphi^{\varepsilon,1}\cdot w^\varepsilon\right| + \left|\int_\Omega w_3^\varepsilon D_3\varphi^{\varepsilon,2}\cdot w^\varepsilon\right| \\
&\leqslant \text{(thanks to (2.10))} \\
&\leqslant \kappa\,|w^\varepsilon|_{L^2}^2 + \left|\int_\Omega D_3 w_3^\varepsilon\varphi^{\varepsilon,1}\cdot w^\varepsilon\right| + \left|\int_\Omega w_3^\varepsilon\varphi^{\varepsilon,1}D_3 w\right| \\
&\leqslant \text{(thanks to Hardy's inequality)} \\
&\leqslant \kappa\,|w^\varepsilon|_{L^2}^2 + 4\,|z\varphi^{\varepsilon,1}|_{L^\infty}\,|\nabla w^\varepsilon|_{L^2}^2.
\end{aligned}
$$

In order to dominate (3.21) by the dissipation term $\varepsilon\,|\nabla w^\varepsilon|^2$, it is sufficient to have

$$|z\varphi^{\varepsilon,1}|_{L^\infty} \leqslant \frac{\varepsilon}{32}. \tag{3.22}$$

Thanks to (2.10), this is equivalent to

$$\|v^0\|_{L^\infty(z=0)}\ \rho\left(\frac{z}{h}\right) e^{-Uz/\varepsilon} z \leqslant \frac{\varepsilon}{32} \tag{3.23}$$

or

$$\frac{\|v^0\|_{L^\infty(z=0)}}{U} \leqslant \frac{e}{32}. \tag{$*$}$$

Hence under the assumption of $(*)$ we have

$$\left|\int_\Omega (w^\varepsilon\cdot\nabla)\,\varphi^\varepsilon\cdot w^\varepsilon\right| \leqslant \kappa\,|w^\varepsilon|^2_{L^2} + \frac{\varepsilon}{8}\,|\nabla w^\varepsilon|^2_{L^2}. \tag{3.24}$$

Combining (3.12)–(3.24) and assumption $(*)$ we find

$$\frac{d}{dt}\,|w^\varepsilon|^2_{L^2} + \varepsilon\,|\nabla w^\varepsilon|^2_{L^2} \leqslant \kappa\,|w^\varepsilon|^2_{L^2} + \kappa\varepsilon^2 \tag{3.25}$$

provided $(*)$ holds.

We then deduce, via the usual Gronwall inequality

$$\begin{cases} \|w^\varepsilon\|_{L^\infty(0,T;H)} \leqslant \kappa\varepsilon, \\ \|w^\varepsilon\|_{L^2(0,T;V)} \leqslant \kappa\varepsilon^{\frac{1}{2}}. \end{cases} \tag{3.26}$$

This implies that the convergence of v^ε to v^0 is true and that the boundary layer has thickness ε/U, as indicated by the form of φ^ε.

We now discuss the validity of the condition $(*)$, or the relative smallness of the tangential slip with respect to the normal flux. We will assume the (local in time in the three dimensional case) well posedness of the inviscid problem. By the compatibility condition that we impose on the data we have for the initial data

$$\frac{\|v_0\|_{L^\infty(z=0)}}{U} = \frac{\max\{\|u_{01}\|_{L^\infty(z=0)},\ \|u_{02}\|_{L^\infty(z=0)}\}}{U} = 0. \tag{3.27}$$

Thus by continuous dependence we deduce that there exists $T_* > 0$, such that

$$\frac{\|v^0(t)\|_{L^\infty(z=0)}}{U} \leqslant \frac{e}{32} \quad \text{for} \quad t \leqslant T_*. \tag{3.28}$$

Hence the convergence (3.26) is true for $T = T_*$. This is a short time result.

THEOREM 3.1. *Let f and u_0 be smooth functions satisfying certain compatibility conditions so that the inviscid problem* (3.4)–(3.5) *is well-posed (at least locally in time). Let u^ε and u^0 be the solution of* (3.1) *and* (3.4)–(3.5). *Then there exists a time $T_* > 0$ so that* (3.28) *is satisfied.*

Moreover, as $\varepsilon \to 0$, $u^\varepsilon - u^0$ is estimated as

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T_*;H)} = \|v^\varepsilon - v^0\|_{L^\infty(0,T_*;H)} \leqslant \kappa\varepsilon^{1/2}, \tag{3.29}$$

$$\|u^\varepsilon - u^0 + \varphi^\varepsilon\|_{L^2(0,T_*;V)} = \|v^\varepsilon - v^0 + \varphi^\varepsilon\|_{L^2(0,T_*;V)} \leqslant \kappa\varepsilon^{1/2}. \tag{3.30}$$

The adjusted difference $w^\varepsilon = u^\varepsilon - u^0 + \varphi^\varepsilon = v^\varepsilon - v^0 + \varphi^\varepsilon$, the corrector φ^ε being given by (2.2), *is estimated by* (2.10).

Remark 3.1. In general for a non-trivial f, condition $(*)$ will be violated after a sufficiently long time. However, in the special case of $f = 0$, is might be possible for $(*)$ to hold for all time provided that it is true at the initial time. Of course this is some what equivalent to a maximum principle on the velocity field for solutions of the 3D Euler equations. We are not sure if this is valid and it will be subject to our further investigation.

Remark 3.2. The estimate in the theorem is optimal so far as the dependence on the viscosity ε is concerned. Indeed we observe that in the case when everything depend on the vertical variable z only the problem reduce to a linear convection-diffusion equation with small diffusive coefficient and it is shown in the thesis of Cheng [5] that the estimates listed in the theorem is optimal with the choice of corrector like φ^ε.

4. UNIFORM ESTIMATES IN SPACE AND TIME IN SPACE DIMENSION TWO

Our aim in this section is to derive estimates of the adjusted *error* $u^\varepsilon - u^0 + \varphi^\varepsilon = w^\varepsilon$ which are *uniform* in space and time, whereas the previous

estimates where in L^2 and Sobolev spaces, namely for mean root square norms.

There is no known entropy function for homogeneous incompressible flows nor maximum principle for the velocity field. Hence the usual way to derive space uniform estimate is to derive estimate in higher Sobolev spaces (in H^k for $k > \frac{n}{2}$ where n is the dimension of the space) and apply Sobolev imbedding (see for instance [7, 9, 30]). However, this technique is not convenient for our boundary layer problem, at least with our choice of corrector in the form of φ^ε. In deed for a one-dimensional linearized convection-diffusion problem (a special case of our problem with dependence on the vertical variable z only) it is shown by Cheng [5] (Indiana University, 1999) that the best possible estimate in H^2 is of the order $\varepsilon^{-1/2}$ and the best possible estimate in H^1 is of the order $\varepsilon^{1/2}$. Hence according to interpolation inequality and Sobolev imbedding in dimension three we only obtain a uniform bound on w^ε of order one which is not good. In space dimension two, the estimates are good enough to derive $L^2(0, T; L^\infty)$ type estimates on w^ε. However, we are not able to derive strong enough time uniform estimates ($L^\infty(0, T; H^1)$ say) which vanishes at vanishing viscosity. Notice $L^\infty(0, T; H^1)$ estimate on w^ε is perhaps the minimum that we need to ask for in order to establish uniform estimates in space and in time provided we use a classical Sobolev imbedding. We suspect that such estimate may not exist due to possible intermittency in time. The remedy is to then to consider an anisotropic Sobolev imbedding that we developed before (see [32, Remark 4.2]) which do not need the whole gradient but derivative in one direction more than other directions. We combine this anisotropic imbedding with our previous idea of better control on the tangential derivative enables us to establish the space-time uniform estimates of the boundary layer. This verifies that our idea of better control on tangential derivative than the whole gradient is true at least in this non-characteristic case. We still need to work on the physically more interesting no-slip boundary case in the future.

In the process of establishing space-time uniform estimates for w^ε we encounter the same kind of difficulty as in the proof of the regularity of weak solution to the Navier–Stokes equations in the three dimensional case. Hence *we will restrict ourself to the two dimensional case for the space-time uniform estimates*.

Our aim is to derive an L^∞ estimate on w^ε both in time and in space. We will rely on our anisotropic Sobolev imbedding derived in [32], and the idea of bounding the tangential derivatives [31–34].

We differentiate the equation for the adjusted difference (3.11) in time and obtain

(4.1)
$$\frac{\partial^2 w^\varepsilon}{\partial t^2} - \varepsilon \Delta \frac{\partial w^\varepsilon}{\partial t} + (v^\varepsilon \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} + \left(\frac{\partial v^\varepsilon}{\partial t} \cdot \nabla\right) w^\varepsilon + \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) v^0 + (w^\varepsilon \cdot \nabla) \frac{\partial v^0}{\partial t}$$
$$- \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) \varphi^\varepsilon - (w^\varepsilon \cdot \nabla) \frac{\partial \varphi^\varepsilon}{\partial t} - U D_3 \frac{\partial w^\varepsilon}{\partial t} + \nabla \frac{\partial}{\partial t} (p^\varepsilon - p^0)$$
$$= \frac{\partial^2 \varphi^\varepsilon}{\partial t^2} - \varepsilon \Delta \frac{\partial v^0}{\partial t} - \frac{\partial}{\partial t} (\varepsilon \Delta \varphi^\varepsilon + U D_3 \varphi^\varepsilon) - \left(\frac{\partial \varphi^\varepsilon}{\partial t} \cdot \nabla\right) \varphi^\varepsilon - (\varphi^\varepsilon \cdot \nabla) \frac{\partial \varphi^\varepsilon}{\partial t}$$
$$+ \left(\frac{\partial \varphi^\varepsilon}{\partial t} \cdot \nabla\right) v^0 + (\varphi^\varepsilon \cdot \nabla) \frac{\partial v^0}{\partial t} + \left(\frac{\partial v^0}{\partial t} \cdot \nabla\right) \varphi^\varepsilon + (v^0 \cdot \nabla) \frac{\partial \varphi^\varepsilon}{\partial t},$$

(4.2)
$$\frac{\partial w^\varepsilon}{\partial t} = -\varepsilon P \Delta v_0, \qquad \text{at} \quad t = 0,$$

(4.3)
$$\operatorname{div} \frac{\partial w^\varepsilon}{\partial t} = 0,$$

(4.4)
$$\frac{\partial w^\varepsilon}{\partial t} = 0 \qquad \text{at} \quad z = 0, h.$$

Next we multiply (4.1) by $\partial w^\varepsilon / \partial t$ and integrate over Ω. Notice that

(4.5)
$$\int_\Omega \frac{\partial^2 w^\varepsilon}{\partial t} \cdot \frac{\partial w^\varepsilon}{\partial t} = \frac{1}{2} \frac{d}{dt} \left| \frac{\partial w^\varepsilon}{\partial t} \right|_{L^2}^2,$$

(4.6)
$$\int_\Omega -\varepsilon \Delta \frac{\partial w^\varepsilon}{\partial t} \cdot \frac{\partial w^\varepsilon}{\partial t} = \varepsilon \left| \nabla \frac{\partial w^\varepsilon}{\partial t} \right|_{L^2}^2,$$

(4.7)
$$\int_\Omega (v^\varepsilon \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} \cdot \frac{\partial w^\varepsilon}{\partial t} = 0,$$

(4.8)
$$\left| \int_\Omega \left(\frac{\partial v^\varepsilon}{\partial t} \cdot \nabla\right) w^\varepsilon \cdot \frac{\partial w^\varepsilon}{\partial t} \right|$$
$$\leqslant \left| \int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) w^\varepsilon \cdot \frac{\partial w^\varepsilon}{\partial t} \right| + \left| \int_\Omega \left(\left(\frac{\partial v^0}{\partial t} + \frac{\partial \varphi^\varepsilon}{\partial t}\right) \cdot \nabla\right) w^\varepsilon \cdot \frac{\partial w^\varepsilon}{\partial t} \right|$$
$$\leqslant |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial w^\varepsilon}{\partial t} \right|_{L^4}^2 + \kappa \, |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial w^\varepsilon}{\partial t} \right|_{L^2}$$
$$\leqslant \text{(Sobolev imbedding and interpolation)}$$

$$\leqslant \kappa\, |\nabla w^\varepsilon|_{L^2} \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} + \kappa\, |\nabla w^\varepsilon|_{L^2} \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}$$

$$\leqslant \frac{\varepsilon}{8}\left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 + \kappa\left(\frac{|\nabla w^\varepsilon|_{L^2}^2}{\varepsilon}+1\right)\left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 + \kappa\, |\nabla w^\varepsilon|_{L^2}^2,$$

$$\left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t}\cdot\nabla\right) v^0\cdot\frac{\partial w^\varepsilon}{\partial t}\right| \leqslant \kappa \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2, \tag{4.9}$$

$$\begin{aligned}\left|\int_\Omega (w^\varepsilon\cdot\nabla)\frac{\partial v^0}{\partial t}\cdot\frac{\partial w^\varepsilon}{\partial t}\right| &\leqslant \kappa\, |w^\varepsilon|_{L^2}\left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 \\ &\leqslant \kappa\varepsilon \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 \\ &\leqslant \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 + \kappa\varepsilon^2\end{aligned} \tag{4.10}$$

(4.11)

$$\begin{aligned}&\left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t}\cdot\nabla\right)\varphi^\varepsilon\cdot\frac{\partial w^\varepsilon}{\partial t}\right| \\ &\quad\leqslant \left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t}\cdot\nabla\right)\frac{\partial w^\varepsilon}{\partial t}\,\varphi^\varepsilon\right| \\ &\quad\leqslant \left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}\left|\frac{1}{z}\,\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} |z\varphi^\varepsilon|_{L^\infty} \\ &\quad\leqslant \text{(by Hardy's inequality and by the smallness assumption (*))} \\ &\quad\leqslant \frac{\varepsilon}{8}\left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega (w^\varepsilon\cdot\nabla)\frac{\partial\varphi^\varepsilon}{\partial t}\cdot\frac{\partial w^\varepsilon}{\partial t}\right| &\leqslant \left|\int_\Omega (w^\varepsilon\cdot\nabla)\frac{\partial w^\varepsilon}{\partial t}\,\frac{\partial \varphi^\varepsilon}{\partial t}\right| \\ &\leqslant \left|\frac{w^\varepsilon}{z}\right|_{L^2}\left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}\left|z\,\frac{\partial\varphi^\varepsilon}{\partial t}\right|_{L^\infty} \\ &\leqslant \kappa\varepsilon\, |\nabla w^\varepsilon|_{L^2}\left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \\ &\leqslant \frac{\varepsilon}{8}\left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 + \kappa\varepsilon\, |\nabla w^\varepsilon|_{L^2}^2,\end{aligned} \tag{4.12}$$

(4.13)
$$\int_{\Omega} UD_3 \frac{\partial w^{\varepsilon}}{\partial t} \frac{\partial w^{\varepsilon}}{\partial t} = 0,$$

(4.14)
$$\int_{\Omega} \nabla \frac{\partial}{\partial t} (p^{\varepsilon} - p^0) \cdot \frac{\partial w^{\varepsilon}}{\partial t} = 0.$$

For the right hand side, we use the same decomposition into a boundary layer part and regular part. Repeating the same procedure as in the previous section we have

(4.15)
$$\left| \int_{\Omega} \mathrm{RHS} \cdot \frac{\partial w^{\varepsilon}}{\partial t} \right| \leqslant \frac{\varepsilon}{8} \left| \nabla \frac{\partial w^{\varepsilon}}{\partial t} \right|_{L^2}^2 + \kappa \varepsilon^2.$$

Hence we have

(4.16)
$$\frac{d}{dt} \left| \frac{\partial w^{\varepsilon}}{\partial t} \right|_{L^2}^2 + \varepsilon \left| \nabla \frac{\partial w^{\varepsilon}}{\partial t} \right|_{L^2}^2 \leqslant \kappa \left(\frac{|\nabla w^{\varepsilon}|_{L^2}^2}{\varepsilon} + 1 \right) \left| \frac{\partial w^{\varepsilon}}{\partial t} \right|_{L^2}^2 + \kappa \, |\nabla w^{\varepsilon}|_{L^2}^2 + \kappa \varepsilon^2,$$

(4.17)
$$\left| \frac{\partial w^{\varepsilon}}{\partial t} \right|_{L^2} = \kappa \varepsilon \qquad \text{at} \qquad t = 0.$$

From this we deduce, after applying the usual Gronwall inequality and the $L^2(0, T; V)$ estimate on w^{ε} (3.26), that

(4.18)
$$\left\| \frac{\partial w^{\varepsilon}}{\partial t} \right\|_{L^{\infty}(0, T; L^2)} \leqslant \kappa \varepsilon^{1/2},$$

(4.19)
$$\left\| \frac{\partial w^{\varepsilon}}{\partial t} \right\|_{L^2(0, T; V)} \leqslant \kappa.$$

Integrating $\partial w^{\varepsilon} / \partial t$ in time we deduce, thanks to (4.19),

(4.20)
$$\|w^{\varepsilon}\|_{L^{\infty}(0, T; V)} \leqslant \kappa.$$

Our second step is to derive estimates on tangential derivatives. For this purpose we apply D_1 to the equation for the adjusted difference (3.15) and obtain

(4.21)
$$\begin{aligned} &\frac{\partial}{\partial t} D_1 w^{\varepsilon} - \varepsilon \Delta D_1 w^{\varepsilon} + (D_1 v^{\varepsilon} \cdot \nabla) \, w^{\varepsilon} + (v^{\varepsilon} \cdot \nabla) \, D_1 w^{\varepsilon} \\ &\qquad + (D_1 w^{\varepsilon} \cdot \nabla) \, v^0 + (w^{\varepsilon} \cdot \nabla) \, D_1 v^0 - (D_1 w^{\varepsilon} \cdot \nabla) \, \varphi^{\varepsilon} \\ &\qquad - (w^{\varepsilon} \cdot \nabla) \, D_1 \varphi^{\varepsilon} - U D_3 D_1 w^{\varepsilon} + \nabla D_1 (p^{\varepsilon} - p^0) \\ &\quad = D_1 (\mathrm{R.H.S.}), \end{aligned}$$

$$\operatorname{div} D_1 w^\varepsilon = 0, \tag{4.22}$$

$$D_1 w^\varepsilon = 0 \qquad \text{at} \quad z = 0, h, \tag{4.23}$$

$$D_1 w^\varepsilon = 0 \qquad \text{at} \quad t = 0. \tag{4.24}$$

Next we multiply (4.21) by $D_1 w^\varepsilon$ and integrate over Ω. We have

$$\int_\Omega \frac{\partial}{\partial t} D_1 w^\varepsilon \cdot D_1 w^\varepsilon = \frac{1}{2} \frac{d}{dt} |D_1 w^\varepsilon|_{L^2}^2, \tag{4.25}$$

$$\int_\Omega -\varepsilon \Delta D_1 w^\varepsilon \cdot D_1 w^\varepsilon = \varepsilon \, |\nabla D_1 w^\varepsilon|_{L^2}^2, \tag{4.26}$$

(4.27)

$$\begin{aligned}
&\left| \int_\Omega (D_1 v^\varepsilon \cdot \nabla) \, w^\varepsilon \cdot D_1 w^\varepsilon \right| \\
&\quad \leqslant \left| \int_\Omega (D_1 w^\varepsilon \cdot \nabla) \, w^\varepsilon \cdot D_1 w^\varepsilon \right| + \left| \int_\Omega ((D_1 v^0 + D_1 \varphi^\varepsilon) \cdot \nabla) \, w^\varepsilon \cdot D_1 w^\varepsilon \right| \\
&\quad \leqslant |\nabla w^\varepsilon|_{L^2} \, |D_1 w^\varepsilon|_{L^4}^2 + \kappa \, |\nabla w^\varepsilon|_{L^2} \, |D_1 w^\varepsilon|_{L^2} \\
&\quad \leqslant \text{(Sobolev imbedding and interpolation)} \\
&\quad \leqslant \kappa \, |\nabla w^\varepsilon|_{L^2} \, |D_1 w^\varepsilon|_{L^2} \, |\nabla D_1 w^\varepsilon|_{L^2} + \kappa \, |\nabla w^\varepsilon|_{L^2} \, |D_1 w^\varepsilon|_{L^2} \\
&\quad \leqslant \frac{\varepsilon}{8} \, |\nabla D_1 w^\varepsilon|_{L^2}^2 + \kappa \left(\frac{|\nabla w^\varepsilon|_{L^2}^2}{\varepsilon} + 1 \right) |D_1 w^\varepsilon|_{L^2}^2 + \kappa \, |\nabla w^\varepsilon|_{L^2}^2,
\end{aligned}$$

$$\int_\Omega (v^\varepsilon \cdot \nabla) \, D_1 w^\varepsilon \cdot D_1 w^\varepsilon = 0, \tag{4.28}$$

$$\left| \int_\Omega (D_1 w^\varepsilon \cdot \nabla) \, v^0 \cdot D_1 w^\varepsilon \right| \leqslant \kappa \, |D_1 w^\varepsilon|_{L^2}^2, \tag{4.29}$$

$$\begin{aligned}
\left| \int_\Omega (w^\varepsilon \cdot \nabla) \, D_1 v^0 \cdot D_1 w^\varepsilon \right| &\leqslant \kappa \, |w^\varepsilon|_{L^2} \, |D_1 w^\varepsilon|_{L^2} \\
&\leqslant |D_1 w^\varepsilon|_{L^2}^2 + \kappa \, |w^\varepsilon|_{L^2}^2 \\
&\leqslant |D_1 w^\varepsilon|_{L^2}^2 + \kappa \varepsilon^2,
\end{aligned} \tag{4.30}$$

(4.31)

$$\begin{aligned}
&\left| \int_\Omega (D_1 w^\varepsilon \cdot \nabla) \, \varphi^\varepsilon \cdot D_1 w^\varepsilon \right| \\
&\quad = \left| \int_\Omega (D_1 w^\varepsilon \cdot \nabla) \, D_1 w^\varepsilon \cdot \varphi^\varepsilon \right|
\end{aligned}$$

$$\leqslant \left|\frac{D_1 w^\varepsilon}{z}\right|_{L^2} |\nabla D_1 w^\varepsilon|_{L^2} |z\varphi^\varepsilon|_{L^\infty}$$

$\leqslant$ (Hardy's inequality and the smallness assumption (*))

$$\leqslant \frac{\varepsilon}{8} |\nabla D_1 w^\varepsilon|_{L^2}^2,$$

(4.32)

$$\left|\int_\Omega (w^\varepsilon \cdot \nabla)\, D_1\varphi^\varepsilon \cdot D_1 w^\varepsilon\right| \leqslant \left|\int_\Omega (w^\varepsilon \cdot \nabla)\, D_1 w^\varepsilon \cdot D_1\varphi^\varepsilon\right|$$

$$\leqslant \left|\frac{w^\varepsilon}{z}\right|_{L^2} |\nabla D_1 w^\varepsilon|_{L^2} |zD_1\varphi^\varepsilon|_{L^\infty}$$

$\leqslant$ (Hardy's inequality and the construction of φ^ε)

$$\leqslant \kappa\varepsilon\, |\nabla w^\varepsilon|_{L^2} |\nabla D_1 w^\varepsilon|_{L^2}$$

$$\leqslant \frac{\varepsilon}{8} |\nabla D_1 w^\varepsilon|_{L^2}^2 + \kappa\varepsilon\, |\nabla w^\varepsilon|_{L^2}^2,$$

(4.33) $$\int_\Omega U D_3 D_1 w^\varepsilon \cdot D_1 w^\varepsilon = 0$$

(4.34) $$\int_\Omega \nabla D_1(p^\varepsilon - p^0) \cdot D_1 w^\varepsilon = 0$$

(4.35) $$\left|\int_\Omega D_1(\text{R.H.S.}) \cdot D_1 w^\varepsilon\right| \leqslant |D_1^2(\text{R.H.S.})|_{L^2} |w^\varepsilon|_{L^2}$$

$$\leqslant \kappa\varepsilon^{3/2}.$$

Combining (4.25)–(4.35) we deduce

(4.36)

$$\frac{d}{dt} |D_1 w^\varepsilon|_{L^2}^2 + \varepsilon\, |\nabla D_1 w^\varepsilon|_{L^2}^2 \leqslant \kappa \left(\frac{|\nabla w^\varepsilon|_{L^2}^2}{\varepsilon} + 1\right) |D_1 w^\varepsilon|_{L^2}^2 + \kappa\, |\nabla w^\varepsilon|_{L^2}^2 + \kappa\varepsilon^{3/2},$$

(4.37) $D_1 w^\varepsilon = 0 \quad$ at $\quad t = 0.$

The usual Gronwall inequality combined with (3.26) implies

(4.38) $$\|D_1 w^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \kappa\varepsilon^{1/2},$$

(4.39) $$\|D_1 w^\varepsilon\|_{L^2(0,T;V)} \leqslant \kappa.$$

Combining (3.26), (4.39) with the anisotropic Sobolev imbedding established in [32, Remark 4.2], we deduce

$$\|w^\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \leqslant \kappa(\|w^\varepsilon\|_{L^2(0,T;V)} + \|w^\varepsilon\|^{1/2}_{L^2(0,T;H)} \|D_1 w^\varepsilon\|^{1/2}_{L^2(0,T;V)}) \leqslant \kappa\varepsilon^{1/2}. \tag{4.40}$$

This confirms that the boundary layer of u^ε is explicitly given by φ^ε.

Remark 4.1. Inequalities (4.38)–(4.39) should be compared to (4.18), (4.19), (4.20) where we observed that we are able to derive better estimates on tangential derivatives.

To obtain uniform in time and in space estimates on w^ε we need to derive uniform in time estimates on $|\nabla D_1 w^\varepsilon|_{L^2}$-(see for instance the anisotropic Sobolev imbedding).

For this purpose we differentiate (4.21) in time and we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} D_1 w^\varepsilon - \varepsilon\Delta \frac{\partial}{\partial t} D_1 w^\varepsilon + \left(\frac{\partial}{\partial t} D_1 v^\varepsilon \cdot \nabla\right) w^\varepsilon + (D_1 v^\varepsilon \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} \\
&\quad + \left(\frac{\partial}{\partial t} v^\varepsilon \cdot \nabla\right) D_1 w^\varepsilon + (v^\varepsilon \cdot \nabla) \frac{\partial}{\partial t} D_1 w^\varepsilon \\
&\quad + \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla\right) v^0 + (D_1 w^\varepsilon \cdot \nabla) \frac{\partial v^0}{\partial t} \\
&\quad + \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) D_1 v^0 + (w^\varepsilon \cdot \nabla) \frac{\partial}{\partial t} D_1 v^0 \\
&\quad - \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla\right) \varphi^\varepsilon - (D_1 w^\varepsilon \cdot \nabla) \frac{\partial \varphi^\varepsilon}{\partial t} \\
&\quad - \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) D_1 \varphi^\varepsilon - (w^\varepsilon \cdot \nabla) \frac{\partial}{\partial t} D_1 \varphi^\varepsilon \\
&\quad - U D_3 \frac{\partial}{\partial t} D_1 w^\varepsilon + \nabla \frac{\partial}{\partial t} D_1 (p^\varepsilon - p^0) \\
&= \frac{\partial}{\partial t} D_1 (\text{R.H.S.}).
\end{aligned} \tag{4.41}$$

$$\frac{\partial}{\partial t} D_1 w^\varepsilon = -\varepsilon P \Delta D_1 v_0, \qquad \text{at} \quad t = 0, \tag{4.42}$$

$$\frac{\partial}{\partial t} D_1 w^\varepsilon = 0 \qquad \text{at} \quad z = 0, h, \tag{4.43}$$

$$\operatorname{div} \frac{\partial}{\partial t} D_1 w^\varepsilon = 0. \tag{4.44}$$

We now multiply (4.41) by $\frac{\partial}{\partial t} D_1 w^\varepsilon$ and integrate over Ω; notice that

$$\int_\Omega \frac{\partial^2}{\partial t^2} D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon = \frac{1}{2} \frac{d}{dt} \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^2}^2, \tag{4.45}$$

$$\int_\Omega -\varepsilon \Delta \frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon = \varepsilon \left| \frac{\partial}{\partial t} \nabla D_1 w^\varepsilon \right|_{L^2}^2, \tag{4.46}$$

(4.47)

$$\begin{aligned}
&\left| \int_\Omega \left(\frac{\partial}{\partial t} D_1 v^\varepsilon \cdot \nabla \right) w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right| \\
&\quad \leqslant \left| \int_\Omega \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla \right) w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right| \\
&\quad\quad + \left| \int_\Omega \left(\frac{\partial}{\partial t} D_1 (v^0 + \varphi^\varepsilon) \cdot \nabla \right) w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right| \\
&\quad \leqslant |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^4}^2 \\
&\quad\quad + \left| \frac{\partial}{\partial t} D_1 (v^0 + \varphi^\varepsilon) \right|_{L^\infty} |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^2} \\
&\quad \leqslant \text{(Sobolev imbedding and interpolation)} \\
&\quad \leqslant \kappa\, |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^2} \left| \frac{\partial}{\partial t} \nabla D_1 w^\varepsilon \right|_{L^2} \\
&\quad\quad + \kappa\, |\nabla w^\varepsilon|_{L^2} \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^2} \\
&\quad \leqslant \text{(Cauchy–Schwarz)} \\
&\quad \leqslant \frac{\varepsilon}{8} \left| \frac{\partial}{\partial t} \nabla D_1 w^\varepsilon \right|_{L^2}^2 + \kappa \left(\frac{|\nabla w^\varepsilon|_{L^2}^2}{\varepsilon} + 1 \right) \left| \frac{\partial}{\partial t} D_1 w^\varepsilon \right|_{L^2}^2 \\
&\quad\quad + \kappa\, |\nabla w^\varepsilon|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&\left| \int_\Omega (D_1 v^\varepsilon \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right| \\
&\quad \leqslant \left| \int_\Omega (D_1 w^\varepsilon \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right| \\
&\quad\quad + \left| \int_\Omega (D_1 (v^0 + \varphi^\varepsilon) \cdot \nabla) \frac{\partial w^\varepsilon}{\partial t} \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon \right|
\end{aligned} \tag{4.48}$$

$$
\begin{aligned}
&\leqslant |D_1 w^\varepsilon|_{L^4} \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^4} \\
&\quad + |D_1(v^0+\varphi^\varepsilon)|_{L^\infty} \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\
&\leqslant \text{(Sobolev imbedding and interpolation)} \\
&\leqslant \kappa\, |D_1 w^\varepsilon|_{L^2}^{1/2} |D_1 w^\varepsilon|_{H^1}^{1/2} \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^{1/2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{H^1}^{1/2} \\
&\quad + \kappa \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\
&\leqslant \text{(thanks to (4.38), (4.39))} \\
&\leqslant \kappa \varepsilon^{\frac{1}{4}} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^{1/2} \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} |D_1 w^\varepsilon|_{H^1}^{1/2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^{1/2} \\
&\quad + \kappa \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\
&\leqslant \text{(thanks to Young's inequality)} \\
&\leqslant \frac{\varepsilon}{8} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2 + \kappa(|D_1 w^\varepsilon|_V^2 + 1) \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa \left|\nabla \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2,
\end{aligned}
$$

(4.49)

$$
\begin{aligned}
&\left|\int_\Omega \left(\frac{\partial}{\partial t} v^\varepsilon \cdot \nabla\right) D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| \\
&\quad \leqslant \left|\int_\Omega \left(\frac{\partial}{\partial t} w^\varepsilon \cdot \nabla\right) D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| \\
&\qquad + \left|\int_\Omega \left(\frac{\partial}{\partial t} (v^0+\varphi^\varepsilon) \cdot \nabla\right) D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| \\
&\quad \leqslant \left|\frac{\partial}{\partial t} w^\varepsilon\right|_{L^4} |\nabla D_1 w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^4} \\
&\qquad + \left|\frac{\partial}{\partial t} (v^0+\varphi^\varepsilon)\right|_{L^\infty} |\nabla D_1 w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\
&\quad \leqslant \text{(using Sobolev imbedding and interpolation)}
\end{aligned}
$$

$$\leqslant \kappa \left|\frac{\partial}{\partial t} w^\varepsilon\right|_{L^2}^{1/2} \left|\frac{\partial}{\partial t} w^\varepsilon\right|_{H^1}^{1/2} |\nabla D_1 w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^{1/2} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^{1/2}$$

$$+\kappa\, |\nabla D_1 w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}$$

$\leqslant$ (thanks to (4.18), (4.19), and Young's inequality)

$$\leqslant \frac{\varepsilon}{8}\left|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\left(\left|\frac{\partial}{\partial t} w^\varepsilon\right|_V^2 + 1\right)\left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|^2 + \kappa\, |\nabla D_1 w^\varepsilon|_{L^2}^2,$$

$$\int_\Omega (v^\varepsilon \cdot \nabla)\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon = 0, \tag{4.50}$$

$$\left|\int_\Omega \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla\right) v^0 \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| \leqslant \kappa \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2, \tag{4.51}$$

$$\begin{aligned}\left|\int_\Omega (D_1 w^\varepsilon \cdot \nabla)\frac{\partial v^0}{\partial t}\cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &\leqslant \kappa\, |D_1 w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\, |D_1 w^\varepsilon|_{L^2}^2 \\ &\leqslant \text{(thanks to (4.38))} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon,\end{aligned} \tag{4.52}$$

$$\begin{aligned}\left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t}\cdot \nabla\right) D_1 v^0 \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &\leqslant \kappa \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa \left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2 \\ &\leqslant \text{(thanks to (4.38))} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + k\varepsilon\end{aligned} \tag{4.53}$$

$$\begin{aligned}\left|\int_\Omega (w^\varepsilon \cdot \nabla)\frac{\partial}{\partial t} D_1 v^0 \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &\leqslant \kappa\, |w^\varepsilon|_{L^2} \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\, |w^\varepsilon|_{L^2}^2 \\ &\leqslant \text{(thanks to (3.26))} \\ &\leqslant \left|\frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon^2,\end{aligned} \tag{4.54}$$

(4.55)

$$
\begin{aligned}
\left|\int_\Omega \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla\right) \varphi^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &= \left|\int_\Omega \left(\frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \nabla\right) \frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \varphi^\varepsilon\right| \\
&\leqslant \left|\frac{1}{z}\, \frac{\partial}{\partial t} D_1 w^\varepsilon\right|_{L^2} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2} |z\varphi^\varepsilon|_{L^\infty} \\
&\leqslant \text{(thanks to Hardy's inequality)} \\
&\leqslant \kappa_h\, |z\varphi^\varepsilon|_{L^\infty} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2 \\
&\leqslant \text{(thanks to the smallness assumption (*))} \\
&\leqslant \frac{\varepsilon}{8} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
(4.56) \qquad \left|\int_\Omega (D_1 w^\varepsilon \cdot \nabla) \frac{\partial \varphi^\varepsilon}{\partial t} \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &= \left|\int_\Omega (D_1 w^\varepsilon \cdot \nabla) \frac{\partial}{\partial t} D_1 w^\varepsilon \cdot \frac{\partial \varphi^\varepsilon}{\partial t}\right| \\
&\leqslant \left|\frac{D_1 w^\varepsilon}{z}\right|_{L^2} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2} \left|z\, \frac{\partial \varphi^\varepsilon}{\partial t}\right|_{L^\infty} \\
&\leqslant \text{(thanks to Hardy's inequality)} \\
&\leqslant \kappa\varepsilon\, |D_1 w^\varepsilon|_V \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2} \\
&\leqslant \frac{\varepsilon}{8} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon\, |D_1 w^\varepsilon|_V^2,
\end{aligned}
$$

$$
\begin{aligned}
(4.57) \qquad \left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t} \cdot \nabla\right) D_1 \varphi^\varepsilon \cdot \frac{\partial}{\partial t} D_1 w^\varepsilon\right| &= \left|\int_\Omega \left(\frac{\partial w^\varepsilon}{\partial t}\, \nabla\right) \frac{\partial}{\partial t} D_1 w^\varepsilon \cdot D_1 \varphi^\varepsilon\right| \\
&\leqslant \left|\frac{1}{z}\, \frac{\partial w^\varepsilon}{\partial t}\right|_{L^2} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2} |z D_1 \varphi|_{L^\infty} \\
&\leqslant \text{(thanks to Hardy's inequality)} \\
&\leqslant \kappa\varepsilon \left|\frac{\partial w^\varepsilon}{\partial t}\right|_V \cdot \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2 \\
&\leqslant \frac{\varepsilon}{8} \left|\frac{\partial}{\partial t} \nabla D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon \left|\frac{\partial w^\varepsilon}{\partial t}\right|_V^2,
\end{aligned}
$$

$$\left|\int_\Omega (w^\varepsilon\cdot\nabla)\frac{\partial}{\partial t}D_1\varphi^\varepsilon\cdot\frac{\partial}{\partial t}D_1 w^\varepsilon\right| = \left|\int_\Omega (w^\varepsilon\cdot\nabla)\frac{\partial}{\partial t}D_1 w^\varepsilon\cdot\frac{\partial}{\partial t}\cdot D_1\varphi^\varepsilon\right| \tag{4.58}$$

$$\leqslant \left|\frac{w^\varepsilon}{z}\right|_{L^2}\left|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right|_{L^2}\left|z\frac{\partial}{\partial t}D_1\varphi^\varepsilon\right|_{L^\infty}$$

$$\leqslant \text{(Hardy's inequality and (2.10))}$$

$$\leqslant \kappa\varepsilon\,|w^\varepsilon|_V\left|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right|_{L^2}$$

$$\leqslant \frac{\varepsilon}{8}\left|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon\,|w^\varepsilon|_V^2,$$

$$\int_\Omega UD_3\frac{\partial}{\partial t}D_1 w^\varepsilon\cdot\frac{\partial}{\partial t}D_1 w^\varepsilon = 0, \tag{4.59}$$

$$\int_\Omega \nabla\frac{\partial}{\partial t}D_1(p^\varepsilon - p^0)\cdot\frac{\partial}{\partial t}D_1 w^\varepsilon = 0 \tag{4.60}$$

$$\left|\int_\Omega \frac{\partial}{\partial t}D_1(\text{R.H.S.})\cdot\frac{\partial}{\partial t}D_1 w^\varepsilon\right| \leqslant \frac{1}{4}\left|\frac{\partial}{\partial t}D_1 w^\varepsilon\right|_{L^2}^2 + \left|\frac{\partial}{\partial t}D_1(\text{R.H.S.})\right|_{L^2}^2 \tag{4.61}$$

$$\leqslant \frac{1}{4}\left|\frac{\partial}{\partial t}D_1 w^\varepsilon\right|_{L^2}^2 + \kappa\varepsilon.$$

Combining (4.45)–(4.61) we deduce

(4.62)

$$\frac{d}{dt}\left|\frac{\partial}{\partial t}D_1 w^\varepsilon\right|_{L^2}^2 + \varepsilon\left|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right|_{L^2}^2 \leqslant \kappa\left(\frac{|\nabla w^\varepsilon|_{L^2}^2}{\varepsilon} + |D_1 w^\varepsilon|_V^2 + 1\right)\left|\frac{\partial}{\partial t}D_1 w^\varepsilon\right|_{L^2}^2$$

$$+\kappa\,|\nabla w^\varepsilon|_{L^2}^2 + \kappa\left|\nabla\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2}^2$$

$$+\kappa\,|\nabla D_1 w^\varepsilon|_{L^2}^2 + \kappa\varepsilon,$$

$$\frac{\partial}{\partial t}D_1 w^\varepsilon = -\varepsilon\Delta D_1 v_0, \qquad \text{at} \quad t = 0. \tag{4.63}$$

Applying the Gronwall inequality and utilizing (3.26), (4.39), (4.19) we deduce that

$$\left\|\frac{\partial}{\partial t}D_1 w^\varepsilon\right\|_{L^\infty(0,T;\,L^2)} \leqslant \kappa, \tag{4.64}$$

$$\left\|\frac{\partial}{\partial t}\nabla D_1 w^\varepsilon\right\|_{L^2(0,T;\,L^2)} \leqslant \kappa\varepsilon^{-1/2}, \tag{4.65}$$

which further implies, via integrating (4.65) in time,

$$\|\nabla D_1 w^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \kappa \varepsilon^{-1/2}. \tag{4.66}$$

Now we apply again the anisotropic Sobolev imbedding from [32], and we obtain

$$\begin{aligned} \|w^\varepsilon\|_{L^\infty((0,T)\times\bar{\Omega})} &\leqslant \kappa(\|w^\varepsilon\|^{1/2}_{L^\infty(0,T;V)} \|D_1 w^\varepsilon\|^{1/2}_{L^\infty(0,T;L^2)} \\ &\quad + \|w^\varepsilon\|^{1/2}_{L^\infty(0,T;L^2)} \|D_1 D_3 w^\varepsilon\|^{1/2}_{L^\infty(0,T;L^2)}) \\ &\leqslant (\text{thanks to (4.20), (4.38), (3.26), (4.66)}), \\ &\leqslant \kappa\varepsilon^{1/4}. \end{aligned} \tag{4.67}$$

Hence we supplement Theorem 3.1 with the following

THEOREM 4.1. *Under the assumptions of Theorem* 3.1, *we have also*

$$\|u^\varepsilon - u^0 + \varphi^\varepsilon\|_{L^\infty((0,T_*)\times\bar{\Omega})} = \|v^\varepsilon - v^0 + \varphi^\varepsilon\|_{L^\infty((0,T_*)\times\bar{\Omega})} \leqslant \kappa\varepsilon^{1/4}. \tag{4.68}$$

Remark. The dependence on the kinematic viscosity ε is not optimal. The optimal convergence rate of ε can be derived via higher order asymptotics.

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