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Bayesian Confidence Limits for the Reliability of Mixed Cascade and Parallel Independent Exponential Subsystems

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***Abstract*—This paper deals with the theoretical problem of deriving Bayesian confidence intervals for the reliability of a system consisting of both cascade and parallel subsystems where each subsystem is independent and has an exponential failure probability density function (pdf). This approach is applicable when test data are available for each individual subsystem and not for the entire system. The Mellin integral transform is used to analyze the system in a step-by-step procedure until the posterior pdf of the system reliability is obtained. The posterior cumulative distribution function is then obtained in the usual manner by integrating the pdf, which serves the dual purpose of yielding system reliability confidence limits while at the same time providing a check on the accuracy of the derived pdf. A computer program has been written in FORTRAN IV to evaluate the confidence limits. An example is presented which uses the computer program.**

Reader Aids:
Purpose: Widen state of the art
Special Math needed for explanations: Theoretical Bayesian statistics
Special Math needed for results: Same
Results useful to: Theoreticians

INTRODUCTION

CONSIDER a system consisting of a mixture of cascade[1] subsystems and parallel networks[2] with each subsystem having a failure pdf that is exponential. The unreliability for a parallel network consisting of M independent subsystems is $Q = Q_1Q_2\cdots Q_M$, while the reliability of N independent cascade subsystems is $R = R_1R_2\cdots R_N$. A Bayesian method which treats the subsystem reliabilities and unreliabilities as random variables and leads to Bayesian confidence limits for the total system reliability is developed in this paper. This method is used to derive the posterior probability distribution for the reliability of a mixture of cascade and parallel independent subsystems with exponential pdf's, and is applicable when test data are available for each individual subsystem but not for the entire system. Such an approach permits the determination for an arbitrary confidence coefficient of a Bayesian type confidence interval whose end points will be denoted as Bayesian confidence limits.

Springer and Thompson [6,7] developed a Bayesian method to obtain Bayesian confidence intervals for independent cascade subsystems where all of the failure pdf's are either exponential or distribution-free. Springer and Byers [5] extended these results to the case where the failure pdf's can be a mixture of exponential and distribution-free. Levy and Moore [4] have obtained confidence limits on the reliability of mixed cascade and parallel subsystems using digital simulation, the authors know of no analytic technique for obtaining system confidence limits for mixed cascade and parallel subsystem, other than the one now being presented. The technique presented in this article requires more computer time than does digital simulation, and thus far has been used for systems with a small number of subsystems. Further research is required to apply it to large scale problems.

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[1] Cascade subsystems are subsystems arranged in series so that the failure of any subsystem results in the failure of the system.
[2] A parallel network (as used here) refers to subsystems, not necessarily identical, in which all must fail to cause a system failure. A system can contain several different parallel networks.

POSTERIOR PDF OF SUBSYSTEM RELIABILITY

In the Bayesian approach, the reliability R_i of subsystem i is regarded as a random variable with a prior pdf $P_i(R_i)$. The posterior pdf for the subsystem reliability is derived by specifying the prior pdf of R_i and using Bayes' theorem. For exponential life pdf's, the posterior pdf of the subsystem reliability is [6]

$$f_i(R_i \mid \hat{T}_i,t_i,r_i) = g_i(\hat{T}_i \mid R_i,r_i,t_i)P_i(R_i) \Big/ \int_0^1 g_i(\hat{T}_i \mid R_i,r_i,t_i)P_i(R_i)\,dR_i \tag{1}$$

where

T_i = total test time for subsystem i
t_i = mission time of subsystem i
r_i = number of failures at which the testing of n_i units of subsystem i is terminated
$g_i(\hat{T}_i \mid R_i,r_i,t_i)$ = conditional pdf of the estimator
$\hat{R}_i = \exp(-r_it_i/\hat{T}_i)$.

The sufficient statistic $\hat{R}_i$ arises from test data which yield a total subsystem test time

$$\hat{T}_i = t_{i1} + \cdots + t_{ir} + (n_i - r_i)t_{ir_i} \tag{2}$$

where $2(\hat{T}_i/t_i)\ln(1/R_i)$ is distributed as chi-square with $2r_i$ degrees of freedom [9]. It follows that $\hat{T}_i$ has the conditional pdf

$$g_i(\hat{T}_i \mid R_i,r_i,t_i) = \frac{(\hat{T}_i/t_i)^{r_i-1}(\ln 1/R_i)^{r_i}R_i^{\hat{T}_i/t_i}}{\Gamma(r_i)t_i}. \tag{3}$$

To complete the derivation of the posterior pdf of R_i, we must specify the prior pdf $P_i(R_i)$. Much discussion in the literature has centered around the selection of the prior pdf. Much of the controversy over the use of Bayesian methods could be eliminated if a prior pdf could be used that would yield the optimum classical bounds. However, Fertig [10] has shown that there do not exist prior pdf's for subsystem reliability that are independent of the current data and still yield the optimum classical bounds.

For exponential subsystems, it is both reasonable and convenient to use the natural conjugate[3] prior pdf

$$P_i(R_i) \equiv \frac{(T_{i0}/t_i + 1)^{r_{i0}+1}}{\Gamma(r_{i0} + 1)} R_i^{T_{i0}/t_i}(\ln 1/R_i)^{r_{i0}}. \tag{4}$$

Here T_{i0} would be identified with an actual or hypothesized total accumulated test time with r_{i0} observed failures in previous experience with the subsystem under consideration, or similar subsystems. With $T_{i0} = 0$ and $r_{i0} = 0$, the prior pdf is uniform over the interval (0,1). As T_{i0} and r_{i0} are increased, the prior pdf is unimodal with the point of zero gradient between $R_i = 0.5$ and 1.0. The greater the values of T_{i0} and r_{i0}, the more the area under the curve is shifted toward $R_i = 1.0$. This type of prior pdf is desirable when prior information reveals that the subsystem has a relatively high reliability.

When $T_{i0} = -1$ and $r_{i0} = -1$ in (4), the posterior pdf coincides with that obtained by the fiducial approach [12]; this prior pdf is "improper"[4], and in terms of R_i is

$$P_i(R_i) = R_i^{-1}(\ln 1/R_i)^{-1}, \quad 0 \leq R_i \leq 1. \tag{5}$$

Berkbigler [11] demonstrated by examples that the results obtained by using (5) and the fiducial approach used by Levy and Moore [4] in their simulation program are the same. Equation (5) is not very realistic from a Bayesian viewpoint, for since it is a U-shaped function it gives more weight to reliabilities close to zero and one.

For the analysis in this paper, the natural conjugate prior pdf (4) is used. The posterior pdf is

$$f_i(R) \equiv f_i(R_i \mid T_i,t_i,r_i) = \frac{(\beta_i + 1)^{r_i+1}}{\Gamma(r_i + 1)} R_i^{\beta_i}(\ln 1/R_i)^{r_i} \tag{6}$$

where

$$\beta_i \equiv (T_i + T_{i0})/t_i$$

$$\gamma_i \equiv r_i + r_{i0}.$$

The posterior pdf for the subsystem unreliability is

$$h_i(Q_i) \equiv f_i(1 - Q_i) \tag{7}$$

and is used to analyze a parallel network.

[3] A family of prior pdf's $P_i(R_i)$ is a natural conjugate for R_i if every member of that family, used as a prior pdf, produces a posterior pdf via (4), which belongs to that family.
[4] The area under the pdf curve is infinite.

ANALYSIS OF MIXED CASCADE AND PARALLEL SUBSYSTEMS

The Mellin transform [13] is a natural tool to use when the pdf of the product of random variables is desired, which is precisely the case when one analyzes system reliability from subsystem test data. The Mellin transform of $f_i(R_i)$ is obtained by evaluating

$$M\{f_i(R_i)\} \equiv \int_0^1 R_i^{s-1} f_i(R_i)\,dR_i \tag{8}$$

where s is a complex variable. The Mellin transform of $g_i(Q_i)$ is obtained in a similar manner. It bears stating that the complexity arises from the fact that no transformation or operation exists by which one can obtain $M\{f_i(R_i)\}$ directly from $M\{g_i(Q_i)\}$.

In order to analyze a network consisting of mixed cascade and parallel independent subsystems, three types of Mellin inversions are required. The configuration of the subsystems for each of the inversions is as follows:

Type I—$N_1 > 1$ cascade subsystems
Type II—N parallel subsystems
Type III—N_1 cascade subsystems plus N_2 parallel networks.

The Mellin transform for the posterior pdf for N_1 cascade subsystems is

$$M\{f_I(R)\} = \prod_{i=1}^{N_1} M\{f_i(R_i)\}, \tag{9}$$

where $f_I(R)$ represents the posterior pdf resulting from a Type I inversion. The Mellin inversion integral is

$$f_I(R) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s} M\{f_I(R)\}. \tag{10}$$

For a Type II inversion, the unreliabilities of the subsystems are used. The Mellin transform of the pdf of the unreliability of N parallel subsystems is

$$M\{h_{II}(Q)\} = \prod_{i=1}^{N} M\{h_i(Q)\} \tag{11}$$

and the associated inversion integral is

$$h_{II}(Q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Q^{-s} M\{h_{II}(Q)\}. \tag{12}$$

The Mellin transform of the posterior pdf resulting from a Type II inversion is required to perform a Type III inversion. Therefore, the transformation,

$$f_{II}(R) = h_{II}(1 - R) \tag{13}$$

is made after a Type II inversion.

The Mellin transform for a Type III inversion is

$$M\{f_{III}(R)\} = \prod_{i=1}^{N_1} M\{f_i(R_i)\} \prod_{j=1}^{N_2} M\{f_j(R_j)\} \tag{14}$$

where there are N_1 cascade subsystems and N_2 parallel networks.

The three types of inversions are used to obtain the posterior pdf for the system reliability, as outlined in detail in the appendix. The posterior cumulative distribution function $F(R)$ for the system reliability can be obtained in either of two ways. In this paper it is obtained using

$$F(R) = \int_0^R f(R)\, dR.$$

The second method uses the Mellin transform of $F(R)$ which is obtained by replacing s by $s + 1$ in $M[f(R)]$ and multiplying the result by $1/s$. Symbolically

$$M\{F(R) \mid s\} = (1/s)M\{f(R) \mid s + 1\}.$$

The Mellin inversion integral is then used to obtain $F(R)$ from $M\{F(R) \mid s\}$. Once the posterior cdf $F(R)$ is known, the Bayesian confidence intervals are obtained using

$$\Pr\{R_L < R < R_U\} \equiv F(R_U) - F(R_L).$$

THE COMPUTER PROGRAM

A computer program using Fortran-based multiple precision subroutines [3] has been written and executed on the UNIVAC 1108. The program is machine independent except for certain constants which must be coded for computers with different word sizes. Using the input data, the program performs each type of inversion at the proper time and determines when the final inversion has been completed. It derives the posterior cdf of the system reliability in tabular form, and the posterior pdf in both functional and tabular form.

When only double precision is used, erroneous results due to roundoff errors may occur. The multiple precision subroutines were used to eliminate the possibility of roundoff errors. The precision level is an input to the computer program and can be changed at will by the analyst. The computer time required to solve a problem depends largely on the configuration of the system and the values of the parameters of the pdf for the subsystems' reliability. The execution time on the UNIVAC 1108 for the problem presented in the next section was six minutes, but can become quite large if several subsystems are involved. This is particularly true if series expansions are required to evaluate the Mellin transforms or if a parallel network contains more than two subsystems.

NUMERICAL EXAMPLE

Consider a system consisting of six independent subsystems as shown in Figure 1. Assume that test data and prior information yield the following parameter values:

$$\hat{\beta}_1 = 20.0, \quad \gamma_1 = 0$$
$$\hat{\beta}_2 = 5.0, \quad \gamma_2 = 1$$
$$\hat{\beta}_3 = 4.0, \quad \gamma_3 = 0$$
$$\hat{\beta}_4 = 5.0, \quad \gamma_4 = 0$$
$$\hat{\beta}_5 = 6.0, \quad \gamma_5 = 1$$

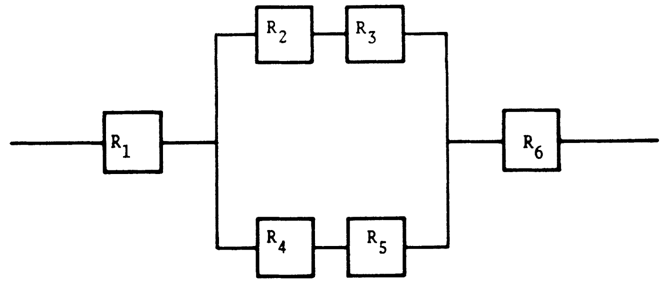

Fig. 1. Block diagram for the example.

$$\hat{\beta}_6 = 25.0, \quad \gamma_6 = 1.$$

The problem is to determine the functional and tabular forms of the posterior pdf $f(R)$, the posterior cdf $F(R)$, and the desired confidence intervals on the system reliability.

The analysis begins by performing two Type I inversions for the two paths in the parallel network, i.e., subsystems 2 and 4 and subsystems 3 and 5. Next a Type II inversion is performed for the parallel network using the results of the Type I inversions. To complete the analysis, a Type III inversion is performed using the results of the Type II inversion and subsystems 1 and 6. The resulting posterior pdf for the system reliability was computed using 16 significant digits and, together with the cdf, is tabulated in Table 1. The intervals of tabulation are entirely up to the analyst. In this example $\Delta R = 0.1$ for $0.10 \leq R \leq 0.80$ and $\Delta R = 0.05$ for $0.8 \leq R \leq 0.95$. If one is interested in a specified lower confidence limit, he can obtain it by interpolating in Table 1. Thus, the lower 95% confidence limit obtained in this way is 0.60.

The program also evaluates the first and second moments of the system reliability from two independent sources, namely, the Mellin transform of the pdf for each subsystem and from the pdf of the system reliability. Numerical agreement between each of these two moments as obtained from these two independent sources verifies the accuracy of the coefficients in the posterior pdf and in the tabular values of $f(R)$ and $F(R)$. For this example the values of the first two moments as calculated from the two methods agreed to within 0.2%, thereby attesting to the accuracy of the derivation of the posterior pdf and of the tabulated values of the pdf and cdf in Table I.

CONCLUSIONS

The analytic solution presented in this paper, hopefully, will stimulate further interest in refining the method. Although the method in its present form is not adaptable to large scale problems, the authors feel that it could be with certain assumptions and refinement. For example, the solution presented here does not assume that parallel subsystems are identical, whereas in most real world situations they are identical. Assuming that parallel subsystems are identical would further simplify the solution, while refinement in the multiple precision

TABLE I
TABULATION OF PDF AND CDF

R	pdf	cdf
0.10	0.000	0.000
0.20	0.000	0.000
0.30	0.001	0.000
0.40	0.019	0.001
0.50	0.165	0.008
0.60	0.858	0.051
0.70	2.725	0.220
0.80	4.445	0.598
0.85	3.741	0.809
0.90	1.833	0.951
0.95	0.250	0.997

programming would certainly reduce the computer time required. These are only two examples of how the present method could be improved.

APPENDIX

The solution for the Type I inversion is obtained by performing the inversion indicated by (9), the result of which is the posterior pdf for N_1 cascade subsystems. The inversion has been performed by Springer and Thompson [6] and is not repeated here.

A Type II inversion is used for N independent parallel subsystems. An expression for the Mellin transform of $h_{II}(Q)$ has been obtained [9] that is applicable to cases 1, 2, or 3 or any combination of the three cases. By letting the first P subsystems have $\gamma_i = 1$ and the next T subsystems have $\gamma_i = 0$, the Mellin transform is

$$M[h_{II}(Q)] = K \sum_{v=1}^{\lambda} C_v \left[\prod_{i=1}^{N} \frac{\Gamma(s + b_{iv})}{\Gamma(s + b_{iv} + \tau_i)} \right] \cdot \prod_{i=1}^{P} [\psi(s + \tau_i) - \psi(\tau_i)] \quad (15)$$

where

$\tau_i = \hat{\beta}_i + 1$

$K = \prod_{i=1}^{N} \frac{\Gamma(\tau_i)\tau_i^{\gamma_i+1}}{\Gamma(\gamma_i + 1)}$

$\lambda = \rho^{N-P-T}$

b_{iv} = portion of the argument for the i-th Gamma function in the v-th term after taking the product of the series. Note that $b_{iv} = 0$ if $\gamma_i \epsilon \{0,1\}$.

C_v = resulting coefficient for the v-th term after taking the product of the series.

The posterior pdf $h_{II}(Q)$ is found by inverting $M[g_{II}(Q)]$ termwise and summing the results; the method of residues is used to perform the inversion. The results of the inversion and the detail steps required are given by Byers [1].

TYPE III INVERSION

The result of a Type II inversion is the posterior pdf for the reliability of cascade and parallel subsystems combined. For constant failure rates, a Type II inversion must be performed before a Type III inversion, since the Mellin transform of $f(R)$ cannot be obtained directly from the Mellin transform of $h(Q)$. We must first invert $M\{h(Q)\}$ to obtain $h(Q)$, make the transformation to $f(R)$ and then obtain $M\{f(R)\}$.

The Mellin transform for a Type III inversion is

$$M\{f_{III}(R)\} = \prod_{i=1}^{N_1} M\{f_i(R_i)\} \prod_{j=1}^{N_2} M\{f_j(R_j)\} \quad (16)$$

where there are N_1 cascade subsystems and N_2 parallel networks. The inversion of (16) yields the posterior pdf for a combination of cascade subsystems and parallel networks. The results of the inversion and the detail steps required are given by Byers [1].

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A Bayesian Reliability Model with a Stochastically Monotone Failure Rate

BEV LITTLEWOOD AND JOHN L. VERRALL

Abstract—**A reliability model is proposed with the usual continuous random variable representation of the working times of the system between failures. The model utilizes a *probabilistic* repair rule, via a Bayesian argument, in order to simulate decay (or improvement) of reliability in time.**

Reader Aids:
Purpose: widen state of the art
Special math needed for explanations: Bayesian probability
Special math needed for results: same
Results useful to: Statisticians and Theoretically inclined engineers

1. INTRODUCTION

WE became interested in the problems posed by monotone reliability models during the search for a model which would adequately describe the growth in reliability of an item of computer software during its development. We quickly became aware that the "repair" action carried out upon the occurrence of a software failure was such that the state of the system after the "repair" was uncertain. This contrasts with the more common interpretation of burn-in testing of complex hardware systems of very many components: when a failure of such a system occurs the offending component can be replaced by one with a reliability known to be high, so producing a predictable, deterministic improvement in reliability. Even in hardware systems, repairs may be unpredictable; the model to be presented here will hopefully deal with cases such as these.

In an earlier paper [1] we have justified assuming such a model for the reliability growth of computer software (which has a convenient lack of natural degradation). In this paper we present general results and indicate how they might be applied to model the decrease in reliability of certain systems of hardware with age.

Consider, as an example, a hardware system which works in continuous time. The "reliability" of such a system would conventionally be discussed in terms of the distribution of time-to-failure, mean time between failures, failure rate (hazard rate) or some similar measure. Assume that the system is complex and has been assembled in the factory by complicated and expensive machinery which it is impractical to utilize for carrying out repairs later in the system's life. Such a system might be expected to go through a period of burn-in testing before leaving the factory, and upon delivery to its ultimate owner it will

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