01 Jan 1971

Bayesian Confidence Limits For The Reliability Of Mixed Exponential And Distribution-Free Cascade Subsystems

Melvin D. Springer

James K. Byers
Missouri University of Science and Technology

Bayesian Confidence Limits for the Reliability of Mixed Exponential and Distribution-Free Cascade Subsystems

MELVIN D. SPRINGER AND JAMES K. BYERS

Abstract—The problem treated here is the theoretical one of deriving exact Bayesian confidence intervals for the reliability of a system consisting of some independent cascade subsystems with exponential failure probability density functions (pdf) mixed with other independent cascade subsystems whose failure pdf's are unknown. The Mellin integral transform is used to derive the posterior pdf of the system reliability. The posterior cumulative distribution function (cdf) is then obtained in the usual manner by integrating the pdf, which serves the dual purpose of yielding system reliability confidence limits while at the same time providing a check on the derived pdf. A computer program written in Fortran IV is operational. It utilizes multiprecision to obtain the posterior pdf to any desired degree of accuracy in both functional and tabular form. The posterior cdf is tabulated at any desired increments to any required degree of accuracy.

Reader Aids:
Purpose: Widen state of the art
Special math needed for explanations: Bayesian probability and statistics
Special math needed for results: Bayesian probability and statistics
Results useful to: Reliability statisticians

Introduction

THE PROBLEM of obtaining Bayesian confidence intervals for the reliability of a system consisting of statistically independent cascade[1] subsystems when all of the failure probability density functions (pdf) are either exponential or unspecified was solved by Springer and Thompson [1], [2]. The problem treated in the present paper is that of obtaining confidence intervals for the reliability of a mixed system composed of N_1 statistically independent cascade subsystems whose failure pdf's are exponential and N_2 independent cascade subsystems whose failure pdf's are unspecified. The latter are classed as good or bad. The parameter to be estimated for each of the N_2 types of subsystems is the fraction of good subsystems in an infinite population of subsystems of this type. This is accomplished by utilizing the Mellin integral transform, together with test data obtained from the subsystems, to derive the posterior pdf and cumulative distribution function (cdf) for the system reliability. A computer program allows the determination of Bayesian confidence intervals for the system reliability from the tabulated posterior cdf, and also presents the posterior pdf in both tabular and functional form. Incorporation of multiple-precision subroutines into the computer program permits the analysis of systems consisting of any number of independent cascade subsystems. The resultant pdf and cdf provide a valuable tool for reliability analysis, since much can be learned about system reliability by studying its pdf and cdf. In particular, lower 100α percent confidence limits may be obtained. An important feature of this model is that it allows the exact posterior pdf of the system reliability to be derived without the assembly and testing of the entire system per se, since only subsystem test data are required. The model can, of course, be used to evaluate system reliability using system test data in the event an entire system has been tested and such data are available. With expensive and complex systems such as space systems, this situation seldom exists. Even when this is possible, it is important to be able to evaluate the reliability of the system before assembly, in the event that subsystem reliabilities must be increased to attain the required level of system reliability. An illustrative example is presented.

Posterior Probability Density Function of System Reliability

In the Bayesian approach, the reliability R_i of the ith subsystem is regarded as a random variable with a prior pdf $p_i(R_i)$. For exponential life pdf's, it is known [1] that the posterior pdf of the subsystem reliability is

$$\text{pdf}\{R_i|T_i, t_i, r_i\} = \frac{g_i(T_i|R_i, r_i, t_i)p_i(R_i)}{\int_0^1 g_i(T_i|R_i, r_i, t_i)p_i(R_i)\, dR_i}, \qquad i = 1, \cdots, N_1, \quad (1)$$

where

T_i	total subsystem test time,
r_i	number of failures at which the testing of n_i units of the ith subsystem is terminated,
t_i	mission time of the ith subsystem,
$g_i(T_i\|R_i, r_i, t_i)$	conditional pdf of the estimator $\hat{R}_i \equiv \exp(-r_i t_i/T_i)$.

For exponential subsystems, it is both logical [1] and convenient to use the natural conjugate[2] prior pdf

Manuscript received October 14, 1970.
M. D. Springer is with the Department of Industrial Engineering, University of Arkansas, Fayetteville, Ark. 72701.
J. K. Byers is with the Department of Computer Sciences, University of Missouri, Rolla, Mo.

[1] Cascade subsystems are subsystems arranged in series so that the failure of any subsystem results in the failure of the system.

[2] A family of prior pdf's $p_i(R_i)$ is a natural conjugate for R_i if every member of that family, used as a prior pdf, produces a posterior pdf via (1), which also belongs to that family.

$$p_i(R_i) \equiv \frac{(T_{i0}/t_i + 1)^{r_{i0}+1}}{\Gamma(r_{i0} + 1)} R_i^{T_{i0}/t_i}(\ln R_i^{-1})^{r_{i0}} \qquad (2)$$

in which case (1) becomes

$$\mathrm{pdf}\{R_i|T_i, t_i, r_i\} = \frac{(T_i/t_i + T_{i0}/t_i + 1)^{r_i + r_{i0} + 1}}{\Gamma(r_i + r_{i0} + 1)} R_i^{T_i/t_i + T_{i0}/t_i}(\ln R_i^{-1})^{r_i + r_{i0}}. \qquad (3)$$

Here T_{i0} would be identified with an actual or hypothesized total accumulated test time with r_{i0} observed failures in previous experience with the subsystem under consideration, or similar subsystems. For convenience let

$$f_{1i}(R_i) \equiv \mathrm{pdf}\{R_i|T_i, t_i, r_i\}$$

$$\beta_i \equiv (T_i + T_{i0})/t_i$$

$$\gamma_i \equiv r_i + r_{i0}$$

so that (3) becomes

$$f_{1i}(R_i) = \frac{(\beta_i + 1)^{\gamma_i + 1}}{\Gamma(\gamma_i + 1)} R_i^{\beta_i}(\ln R_i^{-1})^{\gamma_i}. \qquad (4)$$

That is, the posterior pdf of the reliability of an exponential subsystem is given by (4).

The question now arises as to the form of the posterior pdf of the reliability R_j of a subsystem whose failure pdf is unknown. One then treats each unit as good or bad and concerns himself with the fraction that is good. In this case the pdf of R_j, the reliability of the jth subsystem, may be characterized by the proportion m_j/n_j of the units of the jth subsystem that survived the mission test time t_j. Here R_j is a random variable that denotes, in the Bayesian sense, the binomial parameter of the jth subsystem, $j = 1, 2, \cdots, N_2$. If again the natural conjugate (beta) prior pdf of R_j is used, it is known [2] that the posterior pdf of R_j is

$$\mathrm{pdf}\{R_j\} = \frac{\Gamma(n_j + b_j + 2)}{\Gamma(m_j + a_j + 1)\Gamma(n_j + b_j - m_j - a_j + 1)} R_j^{m_j + a_j}(1 - R_j)^{n_j + b_j - m_j - a_j}, \qquad j = 1, 2, \cdots, N_2. \qquad (5)$$

For convenience, the notation

$$\alpha_j = n_j + b_j$$

$$\phi_j \equiv m_j + a_j$$

will be used, so that (5) becomes

$$f_{2j}(R_j) \equiv \mathrm{pfd}\{R_j\} = \frac{\Gamma(\alpha_j + 2)}{\Gamma(\phi_j + 1)\Gamma(\alpha_j - \phi_j + 1)} R_j^{\phi_j} \cdot (1 - R_j)^{\alpha_j - \phi_j}, \qquad j = 1, 2, \cdots, N_2. \qquad (6)$$

From a knowledge of the posterior pdf's (4) and (6), one can use the Mellin integral transforms to derive the posterior pdf of the $N_1 + N_2$ cascade subsystems. Note first that the Mellin integral transform of (4) is [1]

$$M\{f_{1i}(R_i)\} = \frac{(\beta_i + 1)^{\gamma_i + 1}}{(s + \beta_i)^{\gamma_i + 1}}, \qquad \mathrm{Re}\{s\} > -\beta_i, i = 1, 2, \cdots, N_1. \qquad (7)$$

Similarly, the Mellin transform of (6) is [2]

$$M\{f_{2j}(R_j)\} = \frac{\Gamma(\alpha_j + 2)}{\Gamma(\phi_j + 1)} \frac{\Gamma(s + \phi_j)}{\Gamma(s + \alpha_j + 1)}, \qquad \mathrm{Re}\{s\} > -\phi_j, j = 1, 2, \cdots, N_2. \qquad (8)$$

Since a_j and b_j are integers, $M\{f_{2j}(R_j)\}$ as given in (8) reduces to

$$M\{f_j(R_j)\} = \frac{(\alpha_j + 1)!}{\phi_j!} \left[\frac{1}{(s + \alpha_j) \cdots (s + \phi_j)}\right]. \qquad (9)$$

Note that the condition that a_j and b_j be integers is not a limitation. In fact, a_j and b_j should actually be integers since they are associated with an actual or hypothesized real-world situation. That is, a_j is identified with the number of tested units of the jth subsystem, out of a total number $a_j + b_j$, which have in previous experience been observed to survive the required mission time. If no such data exist, the fact still remains that any hypothetical values of a_j and b_j are *conceptually* associated with a number of "successes" and "total number of trials" in a binomial distribution. When one then proceeds to consider the random variable R_j, the a_j and b_j become parameters in a beta distribution, since the original binomial distribution is now regarded as a pdf in the random variable R_j as a result of the Bayesian structure.

Since the various subsystems are mutually statistically independent, the Mellin transform of the posterior pdf $h(R)$ of the system reliability R is the product [3], [4]

$$\begin{aligned} M\{h(R)\} &= \prod_{i=1}^{N_1} M\{f_i(R_i)\} \prod_{j=1}^{N_2} M\{f_j(R_j)\} \\ &= C\left(\prod_{i=1}^{N_1} \frac{1}{(s + \beta_i)^{\gamma_i + 1}}\right)\left(\prod_{j=1}^{N_2} \frac{1}{(s + \alpha_j) \cdots (s + \phi_j)}\right) \\ &= C \prod_{i=1}^{N_1} \left(\frac{1}{(s + \beta_i)^{\gamma_i + 1}}\right) \cdot \left(\frac{1}{(s + \alpha)^{b_0}(s + \alpha + 1)^{b_1} \cdots (s + \phi)^{b_{\phi - \alpha}}}\right) \end{aligned} \qquad (10)$$

where b_j, $j = 0, 1, \cdots, \phi - \alpha$ is the collective exponent for the factor $(s + \alpha + j)^{b_j}$ since $(s + \alpha + j)$ may occur in several (or none) of the Mellin transforms of the component pdf's and where

$$C \equiv \left(\prod_{i=1}^{N_1} (\beta_i + 1)^{\gamma_i + 1}\right)\left(\prod_{j=1}^{N_2} \frac{(\alpha_j + 1)!}{\phi_j!}\right)$$

$$\alpha \equiv \min\{\alpha_j\}$$

$$\phi \equiv \max\{\phi_j\}, j = 1, 2, \cdots, N_2.$$

The posterior pdf $h(R)$ can now be obtained [2] by evaluating the Mellin inversion integral

$$h(R) \equiv \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} R^{-s} M\{h(R)\}\, ds \qquad (11)$$

where the path of integration is any line parallel to the

imaginary axis and lying to the right of $\operatorname{Re}\{c\} = -d$, $d = \min\{\beta_i, \phi_j\}$, $i = 1, 2, \cdots, N_1$, $j = 1, 2, \cdots, N_2$.

It remains to evaluate the inversion integral (11), which, since Jordan's lemma is satisfied, is accomplished by the method of residues [5]. The poles occur at $s = b$, where

$$b = \begin{cases} -\beta_i & i = 1, 2, \cdots, N_1 \\ -\phi_j - m_j, & m_j = 0, 1, \cdots, \alpha_j - \phi_j, \quad j = 1, 2, \cdots, N_2. \end{cases}$$

If there is no duplication of the β_i with $\phi_j - m_j$, the pole at $s = -\beta_i$ for the exponential subsystems is of order $k_i \equiv \gamma_i + 1$, while the order of the pole at $s = -(\alpha + m)$ for the distribution-free subsystems is $k_m \equiv b_m$, $m = 0, \cdots, \phi - \alpha$. If $\beta_i = \beta_r$, $i \neq r$, $i \in \{1, 2, \cdots, N_1\}$ the pole at $s = -\beta_i$ is of order $k_i = \gamma_i + \gamma_r + 2$. Also if $\beta_i = \alpha + m$, $i \in \{1, 2, \cdots, N_1\}$, i.e., $\beta_i \in \{\alpha, \alpha + 1, \cdots, \phi\}$, the order of the pole at $s = -\beta$ is $k_i \equiv \gamma_i + b_m + 1$. Moreover, since Jordan's lemma is satisfied, the posterior pdf $h(R)$ is obtained by summing the residues over all the poles. That is,

$$h(R) = \sum_i Q(\beta_i, \gamma_i + 1) + \sum_m Q(\alpha + m, b_m), \quad i = 1, 2, \cdots, N_1, \; m = 0, 1, \cdots, \phi - \alpha \tag{12}$$

where

$$Q(u, v + 1) \equiv \frac{1}{v!} \frac{d^v}{ds^v} [R^{-s}(s + u)^{v+1} M\{h(R)\}]|_{s=-u} \tag{13}$$

and where the Q denote the residues at the relevant poles, with the aforementioned modifications on the order of the poles when the values of β_i are not distinct or when there is duplication of the values of β_i with those of $\phi_j + m_j$. Using (10) the residues as given by (12) and (13) may be conveniently evaluated on a computer in a recursive manner in terms of the lower order derivatives. To accomplish this, let

$$V \equiv R^{-s}$$

$$U \equiv \prod_{r=1}^{N_1} \frac{1}{(s + \beta_r)^{\gamma_r + 1}} \cdot \left[\frac{1}{(s + \alpha)^{b_0}(s + \alpha + 1)^{b_1} \cdots (s + \phi)^{b_{\phi-\alpha}}} \right] \tag{14b}$$

$$W \equiv UV. \tag{14c}$$

Then

$$Q(\beta_i, \gamma_i + 1) = \frac{C}{(\gamma_i)!} W^{(\gamma_i)} \Big|_{s=-\beta_i} \tag{15a}$$

and

$$Q(\alpha + m, b_m) = \frac{C}{(b_m - 1)!} W^{(b_m - 1)} \Big|_{s=-(\alpha+m)} \tag{15b}$$

where

$$W^{(k)} \equiv d^k W / ds^k. \tag{15c}$$

Since $W \equiv UV$, the $(k-1)$st derivative of W may be conveniently obtained by Leibniz' rule [6] for the differentiation of a product, from which it immediately follows that

$$W^{(k-1)} = \sum_{c=0}^{k-1} \binom{k-1}{c} V^{(c)} U^{(k-c-1)}. \tag{16}$$

From the definitions of V and U, it is clear that

$$V^{(c)} = (-1)^c R^{-s} (\ln R^{-1})^c \tag{17}$$

and that the application of Leibniz' rule gives

$$U^{(k-c-1)} = \sum_{g=0}^{k-c-1} \binom{k-c-1}{g} X^{(g)} Y^{(k-c-g-1)} \tag{18}$$

where

$$X \equiv \prod_{r=1}^{N_1} (s + \beta_r)^{-(\gamma_r + 1)} \tag{19a}$$

$$Y \equiv ((s + \alpha)^{b_0} (s + \alpha + 1)^{b_1} \cdots (s + \phi)^{b_{\phi-\alpha}})^{-1} \tag{19b}$$

The evaluation of the required derivatives of X and Y may be expedited by differentiating $\ln X$ and $\ln Y$. It can then be shown (somewhat tediously) that

$$X^{(g)} = -\sum_{p=0}^{g-1} \binom{g-1}{p} Z^{(p)} X^{(g-p-1)}, \qquad g > 1 \tag{20}$$

$$Y^{(n)} = \sum_{a=0}^{n-1} \binom{n-1}{a} Y^{(n-a-1)} B^{(a)} \tag{21}$$

where

$$Z^{(p)} \equiv (-1)^p p! \sum_r \frac{\gamma_r + 1}{(s + \beta_r)^{p+1}} \tag{22}$$

$$B^{(a)} \equiv (-1)^{a+1} a! \sum_{m=0}^{\phi-\alpha} b_m (s + \alpha + m)^{-(a+1)}. \tag{23}$$

In particular if one sets $n = k - c - g$ in (21) he obtains the derivative $Y^{(k-c-g-1)}$ needed in (18) to evaluate the various derivatives of U. Having also obtained the derivatives of V from (17), one can evaluate $W^{(k-1)}$ and hence the residues $Q(\beta_i, \gamma_i + 1)$ and $Q(\alpha + m, b_m)$ in (15). Summation of these residues then leads to the posterior pdf $h(R)$ in (12).

The posterior cdf $H(R)$ may be obtained in either of two ways. It may be determined by evaluating the integral

$$H(R) \equiv \int_0^R h(R')\, dR' \tag{24}$$

or by evaluating the inversion integral

$$H(R) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R^{-s} M\{H(R)\}\, ds \tag{25}$$

by the method of residues in the manner already explained. The Mellin transform of $H(R)$ is obtained by replacing s by $s + 1$ in $M\{h(R)|s\}$ and multiplying the result by $1/s$. Symbolically,

$$M\{H(R)|s\} = (1/s) M\{h(R)|s + 1\}.$$

In this paper, $H(R)$ is obtained from (24) since this provides

TABLE I
POSTERIOR PDF FOR EXAMPLE PROBLEM

Exponent of R	Exponent of ln $1/R$	Coefficient a_i		
90.0	2.0	−3 313 870 593	3.975 617 283	9 506 172 824
90.0	1.0	−5 917 758 905	5.263 825 229	3 357 502 814
90.0	0.0	−3 402 599 999	2.058 355 631	6 902 199 115
87.0	1.0	−4 511 118 398	4.648 373 268	0 222 912 442
87.0	0.0	3 380 863 327	9.297 898 389	6 118 699 130
57.0	0.0	7 003 374 048	180.793 814 6	9 011 098 628
58.0	1.0	−4 931 703 645	4 867.485.183	1 896 551 709
58.0	0.0	7 378 069 670	2 055.702 510	0 520 523 656
59.0	0.0	−1 745 673 473	04 652.070 75	9 625 390 213
60.0	1.0	6 904 808 043	0 582.493 091	5 002 794 255
60.0	0.0	8 004 462 657	3 230.816.065	3 318 054 077
61.0	0.0	1 373 886 834	7 897.518 826	7 934 998 011

TABLE II
TABULATION OF $h(R)$ AND $H(R)$ FOR EXAMPLE PROBLEM

R	$h(R)$	$H(R)$
0.50	0.000 000 0	0.000 000 00
0.55	0.000 000 0	0.000 000 05
0.60	0.000 214 9	0.000 002 84
0.65	0.006 386 5	0.000 096 28
0.70	0.106 923 3	0.001 946 52
0.75	1.019 949 9	0.023 178 71
0.80	4.930 241 2	0.155 181 55
0.85	9.591 588 7	0.539 125 99
0.90	4.286 000 6	0.926 907 65
0.92	1.519 146 8	0.982 809 63
0.94	0.258 550 9	0.998 131 07
0.96	0.011 875 4	0.999 950 61
0.98	0.000 022 2	0.999 999 94

a check on the derived pdf $h(R)$. Further checks on $h(R)$ are also possible by comparing the various moments as obtained from the Mellin transform of $h(R)$ with the corresponding moments obtained from the derived pdf $h(R)$. This is, of course, possible since

$$E\{R^k\} = M\{h(R)\}|_{s=k+1} = \int_0^1 R^k h(R)\,dR.$$

The posterior cdf $H(R)$ is of primary interest since it provides the confidence intervals for R; i.e.,

$$\Pr\{R_L < R < R_U\} = H(R_U) - H(R_L).$$

EVALUATION OF $h(R)$ AND $H(R)$

A computer program using Fortran-based multiple-precision subroutines [7] has been written and is operational. The program is machine independent and has been executed on the UNIVAC 1108 and the IBM 7040. It derives the posterior cdf in tabular form, and the posterior pdf in both functional (algebraic) and tabular form. The presentation of the functional form of the posterior pdf permits possible further study of the nature of the system reliability. While desired confidence limits could be obtained directly by numerical inversion or by interpolating between tabular values, the computer program utilizes only the latter method. The lower 100α percent confidence limit, for example, is given by the value $R = R_\alpha$ for which $H(R_\alpha) = 1 - \alpha$.

When only double precision is used, erroneous results due to roundoff may occur. Such erroneous results have a high probability of occurrence when double precision is used with systems consisting of more than three subsystems. To eliminate the possibility of obtaining erroneous results due to roundoff, multiple precision is used. The precision level is an input to the computer program and can be changed at will by the analyst. The computer time required to solve a problem of the type discussed in this paper is within practical limits. For example, the system consisting of six subsystems treated in the numerical example presented in the next section, whose analysis was carried out with calculations involving numbers with thirty significant digits, required one minute on the UNIVAC 1108. The largest system that has been solved using the aforementioned computer program consisted of twelve subsystems. The execution time for this problem on the UNIVAC was three minutes.

NUMERICAL EXAMPLE

Consider a system consisting of six independent cascade subsystems, three of which have exponential life pdf's and three of which are distribution-free. Assume that test data and prior information yield the following parameter values:

1) exponential subsystems

$$\beta_1 = 90, \quad \gamma_1 = 0$$
$$\beta_2 = 90, \quad \gamma_2 = 1$$
$$\beta_3 = 87, \quad \gamma_3 = 1$$

2) distribution-free subsystems

$$\alpha_1 = 60, \quad \phi_1 = 58$$
$$\alpha_2 = 61, \quad \phi_2 = 60$$
$$\alpha_3 = 58, \quad \phi_3 = 57.$$

The problem is to determine 1) the functional and tabular forms of the posterior pdf $h(R)$; 2) the tabular form of the posterior cdf $H(R)$; and 3) the desired confidence intervals on system reliability R.

The posterior pdf is $\Sigma_{i=1}^{12} a_i R^{b_i} (\ln R^{-1})^{c_i}$, where the constants a_i, b_i, and c_i are determined from the ith row of Table I. The tabular forms of the posterior pdf and cdf are given in Table II. The intervals of tabulation are completely arbitrary, but in this example $\Delta R = 0.05$ for $0.50 \leq R \leq 0.90$ and $\Delta R = 0.02$ for $0.90 \leq R \leq 1.00$. If one is interested in a specific lower confidence limit, he can obtain it by interpolating in Table II. Thus, the lower 95 percent confidence limit is 0.760, which is accurate to at least two decimal places. To obtain a more accurate estimate, one

TABLE III

COMPARISON OF FIRST AND SECOND MOMENTS AS COMPUTED FROM DERIVED PDF AND FROM MELLIN INTEGRAL TRANSFORM

Moment About the Origin	Moment Derived from pdf	Moment Derived from Mellin Integral Transform
First	0.842 698 790	0.842 698 723
Second	0.711 912 461	0.711 912 401

can rerun the problem with arbitrary initial value of R, small values of ΔR (say $\Delta R = 0.001$) for $0.75 < R < 0.80$, and an arbitrary final value of $R < 1$. From the resultant calculations, the lower 95 percent confidence limit on R could, by interpolation, be obtained to at least five decimal places with relatively little computer time.

An indication of the accuracy of the coefficients in the posterior pdf and in the tabular values of $h(R)$ and $H(R)$ is given by the fact that the values of the Mellin transform for $s = 2$, $s = 3$ agree to six decimal places with the first and second moments, respectively, of the derived pdf $h(R)$. This comparison is given in Table III.

REFERENCES

[1] M. D. Springer and W. E. Thompson, "Bayesian confidence limits for the reliability of cascade exponential subsystems," *IEEE Trans. Rel.*, vol. R-16, Sept. 1967, pp. 86–89.

[2] ——, "Bayesian confidence limits for the product of N binomial parameters," *Biometrika*, vol. 53, 1966, pp. 611–613.

[3] ——, "The distribution of products of independent random variables," *J. Soc. Ind. Appl. Math.*, vol. 14, May 1966, pp. 511–526.

[4] B. Epstein, "Some applications of the Mellin transform in statistics," *Annu. Math. Statist.*, vol. 19, 1948, pp. 370–379.

[5] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge, England: Cambridge Univ. Press, 1940, ch. 6.

[6] P. Franklin, *A Treatise on Advanced Calculus*. New York: Wiley, 1940, p. 111.

[7] G. Kacin, "A set of multiple precision subroutines," *Share Program Library IBM*, Hawthorne, N.Y., SDA 3477, Feb. 3, 1967.

Simulation Model for the Maintenance of a Deteriorating Component System

DAVID B. BROWN AND H. F. MARTZ, JR.

Abstract—**The model finds the optimal time-independent component replacement policy for a system composed of components subject to probabilistic deterioration. According to this policy the system is reviewed periodically and components are replaced such that the system will operate at minimum expected cost. The final policy is easy to implement since decisions depend only upon the observed state of the system; hence, no component monitoring is required. Both the cost of operating the system in a degraded state and component replacement costs are considered. The model assumes that all past decisions affect the current system state as well as the system improvement cost. All time-independent policies are simulated and the one yielding the lowest expected cost is chosen. This paper presents the general mathematical development of the model with a description of the simulation procedure and a simple example of a six-component system.**

Reader Aids:
Purpose: Widen state of the art
Special math needed for explanations: Matrix notation
Special math needed for results: Matrix notation
Results useful to: Theoretically inclined reliability engineers, system analysts

Manuscript received October 26, 1970; revised December 8, 1970.
D. B. Brown is with the Department of Mechanical and Aerospace Engineering, Rutgers University, New Brunswick, N.J.
H. F. Martz, Jr., is with the Department of Industrial Engineering, Texas Technological University, Lubbock, Tex. 79409.

I. INTRODUCTION

THE HIGH cost involved in maintaining complex systems has led to extensive investigation into the improvement of maintenance policies. Some important work has applied to systems that are observed periodically and classified into one of a number of mutually exclusive states. After each observation, a decision is made depending upon the observed state of the system or components. This decision determines the chance laws that affect the future course of the system as well as some immediate return or cost.

Derman [4] has studied such systems in which the maintenance decision, whether to replace a component or not, depends upon the state of the component. According to the adaption made by Barlow and Proschan [1], two costs—the cost if the unit is replaced before becoming inoperative and the cost incurred if it fails and then has to be replaced—were balanced. Eppen [6] used system information to form a one-step-down transition matrix, which was useful in determining maintenance decisions. Other papers [3], [5], [8] also consider similar types of maintenance models.