01 Sep 2007

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Dedicated to Prof. Peter Lax with great admiration and sincere gratitude on the occasion of his 80th birthday

Abstract

We study asymptotic behavior of the global attractors to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. In particular, we show that the global attractors to the Boussinesq system for Rayleigh-Bénard convection converge to that of the infinite-Prandtl-number model for convection as the Prandtl number approaches infinity. This offers partial justification of the infinite-Prandtl-number model for convection as a valid simplified model for convection at large Prandtl number even in the long-time regime. © 2006 Wiley Periodicals, Inc.

1 Introduction

One of the fundamental systems in fluid dynamics is the following *Boussinesq system for Raleigh-Bénard convection* (*nondimensional*):

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p = \Delta\mathbf{u} + \mathrm{Ra}\,\mathbf{k}T, \quad \nabla\cdot\mathbf{u} = 0, \tag{1.1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \Delta T, \tag{1.2}$$

$$\mathbf{u}|_{z=0,1} = 0, \tag{1.3}$$

$$T|_{z=0} = 1, \quad T|_{z=1} = 0, \tag{1.4}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad T|_{t=0} = T_0, \tag{1.5}$$

where $\mathbf{u}$ is the fluid velocity field, p is the modified pressure, T is the temperature field, and $\mathbf{k}$ is the unit upward vector.

The system is a model for convection, i.e., fluid motion induced by differential heating, of a layer of fluids bounded by two horizontal parallel plates a distance

Communications on Pure and Applied Mathematics, Vol. LX, 1293–1318 (2007)

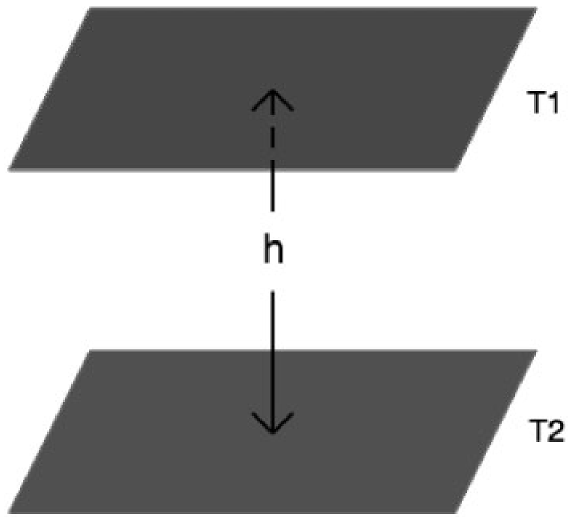

FIGURE 1.1. Rayleigh-Bénard convection.

h apart in the Rayleigh-Bénard setting [22, 43] with the bottom plate heated at temperature T_2 and the top plate cooled at temperature T_1 ($T_1 < T_2$).

We assume that the fluids occupy the (nondimensionalized) region

$$\Omega = [0, L_x] \times [0, L_y] \times [0, 1] \tag{1.6}$$

with periodicity in the horizontal directions assumed for simplicity. The parameters of the system are thus absorbed into the geometry of the domain plus two nondimensional numbers: the Rayleigh number

$$\mathrm{Ra} = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa} \tag{1.7}$$

measuring the ratio of overall buoyancy force to the damping coefficients, and the Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\kappa} \tag{1.8}$$

measuring the relative importance of kinematic viscosity over thermal diffusivity. Here ν and κ are the kinematic viscosity and thermal diffusive coefficient, respectively, α is the thermal expansion coefficient of the fluid, g is the gravitational constant, h is the distance between the two plates confining the fluid, and $T_2 - T_1$ is the temperature difference between the bottom and top plates. We have taken the distance between the plates, h, as the typical length scale, and the thermal diffusive time as the typical time scale; the temperature is scaled so that the top plate is set to 0 while the bottom plate is set to 1 (see [43] among others).

The Boussinesq system exhibits extremely rich phenomena from pure conduction at low Rayleigh number, to Bénard cells at first bifurcation, spatial-temporal patterns and chaos at intermediate Rayleigh number, all the way to convective turbulence at high Rayleigh number (see, for instance, [22, 43], and the recent review by [3, 26, 38]). In fact, the Boussinesq system is considered a fundamental paradigm for nonlinear dynamics including instabilities and bifurcations, pattern formation, chaotic dynamics, and fully developed turbulence [26]. On the other hand, we have very limited mathematical knowledge on the system. Even the issue of the existence of regular enough solutions is unresolved. Indeed, the velocity equation is exactly the Navier-Stokes system (forced by a buoyancy term) whose regularity of solutions is one of the million-dollar mathematical problems of the new millennium (`www.claymath.org/Millennium_Prize_Problems/`). For such a complex system, simplification is highly desirable. A simpler model can be obtained if we consider the regime of large Prandtl number. If we formally set the Prandtl number to infinity, we arrive at the following *infinite-Prandtl-number model* (*nondimensional*)

$$\nabla p^0 = \Delta \mathbf{u}^0 + \mathrm{Ra}\, \mathbf{k} T^0, \quad \nabla \cdot \mathbf{u}^0 = 0, \tag{1.9}$$

$$\frac{\partial T^0}{\partial t} + \mathbf{u}^0 \cdot \nabla T^0 = \Delta T^0, \tag{1.10}$$

$$\mathbf{u}^0|_{z=0,1} = 0, \tag{1.11}$$

$$T^0|_{z=0} = 1, \quad T^0|_{z=1} = 0. \tag{1.12}$$

(see, for instance, [3, 4, 6, 8, 23, 43, 44, 45] among others), which is relevant for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure. One observes that the Navier-Stokes equations in the Boussinesq system have been replaced by the Stokes system in the infinite-Prandtl-number model.

The fact that the velocity field is linearly "slaved" by the temperature field has been exploited in several recent, very interesting works on rigorous estimates on the rate of heat convection in this infinite-Prandtl-number setting (see [8, 10, 12, 14] and the references therein, as well as the work of [4]).

An important natural question is whether such an approximation is valid.

The mathematical justification of the infinite-Prandtl-number model on any fixed time interval can be found in [44]. Encouraged by the finite-time convergence, we naturally inquire if the solutions of the Boussinesq system and solutions of the infinite-Prandtl-number model remain close on a large time interval for large Prandtl number. In general, we should not expect long-time proximity of each individual orbit. Such a long-time orbital stability result shouldn't be expected for such complex systems where turbulent/chaotic behavior abound. Instead, the statistical properties for such systems are much more important and physically relevant, and hence it is natural to ask if the statistical properties (in terms of invariant measures) as well as global attractors (which contains the support of any invariant measures if they exist) remain close.

The first obstacle in studying long-time behavior is the well-posedness of the Boussinesq system global in time. This is closely related to the well-known problem related to the three-dimensional Navier-Stokes equations ([9, 32, 41] among others). This is partially resolved by considering suitable weak solutions (see Section 2). Indeed, we are able to show the eventual regularity for suitably defined weak solutions to the Boussinesq system that exists for all time [46].

Recall that the global attractor of a given dynamical system is a compact invariant set that attracts all bounded sets in the phase space ([24, 40] among others). In particular, the global attractor is maximal in the sense that any compact invariant set must be a subset of the global attractor. The global attractor is also minimal in the sense that any bounded set that attracts an arbitrary bounded set in the phase space must contain the global attractor. Therefore the closeness of global attractors, if they exist, would be a good measure of closeness of long-time behavior. It also provides positive indication on the closeness of statistical properties since the invariant measures are supported on the global attractors.

Although the dynamics of the Boussinesq system may not be well-defined due to the well-known regularity problem, all properly defined weak solutions become regular after a transitional time at large Prandtl number [46]. The dynamics is also well-defined if solutions start from a bounded set in a (smaller) subspace of the phase space. Moreover, such a bounded set is in fact absorbing in the sense that all suitably defined weak solutions will enter this bounded set in finite time [46]. Furthermore, the system possesses a global attractor that is regular in space and in time, and attracts all suitably defined weak solutions [46]. It is also known that the infinite-Prandtl-number model for convection possesses a global attractor. Hence it makes sense to discuss the closeness of the global attractors.

Another issue that we encounter here is the difference in natural phase spaces: for the Boussinesq system we need both velocity and temperature, while only the temperature field is needed for the infinite-Prandtl-number model since the velocity field is linearly slaved. There are two ways of handling the discrepancy in phase space:

(1) project the phase space of the Boussinesq system down to the temperature field only, and
(2) lift the phase space of the infinite-Prandtl-number model to the product space of velocity and temperature.

We will see that the comparison of global attractors after projection is relatively easy and is similar to the upper semicontinuity of global attractors for dynamical systems (see, for instance, [24, 40]). The comparison of global attractors after lifting the phase space of the infinite-Prandtl-number model is a little bit more involved. Here we view the Boussinesq system as a small perturbation of the infinite-Prandtl-number model. The proximity of global attractors follows from appropriate a priori estimates (uniform in Prandtl number) on the material derivative of the velocity field after the initial layer. The convergence result (Corollary 3.2) was announced earlier [44, 45].

The rest of the manuscript is organized as follows: In Section 2 we derive a few a priori estimates needed in the proof in Section 3. These estimates refine previous estimates [46]. In Section 3 we present our main results on the convergence of the global attractors of the Boussinesq system to that of the infinite-Prandtl-number model. The proof of the main results are sketched as well. In Section 4 we make concluding remarks. In particular, we discuss if similar results are valid for other systems with multiple time scales.

Throughout this manuscript, we assume the physically important assumption of the domain having a large aspect ratio, i.e.,

$$L_x \geq 1, \quad L_y \geq 1, \quad \text{and hence} \quad |\Omega| \geq 1. \tag{1.13}$$

Likewise, we also assume the physically important case of a high Rayleigh number

$$\mathrm{Ra} \geq 1 \tag{1.14}$$

so that we may have nontrivial dynamics.

We also follow the mathematical tradition of denoting our small parameter as ε, i.e.,

$$\varepsilon = \frac{1}{\mathrm{Pr}}. \tag{1.15}$$

2 A Priori Estimates

Here we establish a few a priori estimates on suitable weak solutions to the Boussinesq system needed in the proof of the convergence of the global attractors. The estimates given here refine previous estimates [46] and are uniform in terms of the (large) Prandtl number, or equivalently, the small parameter ε. They are uniform in time as well as modulo an initial layer. An initial time layer has to be neglected here since the time derivative is proportional to $1/\varepsilon$ within a certain initial layer [44]. Estimates in higher-order Sobolev spaces can be derived as well.

Following a traditional approach, we recast the Boussinesq system in terms of the perturbative variable

$$(\mathbf{u}, \theta) = (\mathbf{u}, T - (1 - z)) \tag{2.1}$$

(perturbation away from the pure conduction state $(0, 1-z)$). Thus we have the *nondimensional Boussinesq system for Raleigh-Bénard convection in a perturbative variable*:

$$\frac{1}{\Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p = \Delta\mathbf{u} + \mathrm{Ra}\,\mathbf{k}\theta, \quad \nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta - u_3 = \Delta\theta, \tag{2.3}$$

$$\mathbf{u}|_{z=0,1} = 0, \tag{2.4}$$

$$\theta|_{z=0} = 1, \quad \theta|_{z=1} = 0, \tag{2.5}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \theta|_{t=0} = \theta_0. \tag{2.6}$$

The infinite-Prandtl-number model for convection can be cast in terms of the perturbative variable in a similar fashion. Thus we have the *nondimensional infinite-Prandtl-number model in a perturbative variable*:

$$\nabla p^0 = \Delta\mathbf{u}^0 + \mathrm{Ra}\,\mathbf{k}\theta^0, \quad \nabla\cdot\mathbf{u}^0 = 0, \tag{2.7}$$

$$\frac{\partial\theta^0}{\partial t} + \mathbf{u}^0\cdot\nabla\theta^0 - u_3^0 = \Delta\theta^0, \tag{2.8}$$

$$\mathbf{u}^0|_{z=0,1} = 0, \tag{2.9}$$

$$\theta^0|_{z=0,1} = 0. \tag{2.10}$$

Next, we recall the definition of suitable weak solutions [46]:

DEFINITION 2.1 (Suitable Weak Solution) The function $(\mathbf{u}, \theta)$ is called a *suitable weak solution to the Boussinesq equations on the time interval* $[0, T^*]$ *with given initial data* $(\mathbf{u}_0, \theta_0)$ if the following hold:

$$\mathbf{u} \in L^\infty(0, T^*; H) \cap L^2(0, T^*; V) \cap C_w([0, T^*]; H), \tag{2.11}$$

$$\mathbf{u}' \in L^{4/3}(0, T^*; V'), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{2.12}$$

$$\theta \in L^\infty(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; H_0^1(\Omega)) \cap C_w([0, T^*]; L^2(\Omega)), \tag{2.13}$$

$$\theta' \in L^{4/3}(0, T^*; H^{-1}(\Omega)), \quad \theta(0) = \theta_0, \tag{2.14}$$

$$\varepsilon\left(\frac{d}{dt}\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\right) + \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} = \mathrm{Ra}\int_\Omega \theta v_3 \quad \forall\mathbf{v}\in V, \tag{2.15}$$

$$\frac{d}{dt}\int_\Omega \theta\eta + \int_\Omega \mathbf{u}\cdot\nabla\theta\,\eta + \int_\Omega \nabla\theta\cdot\nabla\eta = \int_\Omega u_3\,\eta \quad \forall\eta\in H_0^1, \tag{2.16}$$

$$\frac{\varepsilon}{2}\frac{d}{dt}|\mathbf{u}(t)|_{L^2}^2 + |\nabla\mathbf{u}(t)|_{L^2}^2 \le \mathrm{Ra}\,\theta(t)u_3(t), \tag{2.17}$$

$$\frac{d}{dt}|(T-1)^+(t)|_{L^2}^2 + 2|\nabla(T-1)^+(t)|_{L^2}^2 \le 0, \tag{2.18}$$

$$\frac{d}{dt}|T^-(t)|_{L^2}^2 + 2|\nabla T^-(t)|_{L^2}^2 \le 0. \tag{2.19}$$

Here we have used standard notation on function spaces used in the study of incompressible fluids (see, for instance, [9, 13, 20, 41] among others). In particular, the energy inequalities should be understood in the weak sense with the initial value taken into consideration.

Recall our goal here is to derive estimates that are uniform in large Prandtl number Pr or small ε that are also uniform in time after neglecting an initial transitional time interval.

We will start with the easy L^2-estimates for the temperature and velocity fields. In the second stage we derive uniform H^1-estimates for the solutions. These estimates imply the eventual regularity of solutions to the Boussinesq system for convection and the existence of global attractors [46]. In the third stage, we derive uniform estimates on the time derivative of the solutions. In the fourth stage, we utilize the uniform estimates on the time derivative to derive uniform estimates in H^2.

We start with L^2-estimates on the temperature field. Thanks to (2.18), (2.19), and the Poincaré inequality, we see that

$$|(T-1)^+(t)|_{L^2}^2 \le e^{-2t}|(T_0-1)^+|_{L^2}^2 \le e^{-2t}|T_0|_{L^2}^2, \tag{2.20}$$

$$|T^-(t)|_{L^2}^2 \le e^{-2t}|T_0|_{L^2}^2. \tag{2.21}$$

Therefore,

$$\begin{aligned} |T(t)|_{L^2} &\le |T^-(t)|_{L^2} + |T^+(t)|_{L^2} \\ &\le |T^-(t)|_{L^2} + |(T-1)^+(t)|_{L^2} + |\Omega| \\ &\le 2e^{-t}|T_0|_{L^2} + |\Omega|. \end{aligned} \tag{2.22}$$

Hence, for

$$t_1 = \ln \frac{2|T_0|_{L^2}}{|\Omega|} \tag{2.23}$$

we have

$$|T(t)|_{L^2} \le 2|\Omega| \quad t \ge t_1. \tag{2.24}$$

With the L^2-estimate in temperature, the L^2-estimate of the perturbative variable θ is obvious. Indeed,

$$\begin{aligned} |\theta(t)|_{L^2} &\le |T(t)|_{L^2} + |1-z|_{L^2} \\ &\le 2e^{-t}|T_0|_{L^2} + 2|\Omega| \\ &\le 3|\Omega| \quad \forall t \ge t_1. \end{aligned} \tag{2.25}$$

This, together with the energy inequality for the velocity field (2.17), implies

$$\frac{\varepsilon}{2}\frac{d}{dt}|\mathbf{u}(t)|_{L^2}^2 + |\nabla \mathbf{u}(t)|_{L^2}^2 \le \mathrm{Ra}(2e^{-t}|T_0|_{L^2} + 2|\Omega|)|u_3(t)|_{L^2}, \tag{2.26}$$

which further implies, by Poincaré and the Cauchy-Schwarz inequality,

$$\varepsilon\frac{d}{dt}|\mathbf{u}(t)|_{L^2}^2 + |\mathbf{u}(t)|_{L^2}^2 \le 4\,\mathrm{Ra}^2(e^{-t}|T_0|_{L^2} + |\Omega|)^2. \tag{2.27}$$

Therefore, by the Gronwall inequality,

$$\begin{aligned} |\mathbf{u}(t)|_{L^2}^2 &\le |\mathbf{u}_0|_{L_2}^2 e^{-\frac{t}{\varepsilon}} \\ &\quad + 4\,\mathrm{Ra}^2\left(\frac{1}{1-2\varepsilon}\,e^{-2t}|T_0|_{L^2}^2 + \frac{2}{1-\varepsilon}\,e^{-t}|T_0|_{L^2}|\Omega| + |\Omega|^2\right). \end{aligned} \tag{2.28}$$

Hence, for

$$t_2 = \max\left\{t_1, \frac{1}{2}\ln\left(\frac{|\mathbf{u}_0|_{L^2}}{|\Omega|\,\mathrm{Ra}}\right), \ln\left(\frac{16|T_0|_{L^2}}{3|\Omega|}\right)\right\} \tag{2.29}$$

and $\varepsilon < \frac{1}{4}$, we have

$$|\mathbf{u}(t)|_{L^2} \le 3\,\mathrm{Ra}\,|\Omega| \quad \forall t \ge t_2. \tag{2.30}$$

This completes the uniform estimates in the L^2-space.

Next, we focus on the uniform estimates in the H^1-space. We first derive a uniform H^1-bound for the velocity.

Following [46], we show that a ball of radius $R_1 = c_1\,\mathrm{Ra}$ is absorbing after time t_1 for suitable c_1 (2.40).

Indeed, multiplying the velocity equation in the Boussinesq equation (2.2) by $A\mathbf{u}$, where A is the Stokes operator [9, 13, 20, 41], integrating over Ω, and applying Cauchy-Schwarz, Agmon's inequality, and the uniform estimate on θ (2.25), we have, for $t \ge t_1$,

$$\begin{aligned} \frac{\varepsilon}{2}\frac{d}{dt}|\nabla\mathbf{u}|_{L^2}^2 + |A\mathbf{u}|_{L^2}^2 &\le Ra|\theta|_{L^2}|A\mathbf{u}|_{L^2} + \varepsilon|\nabla\mathbf{u}|_{L^2}|A\mathbf{u}|_{L^2}|\mathbf{u}|_{L^\infty} \\ &\le Ra|\theta|_{L^2}|A\mathbf{u}|_{L^2} + c_2\varepsilon|\nabla\mathbf{u}|_{L^2}^{3/2}|A\mathbf{u}|_{L^2}^{3/2} \\ &\le \frac{1}{2}|A\mathbf{u}|_{L^2}^2 + Ra^2|\theta|_{L^2}^2 + 64c_2^4\varepsilon^4|\nabla\mathbf{u}|_{L^2}^6 \\ &\le \frac{1}{2}|A\mathbf{u}|_{L^2} + 9|\Omega|^2Ra^2 + 64c_2^4\varepsilon^4|\nabla\mathbf{u}|_{L^2}^6. \end{aligned} \tag{2.31}$$

Hence,

$$\varepsilon\frac{d}{dt}|\nabla\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}|_{L^2}^2 \le 18|\Omega|^2\mathrm{Ra}^2 + 128c_2^4\varepsilon^4|\nabla\mathbf{u}|_{L^2}^6. \tag{2.32}$$

Consequently, we have

$$\frac{d}{dt}|\nabla\mathbf{u}|_{L^2}^2 \le 0 \tag{2.33}$$

provided the following hold simultaneously:

$$\frac{1}{2}|\nabla \mathbf{u}|_{L^2}^2 \geq 18|\Omega|^2 \mathrm{Ra}^2, \tag{2.34}$$

$$\frac{1}{2}|\nabla \mathbf{u}|_{L^2}^2 \geq 128 c_2^4 \varepsilon^4 |\nabla \mathbf{u}|_{L^2}^6, \tag{2.35}$$

or equivalently,

$$|\nabla \mathbf{u}|_{L^2}^2 \geq 36|\Omega|^2 \mathrm{Ra}^2, \tag{2.36}$$

$$|\nabla \mathbf{u}|_{L^2}^2 \leq \frac{1}{16 c_2^2 \varepsilon^2}. \tag{2.37}$$

Hence we need

$$36|\Omega|^2\,\mathrm{Ra}^2 \leq \frac{1}{16 c_2^2 \varepsilon^2}, \tag{2.38}$$

i.e.,

$$\varepsilon\,\mathrm{Ra} = \frac{\mathrm{Ra}}{\mathrm{Pr}} \leq \frac{1}{24 c_2 |\Omega|}. \tag{2.39}$$

This is the exact condition of large Prandtl number that we need, as was discovered earlier [46].

Now we set

$$c_1 = 6 \tag{2.40}$$

and we observe that the ball of radius

$$R_1 = c_1\,\mathrm{Ra}\,|\Omega| = 6\,\mathrm{Ra}\,|\Omega| \tag{2.41}$$

in H^1 centered at the origin is invariant for the velocity field after t_1 under the large-Prandtl-number assumption (2.39) since $\frac{d}{dt}|\nabla \mathbf{u}|_{L^2}^2 < 0$ at the boundary of the ball for $t \geq t_1$.

In order to show that this ball is absorbing, we need to show that the velocity field must enter this ball within a set period of time after t_1. For this purpose, we go back to the energy inequality for the velocity (2.26), and we can deduce

$$\begin{aligned} &\frac{1}{t-t_2}\int_{t_2}^{t} |\nabla \mathbf{u}(s)|_{L^2}^2\,ds \\ &\quad\leq \frac{\varepsilon}{t-t_2}|\mathbf{u}(t_2)|_{L^2}^2 + \frac{4\,\mathrm{Ra}^2}{t-t_2}\left(\frac{1}{2}|T_0|_{L^2}^2 + 2|T_0|_{L^2}|\Omega| + (t-t_2)|\Omega|^2\right). \end{aligned} \tag{2.42}$$

For the given $c_1 = 6$ and $\varepsilon < \frac{1}{4}$, we define

$$t_3 = \max\left\{t_2 + \frac{9}{4},\, t_2 + \frac{2|T_0|_{L^2}^2}{|\Omega|^2},\, t_2 + \frac{2|T_0|_{L^2}}{|\Omega|}\right\}. \tag{2.43}$$

We then have

$$\frac{1}{t-t_2}\int_{t_2}^{t}|\nabla u(s)|_{L^2}^2\,ds \le \frac{c_1^2|\Omega|^2\,\mathrm{Ra}^2}{2} = 18|\Omega|^2\mathrm{Ra}^2, \quad t \ge t_3, \tag{2.44}$$

which implies the existence of $t^* \in [t_2, t_3]$ such that

$$|\nabla \mathbf{u}(t^*)|_{L^2} \le c_1|\Omega|\,\mathrm{Ra} = R_1. \tag{2.45}$$

Therefore,

$$|\nabla \mathbf{u}(t)|_{L^2} \le 6|\Omega|\,\mathrm{Ra} \quad \forall t \ge t_3. \tag{2.46}$$

Now we see that a ball of radius $R_1 = c_1|\Omega|\,\mathrm{Ra} = 6|\Omega|\,\mathrm{Ra}$ in H^1 is absorbing for the velocity field.

This uniform estimate in H^1 for the velocity field implies a similar H^1-estimate for the perturbative temperature field. Indeed, multiplying the temperature equation (2.3) by θ, integrating over Ω, and applying the Poincaré inequality, we have

$$\frac{d}{dt}|\theta|_{L^2}^2 + |\theta|_{L^2}^2 + |\nabla\theta|_{L^2}^2 \le 2|u_3|_{L^2}|\theta|_{L^2} \le 9\,\mathrm{Ra}\,|\Omega|^2 \quad \text{for } t \ge t_2. \tag{2.47}$$

This implies that for any $t \ge t^* \ge t_2$

$$\begin{aligned} e^{-t}\int_{t^*}^{t} e^{s}|\nabla\theta(s)|_{L^2}^2\,ds &\le 9\,\mathrm{Ra}\,|\Omega|^2 + e^{-(t-t^*)}|\theta(t^*)|_{L^2}^2 \\ &\le 9\,\mathrm{Ra}\,|\Omega|^2 + 9e^{-(t-t^*)}|\Omega|^2 \\ &\le 10\,\mathrm{Ra}\,|\Omega|^2. \end{aligned} \tag{2.48}$$

We also have, for $t \ge t_3 + 1$,

$$\begin{aligned} \frac{1}{t-t_3}\int_{t_3}^{t}|\nabla\theta(s)|_{L^2}^2 &\le \frac{|\theta(t_3)|_{L^2}^2}{t-t_3} + 9a\mathrm{Ra}|\Omega|^2 \\ &\le 9|\Omega|^2\left(\mathrm{Ra} + \frac{1}{t-t_3}\right) \\ &\le 10|\Omega|^2\,\mathrm{Ra} \quad \text{for } t \ge t_3 + 1. \end{aligned} \tag{2.49}$$

This implies there exists $t^* \in [t_3, t_3 + 1]$ such that

$$|\nabla\theta(t^*)|_{L^2}^2 \le 10|\Omega|^2\,\mathrm{Ra}\,. \tag{2.50}$$

Next, we multiply the perturbative temperature equation (2.3) by $-\Delta\theta$ and integrate over Ω; we then have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\nabla\theta|_{L^2}^2 + |\Delta\theta|_{L^2}^2 &\le |u_3|_{L^2}\,|\Delta\theta|_{L^2} + |\mathbf{u}|_{L^6}\,|\nabla\theta|_{L^3}\,|\Delta\theta|_{L^2} \\ &\le \frac{1}{4}|\Delta\theta|_{L^2}^2 + |u_3|_{L^2}^2 + c_3|\nabla\mathbf{u}|_{L^2}\,|\nabla\theta|_{L^2}^{1/2}\,|\Delta\theta|_{L^2}^{3/2} \\ &\le \frac{1}{2}|\Delta\theta|_{L^2}^2 + |u_3|_{L^2}^2 + c_4|\nabla\mathbf{u}|_{L^2}^4\,|\nabla\theta|_{L^2}^2. \end{aligned} \tag{2.51}$$

Hence, after applying Poincaré inequality, we have, for $t \geq t_3$,

$$\frac{d}{dt}|\nabla\theta|_{L^2}^2 + |\nabla\theta|_{L^2}^2 \leq 2|u_3|_{L^2}^2 + 2c_4|\nabla\mathbf{u}|_{L^2}^4\,|\nabla\theta|_{L^2}^2 \tag{2.52}$$
$$\leq 18\,\mathrm{Ra}^2\,|\Omega|^2 + 2c_4 6^4|\Omega|^4\,\mathrm{Ra}^4\,|\nabla\theta|_{L^2}^2.$$

This implies, with Gronwall's inequality and t^* chosen in (2.50) and the intermediate estimate (2.48),

$$\begin{aligned}|\nabla\theta(t)|_{L^2}^2 &\leq e^{-(t-t^*)}|\nabla\theta(t^*)|_{L^2}^2 + 18\,\mathrm{Ra}^2\,|\Omega|^2 + 20c_4 6^4|\Omega|^6\,\mathrm{Ra}^5\\ &\leq 10|\Omega|^2\,\mathrm{Ra} + 18|\Omega|^2\,\mathrm{Ra}^2 + 20c_4 6^4|\Omega|^6\,\mathrm{Ra}^5\\ &\leq c_5|\Omega|^6\,\mathrm{Ra}^5 \quad \forall t \geq t_3 + 1.\end{aligned} \tag{2.53}$$

These uniform H^1-norm estimates after neglecting a transitional time period (depending on initial data) imply the existence of a global attractor for the Boussinesq system at large Prandtl number (2.39).

This completes our uniform H^1-estimates.

Next, we estimate the time derivatives that are needed in order to view the Boussinesq system as a perturbation of the infinite-Prandtl-number model.

We first observe that, according to the perturbative temperature equation (2.3) and the uniform estimates (2.53), (2.46), (2.25), and (2.30),

$$\begin{aligned}\left|\frac{\partial\theta}{\partial t}\right|_{H^{-1}} &\leq |\Delta\theta|_{H^{-1}} + |\mathbf{u}\cdot\nabla\theta|_{H^{-1}} + |u_3|_{H^{-1}}\\ &\leq |\nabla\theta|_{L^2} + |\mathbf{u}|_{L^6}\,|\theta|_{L^3} + |u_3|_{L^2}\\ &\leq |\nabla\theta|_{L^2} + c_6|\nabla\mathbf{u}|_{L^2}\,|\theta|_{L^2}^{1/2}\,|\nabla\theta|_{L^2}^{1/2} + |u_3|_{L^2}\\ &\leq c_7|\Omega|^3\,\mathrm{Ra}^{5/2} \quad \forall t \geq t_3 + 1.\end{aligned} \tag{2.54}$$

Next we differentiate the velocity equation (2.2) in time and deduce

$$\varepsilon\left(\frac{\partial^2\mathbf{u}}{\partial t^2} + \left(\frac{\partial\mathbf{u}}{\partial t}\cdot\nabla\right)\mathbf{u} + (\mathbf{u}\cdot\nabla)\frac{\partial\mathbf{u}}{\partial t}\right) + \nabla\frac{\partial p}{\partial t} = \Delta\frac{\partial\mathbf{u}}{\partial t} + \mathrm{Ra}\,\mathbf{k}\,\frac{\partial\theta}{\partial t}. \tag{2.55}$$

Multiplying this equation by $\partial\mathbf{u}/\partial t$ and integrating over Ω, we deduce, for $t \geq t_3 + 1$,

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{d}{dt}\left|\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}^2 + \left|\nabla\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}^2\\ &\quad\leq \mathrm{Ra}\left|\frac{\partial\theta}{\partial t}\right|_{H^{-1}}\left|\nabla\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2} + \varepsilon|\nabla\mathbf{u}|_{L^2}\left|\frac{\partial\mathbf{u}}{\partial t}\right|_{L^4}^2\\ &\quad\leq \mathrm{Ra}\left|\frac{\partial\theta}{\partial t}\right|_{H^{-1}}\left|\nabla\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2} + c_8\varepsilon|\nabla\mathbf{u}|_{L^2}\left|\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}^{1/2}\left|\nabla\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}^{\frac{3}{2}}\end{aligned}$$

$$\leq \frac{1}{4}\left|\nabla \frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2 + 2\,\mathrm{Ra}^2 \left|\frac{\partial \theta}{\partial t}\right|_{H^{-1}}^2 + c_9\varepsilon^4 |\nabla \mathbf{u}|_{L^2}^4 \left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2$$

$$\leq \frac{1}{4}\left|\nabla \frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2 + 2c_7^2|\Omega|^6\,\mathrm{Ra}^7 + c_9 6^4\varepsilon^4|\Omega|^4\,\mathrm{Ra}^4 \left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2 \tag{2.56}$$

where we have utilized (2.54) and (2.46).

Therefore,

$$\varepsilon \frac{d}{dt}\left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2 + \left|\nabla \frac{\partial \mathbf{u}}{\partial t}\right|_{L^2}^2 \leq 4c_7^2|\Omega|^6\,\mathrm{Ra}^7, \quad t \geq t_3 + 1, \tag{2.57}$$

provided $\mathrm{Pr} = 1/\varepsilon$ is large enough so that

$$c_9 6^4\varepsilon^4|\Omega|^4\,\mathrm{Ra}^4 = c_9 6^4|\Omega|^4\left(\frac{\mathrm{Ra}}{\mathrm{Pr}}\right)^4 \leq \frac{1}{4}. \tag{2.58}$$

This is again large Prandtl number condition as in (2.39).

Applying the Poincaré and Gronwall inequalities, we deduce, for $t \geq t_* \geq t_3 + 1$,

$$\left|\frac{\partial \mathbf{u}}{\partial t}(t)\right|_{L^2}^2 \leq e^{-\frac{t-t^*}{\varepsilon}}\left|\frac{\partial \mathbf{u}}{\partial t}(t_*)\right|_{L^2}^2 + 4c_7^2|\Omega|^6\,\mathrm{Ra}^7$$

$$\leq \frac{e^{-\frac{t-t_*}{\varepsilon}}}{\varepsilon^2}\left(|\Delta \mathbf{u}(t_*)|_{L^2} + \mathrm{Ra}\,|\theta(t_*)| + \varepsilon|(\mathbf{u}(t_*)\cdot\nabla)\mathbf{u}(t_*)|_{L^2}\right)^2$$

$$+ 4c_7^2|\Omega|^6\,\mathrm{Ra}^7\,. \tag{2.59}$$

Thanks to (2.31), (2.46), and (2.39), we have

$$\varepsilon \frac{d}{dt}|\nabla \mathbf{u}|_{L^2}^2 + |A\mathbf{u}|_{L^2}^2 \leq 18|\Omega|^2\,\mathrm{Ra}^2 + 128c_2^4\varepsilon^4|\nabla \mathbf{u}|_{L^2}^6$$

$$\leq 18|\Omega|^2\,\mathrm{Ra}^2 + 128c_2^4\varepsilon^4 6^6|\Omega|^6\,\mathrm{Ra}^6$$

$$\leq 36|\Omega|^2\,\mathrm{Ra}^2 \quad \forall t \geq t_3. \tag{2.60}$$

This implies, for $t > t_3 + 1$,

$$\frac{1}{t - t_3 - 1}\int_{t_3+1}^{t} |A\mathbf{u}(s)|_{L^2}^2\,ds \leq 36|\Omega|^2\,\mathrm{Ra}^2 + \varepsilon|\nabla \mathbf{u}(t_3+1)|_{L^2}^2$$

$$\leq 72|\Omega|^2\,\mathrm{Ra}^2 \tag{2.61}$$

where we have applied the H^1 uniform estimate for the velocity (2.46).

Hence there exists $t^* \in [t_3 + 1, t_3 + 2]$ such that

$$|A\mathbf{u}(t^*)|_{L^2}^2 \leq 72|\Omega|^2\,\mathrm{Ra}^2, \tag{2.62}$$

and thus by elliptic regularity,

$$|\Delta \mathbf{u}(t^*)|_{L^2}^2 \leq c_{10}|\Omega|^2\,\mathrm{Ra}^2, \tag{2.63}$$

which further implies

$$|(\mathbf{u}(t^*)\cdot\nabla)\mathbf{u}(t^*)|_{L^2} \le c_{11}|\nabla\mathbf{u}(t^*)|_{L^2}^{3/2}\,|\Delta\mathbf{u}(t^*)|_{L^2}^{1/2} \le c_{12}|\Omega|^2\,\mathrm{Ra}^2\,. \tag{2.64}$$

Combining this estimate with (2.59), (2.53), (2.63), and (2.39), we have

$$\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2}^2 \le \frac{c_{13}}{\varepsilon}\varepsilon^2 e^{-\frac{t-t_3-2}{\varepsilon}}(|\Omega|^2\,\mathrm{Ra}^2+\varepsilon^2|\Omega|^4\,\mathrm{Ra}^4)+4c_7^2|\Omega|^6\,\mathrm{Ra}^7 \le c_{14}|\Omega|^6\,\mathrm{Ra}^7 \quad \forall t\ge t_3+3 \tag{2.65}$$

since $\frac{1}{\varepsilon}e^{-1/\varepsilon}\le 4e^{-2}$.

Next, we differentiate the temperature equation (2.3) in time and deduce

$$\frac{\partial^2\theta}{\partial t^2}+\mathbf{u}\cdot\nabla\frac{\partial\theta}{\partial t}+\frac{\partial\mathbf{u}}{\partial t}\cdot\nabla\theta-\frac{\partial u_3}{\partial t}=\Delta\frac{\partial\theta}{\partial t}. \tag{2.66}$$

Multiplying this equation by $\partial\theta/\partial t$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial\theta}{\partial t}\right|_{L^2}^2+\left|\nabla\frac{\partial\theta}{\partial t}\right|_{L^2}^2 \le \left|\frac{\partial u_3}{\partial t}\right|_{L^2}\left|\frac{\partial\theta}{\partial t}\right|_{L^2}+\left|\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}|\nabla\theta|_{L^6}\left|\frac{\partial\theta}{\partial t}\right|_{L^3} \le \frac{1}{2}\left|\nabla\frac{\partial\theta}{\partial t}\right|_{L^2}^2+\left|\frac{\partial u_3}{\partial t}\right|_{L^2}^2+c_{15}\left|\frac{\partial\mathbf{u}}{\partial t}\right|_{L^2}^2|\Delta\theta|_{L^2}^2. \tag{2.67}$$

Combining this with (2.65) and the Poincaré inequality, we have, for $t\ge t_3+3$,

$$\frac{d}{dt}\left|\frac{\partial\theta}{\partial t}\right|_{L^2}^2+\left|\frac{\partial\theta}{\partial t}\right|_{L^2}^2 \le c_{16}|\Omega|^6\,\mathrm{Ra}^7(1+|\Delta\theta|_{L^2}^2). \tag{2.68}$$

On the other hand, thanks to (2.51) and (2.53), we have

$$\frac{d}{dt}|\nabla\theta|_{L^2}^2+|\nabla\theta|_{L^2}^2+|\Delta\theta|_{L^2}^2 \le c_{17}|\Omega|^{10}\,\mathrm{Ra}^9 \quad \forall t\ge t_3+1. \tag{2.69}$$

This implies, together with (2.53) and a Gronwall-type argument,

$$e^{-t}\int_{t_3+3}^t e^s|\Delta\theta(s)|_{L^2}^2\,ds \le e^{-(t-t_3-3)}|\nabla\theta(t_3+3)|_{L^2}^2+c_{17}|\Omega|^{10}\,\mathrm{Ra}^9 \le c_{18}|\Omega|^{10}\,\mathrm{Ra}^9, \tag{2.70}$$

and

$$\int_{t_3+3}^t|\Delta\theta(s)|_{L^2}^2\,ds \le c_{17}|\Omega|^{10}\,\mathrm{Ra}^9(t-t_3-3)+|\nabla\theta(t_3+3)|_{L^2}^2 \le c_{19}|\Omega|^{10}\,\mathrm{Ra}^9(t-t_3-2). \tag{2.71}$$

Therefore, there exists $t^*\in[t_3+3,t_3+4]$ such that

$$|\Delta\theta(t^*)|^2\le 2c_{19}|\Omega|^{10}\,\mathrm{Ra}^9\,. \tag{2.72}$$

Hence

$$\begin{aligned}\left|\frac{\partial\theta}{\partial t}(t^*)\right|_{L^2} &\le |\Delta\theta(t^*)|_{L^2} + |u_3(t^*)|_{L^2} + |\mathbf{u}(t^*)|_{L^6}\,|\nabla\theta(t^*)|_{L^3}\\ &\le |\Delta\theta(t^*)|_{L^2} + |u_3(t^*)|_{L^2} + c_{20}|\nabla\mathbf{u}(t^*)|_{L^2}\,|\nabla\theta(t^*)|_{L^2}^{1/2}\,|\Delta\theta(t^*)|_{L^2}^{1/2}\\ &\le c_{21}|\Omega|^5\,\mathrm{Ra}^{5/4}\,. \end{aligned} \tag{2.73}$$

We now apply the Gronwall inequality to (2.68) and utilize (2.70) and (2.73) to deduce

$$\begin{aligned}\left|\frac{\partial\theta}{\partial t}(t)\right|_{L^2}^2 &\le e^{-(t-t^*)}\left|\frac{\partial\theta}{\partial t}(t^*)\right|_{L^2}^2 + c_{22}|\Omega|^{16}\,\mathrm{Ra}^{16}\\ &\le c_{23}|\Omega|^{16}\,\mathrm{Ra}^{16} \quad \forall t \ge t_3 + 4. \end{aligned} \tag{2.74}$$

Thus we have completed the uniform L^2-estimates of the time derivatives.

We are left with the uniform H^2-estimates. For this purpose, we multiply the velocity equation (2.2) by $A\mathbf{u}$ and integrate over Ω and deduce

$$\begin{aligned}|A\mathbf{u}(t)|_{L^2}^2 &\le \mathrm{Ra}\,|A\mathbf{u}(t)|_{L^2}|\theta(t)|_{L^2} + \varepsilon\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2}|A\mathbf{u}(t)|_{L^2}\\ &\quad + \varepsilon|\mathbf{u}(t)|_{L^\infty}\,|\nabla\mathbf{u}(t)|_{L^2}\,|A\mathbf{u}(t)|_{L^2}\\ &\le \mathrm{Ra}\,|A\mathbf{u}(t)|_{L^2}\,|\theta(t)|_{L^2} + \varepsilon\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2}|A\mathbf{u}(t)|_{L^2}\\ &\quad + c_{24}\varepsilon|\nabla\mathbf{u}(t)|_{L^2}^{3/2}|A\mathbf{u}(t)|_{L^2}^{3/2}\\ &\le \frac{1}{2}|A\mathbf{u}(t)|_{L^2}^2 + 4\,\mathrm{Ra}^2\,|\theta(t)|_{L^2}^2 + 4\varepsilon^2\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2}^2 + c_{25}\varepsilon^4|\nabla\mathbf{u}(t)|_{L^2}^6. \end{aligned} \tag{2.75}$$

Hence, for $t \ge t_3 + 3$, we have

$$\begin{aligned}|A\mathbf{u}(t)|_{L^2}^2 &\le 8\,\mathrm{Ra}^8\,|\theta(t)|_{L^2}^2 + 8\varepsilon^2\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2}^2 + 2c_{25}\varepsilon^4|\nabla\mathbf{u}(t)|_{L^2}^6\\ &\le 72\,\mathrm{Ra}^2\,|\Omega|^2 + 8\varepsilon^2 c_{14}|\Omega|^6\,\mathrm{Ra}^7 + c_{26}\varepsilon^4|\Omega|^6\,\mathrm{Ra}^6\\ &\le c_{27}|\Omega|^4\,\mathrm{Ra}^5, \end{aligned} \tag{2.76}$$

where we have used the large-Prandtl-number assumption (2.39) and the large-Rayleigh-number and aspect-ratio assumption.

Elliptic regularity then implies

$$|\mathbf{u}(t)|_{H^2} \le c_{28}|\Omega|^2 \operatorname{Ra}^{5/2} \quad \forall t \ge t_3 + 3.. \tag{2.77}$$

As for the H^2-estimate for the temperature field, we have

$$\begin{aligned} |\Delta\theta(t)|_{L^2} &\le \left|\frac{\partial\theta}{\partial t}(t)\right|_{L^2} + |u_3(t)|_{L^2} + |\mathbf{u}(t)|_{L^\infty}|\nabla\theta(t)|_{L^2} \\ &\le \left|\frac{\partial\theta}{\partial t}(t)\right|_{L^2} + |u_3(t)|_{L^2} + c_{29}|\nabla\mathbf{u}(t)|_{L^2}^{1/2}|\mathbf{u}(t)|_{H^2}^{1/2}|\nabla\theta(t)|_{L^2} \\ &\le c_{30}|\Omega|^8 \operatorname{Ra}^8 \end{aligned} \tag{2.78}$$

where we have applied the uniform estimates (2.74), (2.30), (2.46), (2.77), and (2.53).

This completes our uniform estimates in H^2.

To summarize, we have the following:

LEMMA 2.2 (Uniform A Priori Estimates) *Let* Ra *be an arbitrary large but fixed Rayleigh number. Suppose the Prandtl number* Pr *is large enough so that conditions* (2.39) *and* (2.58) *are satisfied. Then for any given suitable weak solutions* $(\mathbf{u}(t), \theta(t))$ *of the Boussinesq system, there exists a time* $t_3 = t_3(|\mathbf{u}_0|_{L^2}, |T_0|_{L^2}, |\Omega|)$ *given explicitly in* (2.43)*, and a constant* c_{31} *independent of* Pr *and* Ra *such that the following hold when* $t \ge t_3 + 4$*:*

$$|\mathbf{u}(t)|_{H^2} \le c_{31}|\Omega|^2 \operatorname{Ra}^{5/2}, \tag{2.79}$$

$$|\theta(t)|_{H^2} \le c_{31}|\Omega|^8 \operatorname{Ra}^8, \tag{2.80}$$

$$\left|\frac{\partial\mathbf{u}}{\partial t}(t)\right|_{L^2} \le c_{31}|\Omega|^3 \operatorname{Ra}^{7/2}, \tag{2.81}$$

$$\left|\frac{\partial\theta}{\partial t}(t)\right|_{L^2} \le c_{31}|\Omega|^8 \operatorname{Ra}^8 . \tag{2.82}$$

In particular, solutions on any of the global attractors must satisfy these estimates.

3 Convergence of the Global Attractors

We now show the convergence of the global attractors of the Boussinesq system to that of the infinite-Prandtl-number model as the Prandtl number approaches infinity.

As we mentioned earlier, the natural phase space for the Boussinesq system and the infinite-Prandtl-number model are different; the Boussinesq system requires both the velocity and the temperature field while the infinite-Prandtl-number model has only the temperature field (or velocity field).

There are two natural approaches to handle this discrepancy in phase space. We either project the phase space of the Boussinesq system down to the temperature field only, or we lift the phase space for the infinite-Prandtl-number model to the product space of velocity and temperature. We will see that the comparison of global attractors after projection is relatively easy and is similar to the upper semicontinuity of global attractors for dynamical systems (see, for instance, [24, 37, 40]). The comparison of global attractors after lifting the phase space of the infinite-Prandtl-number model is a little bit more involved. The proof utilizes a priori estimates (uniform in Prandtl number) on the material derivative of the velocity field after the initial layer that we derived in the previous section.

It is our belief that the techniques developed here can be applied to more general dynamical systems with two explicitly separated time scales (see Section 4 as well). Therefore, we formulate our result in a more general fashion and view the case of convection at large Prandtl number as a special application.

The reader is referred to [28] for rudiments of functional analysis.

THEOREM 3.1 (Convergence of Global Attractors) *Consider a generalized dynamical system on* $X_1 \times X_2$ *with two explicitly separated time scales*

$$\varepsilon\left(\frac{dx_1}{dt} + g(x_1, x_2)\right) = f_1(x_1, x_2), \quad x_1(0) = x_{10}, \tag{3.1}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2), \quad x_2(0) = x_{20}, \tag{3.2}$$

where X_1, X_2 *are two Banach spaces.*

Let

$$0 = f_1(x_1^0, x_2^0), \tag{3.3}$$

$$\frac{dx_2^0}{dt} = f_2(x_1^0, x_2^0), x_2(0) = x_{20}, \tag{3.4}$$

be the limit system at $\varepsilon = 0$.

We postulate the following assumptions:

(H1) (*Uniform Dissipativity of the Perturbed System*) *The two-time-scale system* (3.1)–(3.2) *possesses a global attractor* $\mathcal{A}_\varepsilon$ *for all small positive* ε. *We also assume that the global attractors are regular and uniformly bounded in the sense that there exist Banach spaces* Y_j, $j = 1, 2$, *which are continuously imbedded in the* X_j, $j = 1, 2$, *respectively, and there exists a constant* R_0 *such that*

$$\|x_1\|_{Y_1} + \|x_2\|_{Y_2} \le R_0 \quad \forall (x_1, x_2) \in \mathcal{A}_\varepsilon \quad \forall \varepsilon. \tag{3.5}$$

(H2) (*Dissipativity of the Limit System*) *The limit system is well-posed and possesses a global attractor* $\mathcal{A}_0$ *in* X_2.

(H3) (*Convergence of the Slow Variable*) *The slow variable of the solutions of the two-time-scale system converges uniformly on bounded sets in* $Y_1 \times Y_2$

to that of the limit system after neglecting a transitional time period; *i.e., for any $R > 0$, there exists a $t_0 > 0$ such that for any $t > t_0$,*

$$\lim_{\varepsilon \to 0} \sup_{\|x_{10}\|_{Y_1} + \|x_{20}\|_{Y_2} \le R} \|x_2(t) - x_2^0(t)\|_{X_2} = 0 \quad \forall t \ge t_0. \tag{3.6}$$

Then the global attractors $\mathcal{A}_\varepsilon$ of the two-time-scale system converge to $\mathcal{A}_0$ after projection, i.e.,

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{X_2} (\mathcal{P}_2 \mathcal{A}_\varepsilon, \mathcal{A}_0) = 0, \tag{3.7}$$

where $\mathcal{P}_2$ is the projection from $X_1 \times X_2$ to X_2 defined as

$$\mathcal{P}_2(x_1, x_2) = x_2. \tag{3.8}$$

Furthermore, let us assume the following:

(H4) (*Smallness of the Perturbation*) *The two-time-scale problem* (3.1)–(3.2) *is a uniformly small perturbation of the limit problem* (3.3)–(3.4) *when confined to the global attractors, i.e.,*

$$\lim_{\varepsilon \to 0} \sup_{(x_1, x_2) \in \mathcal{A}_\varepsilon} \left\| \varepsilon \left(\frac{dx_1}{dt} + g(x_1, x_2) \right) \right\|_{X_1} = 0. \tag{3.9}$$

(H5) (*Continuity of the Slave Relation*) *The first equation in the limit system* (3.3) *can be solved continuously for x_1^0 with given x_2^0 and a nontrivial left-hand side*; *i.e., there exists a continuous function $F_1 : X_2 \times X_1 \to X_1$ such that*

$$y = f_1(F_1(x_2, y), x_2). \tag{3.10}$$

Moreover, we assume F_1 is uniformly continuous for $y = 0$ and x_2 in bounded sets in Y_2.

Then the attractors $\mathcal{A}_\varepsilon$ of the two-time-scale system converge to $\mathcal{A}_0$ after lift, i.e.,

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{X_1 \times X_2} (\mathcal{A}_\varepsilon, \mathcal{L}\mathcal{A}_0) = 0, \tag{3.11}$$

where $\mathcal{L}$ is the lift from X_2 to $X_1 \times X_2$ defined by

$$\mathcal{L}(x_2) = (F_1(x_2, 0), x_2). \tag{3.12}$$

PROOF: We borrow ideas from the proof of upper semicontinuity of global attractors for dissipative dynamical systems [24, 37, 40].

Recall that the Hausdorff semidistance between two sets A_1 and A_2 in a Banach space is defined as

$$\operatorname{dist}_X(A_1, A_2) = \sup_{x_1 \in A_1} \inf_{x_2 \in A_2} \|x_1 - x_2\|_X. \tag{3.13}$$

We first prove the convergence in the projected sense, i.e., (3.7).

Let $\delta > 0$ be fixed. Since the limit system possesses a global attractor $\mathcal{A}_0$ (H2), and since a bounded ball in Y_2 is bounded in X_2 by the continuous imbedding

(H1), and since the global attractor attracts all bounded sets in the phase space, there exists a $T = T(\delta) > 0$ such that

$$\operatorname{dist}_{X_2}\big(S^0(t)B_{R_0}(Y_2), \mathcal{A}_0\big) \le \frac{\delta}{3} \quad \forall t \ge T, \tag{3.14}$$

where $S^0(t)$ denotes the solution semigroup associated with the limit system (3.3)–(3.4) and $B_{R_0}(Y_2)$ denotes the ball in Y_2 with radius R_0 centered at the origin.

On the other hand, utilizing (H3) with $R = R_0$, we see that there exists an $\varepsilon(\delta)$ such that

$$\sup_{(x_{10},x_{20})\in B_{R_0}(Y_1\times Y_2)} \|x_2(T) - x_2^0(T)\|_{X_2} \le \frac{\delta}{2} \quad \forall \varepsilon \le \varepsilon(\delta). \tag{3.15}$$

Now for $(y_1, y_2) \in \mathcal{A}_\varepsilon$ with $\varepsilon \le \varepsilon(\delta)$, there exists $(x_{10}, x_{20}) \in \mathcal{A}_\varepsilon \subset B_{R_0}(Y_1 \times Y_2)$ so that

$$(y_1, y_2) = (x_1(T), x_2(T)) \tag{3.16}$$

since $\mathcal{A}_\varepsilon$ is invariant. Therefore, thanks to (3.15),

$$\|y_2 - x_2^0(T)\|_{X_2} = \|x_2(T) - x_2^0(T)\|_{X_2} \le \frac{\delta}{2}. \tag{3.17}$$

On the other hand, since $x_2^0(T) = S^0(T)x_{20} \in S^0(T)B_{R_0}(Y_2)$, there exists a $x_{2\infty}^0 \in \mathcal{A}_0$ such that

$$\|x_2^0(T) - x_{2\infty}^0\|_{X_2} \le \frac{\delta}{2}, \tag{3.18}$$

by the attracting property of $\mathcal{A}_0$ (3.14).

Hence we deduce, by the triangle inequality,

$$\|y_2 - x_{2\infty}^0\|_{X_2} \le \delta. \tag{3.19}$$

This further implies

$$\operatorname{dist}_{X_2}(y_2, \mathcal{A}_0) \le \delta. \tag{3.20}$$

Since y_2 is an arbitrary element in $\mathcal{P}_2\mathcal{A}_\varepsilon$, $\varepsilon \le \varepsilon(\delta)$, we have

$$\operatorname{dist}_{X_2}(\mathcal{P}_2\mathcal{A}_\varepsilon, \mathcal{A}_0) \le \delta \quad \forall \varepsilon \le \varepsilon(\delta). \tag{3.21}$$

This ends the proof of the convergence of the global attractors in the projected sense, i.e., (3.7).

Next we discuss convergence in the lifted sense as defined in (3.11). Thanks to (H5), we can rewrite the fast equation (3.1) as

$$x_1 = F_1\left(x_2, \varepsilon\left(\frac{dx_1}{dt} + g(x_1, x_2)\right)\right). \tag{3.22}$$

We also notice that for any fixed $\delta > 0$, there exists an $\eta = \eta(\delta) > 0$ such that

$$\|F_1(x_2, y) - F_1(x_2^0, 0)\|_{X_1} \le \frac{\delta}{2} \tag{3.23}$$

provided that

$$\|x_2 - x_2^0\|_{X_2} + \|y\|_{X_1} \le \eta \tag{3.24}$$

by the uniform continuity of F_1 (H5).

Thanks to (H4), we have, for some $\varepsilon_1 = \varepsilon_1(\delta)$,

$$\sup_{(x_1,x_2)\in\mathcal{A}_\varepsilon} \left\| \varepsilon\left(\frac{dx_1}{dt} + g(x_1, x_2)\right)\right\|_{X_1} \le \frac{\eta}{2} \quad \forall \varepsilon \le \varepsilon_1. \tag{3.25}$$

We also have, thanks to the first part of the theorem, that there exists an $\varepsilon_2 = \varepsilon_2(\delta)$ such that

$$\mathrm{dist}_{X_2}(\mathcal{P}_2\mathcal{A}_\varepsilon, \mathcal{A}_0) \le \min\left(\frac{\delta}{3}, \frac{\eta}{3}\right) \quad \forall \varepsilon \le \varepsilon_2. \tag{3.26}$$

Therefore, for any given $(x_1, x_2) \in \mathcal{A}_\varepsilon$, $\varepsilon \le \min(\varepsilon_1, \varepsilon_2)$, there exists an $x_2^0 \in \mathcal{A}_0$ such that

$$\|x_2 - x_2^0\|_{X_2} \le \min\left(\frac{\delta}{2}, \frac{\eta}{2}\right). \tag{3.27}$$

Consequently, for $\varepsilon \le \min(\varepsilon_1, \varepsilon_2)$, we have

$$\|x_2 - x_2^0\|_{X_2} + \left\| \varepsilon\left(\frac{dx_1}{dt} + g(x_1, x_2)\right)\right\|_{X_1} \le \eta \tag{3.28}$$

and hence

$$\begin{aligned}
&\|x_1 - x_1^0\|_{X_1} + \|x_2 - x_2^0\|_{X_2} \\
&\quad = \left\| F_1\left(x_2, \varepsilon\left(\frac{dx_1}{dt} + g(x_1, x_2)\right)\right) - F_1(x_2^0, 0)\right\|_{X_1} + \|x_2 - x_2^0\|_{X_2} \\
&\quad \le \delta.
\end{aligned}$$

Henceforth

$$\mathrm{dist}_{X_1\times X_2}((x_1, x_2), \mathcal{L}\mathcal{A}_0) \le \delta. \tag{3.29}$$

This ends the proof of the theorem. □

An immediate consequence of the theorem is its application to the Boussinesq system for convection at large Prandtl number. We have the following:

COROLLARY 3.2 (Application to Large-Prandtl-Number Convection) *For every large but fixed Rayleigh number* Ra*, the global attractors $\mathcal{A}_\varepsilon$ of the Boussinesq system for convection with Prandtl number* $\mathrm{Pr} = 1/\varepsilon$ *converge to the attractor $\mathcal{A}_0$ of the infinite-Prandtl-number model for convection in both the projected sense of* (3.7) *and the lifted sense of* (3.11) *as ε approaches* 0*.*

PROOF: We need to verify the assumptions in the theorem.

We first identify

$$X_1 = \{\mathbf{u} \in \mathbf{L}^2 : u_3|_{z=0,1} = 0, \nabla \cdot \mathbf{u} = 0, \text{ periodic in } x \text{ and } y\}, \tag{3.30}$$

$$X_2 = L^2, \tag{3.31}$$

$$Y_1 = \{\mathbf{u} \in \mathbf{H}^2 : \mathbf{u}|_{z=0,1} = 0, \nabla \cdot \mathbf{u} = 0, \text{ periodic in } x \text{ and } y\}, \tag{3.32}$$

$$Y_2 = \{\theta \in H^2 : \theta|_{z=0,1} = 0, \text{ periodic in } x \text{ and } y\}. \tag{3.33}$$

It is then obvious that the Hilbert spaces Y_j, $j = 1, 2$, are continuously imbedded in the Hilbert spaces X_j, $j = 1, 2$, respectively.

Hypothesis (H1) is clear thanks to Lemma 2.2 and [46], where we have shown the existence of global attractors for the Boussinesq system for convection.

Hypothesis (H2) is evident with our infinite-Prandtl-number model.

Hypothesis (H3) is verified thanks to theorem 2 of [44], where we proved the convergence of solutions of the Boussinesq system to that of the infinite-Prandtl-number model on a finite time interval after neglecting a transitional time period and with initial data in $Y_1 \times Y_2$. The uniformity of convergence is clear since the constants depend on the $Y_1 \times Y_2$ norm only.

As for Hypothesis (H4), we have

$$\begin{aligned} \left| \varepsilon\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)\right|_{L^2} &\le \varepsilon\left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2} + |\mathbf{u}|_{L^\infty}\,|\nabla\mathbf{u}|_{L^2}\right) \\ &\le \varepsilon\left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2} + c_{32}|\mathbf{u}|_{H^2}\,|\nabla\mathbf{u}|_{L^2}\right) \\ &\to 0 \end{aligned} \tag{3.34}$$

according to Lemma 2.2. Thus Hypothesis (H4) is verified.

In terms of Hypothesis (H5), we have, for $(\mathbf{v}, \theta) \in Y_1 \times Y_2$,

$$F_1(\theta, \mathbf{v}) = A^{-1}(\mathrm{Ra}\, P(\mathbf{k}\theta) - \mathbf{v}) \tag{3.35}$$

where A is the Stokes operator and P is the Leray-Hopf projection from $\mathbf{L}^2$ to X_1 [9, 13, 20, 41]. It is then clear that Hypothesis (H5) is satisfied.

This ends the proof of the corollary. □

Remark 3.3. It is worthwhile to reiterate that the global attractors to the Boussinesq system for convection and the infinite-Prandtl-number model for convection are nontrivial. It is easy to see that the pure conduction state belongs to each of the global attractors. It is well-known that the pure conduction state becomes unstable at high enough Rayleigh number and Bénard cells emerge [25, 30, 36]. Hence the attractors must be nontrivial. Indeed, numerical results indicate convection at large Prandtl number could be very complicated with thermal plumes emerging at a seemingly random time and location [43].

4 Concluding Remarks

In this manuscript we have demonstrated that the global attractors of the Boussinesq system (which exist at large Prandtl number [46]) converge to that of the infinite-Prandtl-number model for convection. This complements our earlier result on the convergence of suitable weak solutions to the Boussinesq system to that of the infinite-Prandtl-number model on a finite interval modulo an initial transitional layer [44]. These two results provide us with confidence for using the infinite-Prandtl-number model as a simplified model for convection at large Prandtl number, both on short time and on long time since the global attractors embody all long-time behavior. This also provides positive indication that the statistical properties may be close.

On the other hand, convergence in the symmetric Hausdorff distance sense, i.e., replacing $\mathrm{dist}(A_1, A_2)$ by $\mathrm{dist}(A_1, A_2) + \mathrm{dist}(A_2, A_1)$ for two sets in a Banach or metric space, usually requires some hyperbolicity [1, 37, 40] and may be much harder or even invalid as we can see from simple bifurcation examples such as

$$\frac{du}{dt} = u(-1 + 2u^2 - u^4 - \epsilon). \tag{4.1}$$

There are two immediate questions that naturally come to mind. First, are there any other good measures of the (long-time) validity of the infinite-Prandtl-number model? If yes, what can we say about the the validity under these measures? Second, is the result here special or could it be applied to more general (generalized) dynamical systems with two explicitly separated time scales?

Regarding the first question, there are many other measures of (long-time) validity of the infinite-Prandtl-number model. For instance, it is easy to show that the Hausdorff/fractal dimension of the global attractor to the Boussinesq system at large Prandtl number and that of the infinite Prandtl number are finite. Thus we may ask if the dimension of the global attractors to the Boussinesq system converge to that of the infinite-Prandtl-number model. This could be very hard since we are usually only able to estimate upper bounds for the dimension of attractors. We can then ask whether there is a bound that is uniform in (large) Prandtl number on the dimension of attractors for the Boussinesq system. The direct application of the Constantin-Foias-Temam version of the Kaplan-Yorke formula [40] does not seem to work and may need to be revised due to the two time scales. The theorem that we have in Section 3 supports an affirmative answer to the question of existence of uniform bounds on the dimension of the global attractors. However, it is not conclusive since two very close sets could have very different fractal dimensions due to local oscillations. Therefore, it also makes sense to discuss convergence of trajectories and attractors in spaces with more regularity (which measures oscillation). An alternative approach is to use the crude estimates using squeezing properties [27, 39, 40]. One may also consider bounds on the number of determining modes, nodes, volumes, etc. (see, for instance, [20] among others).

Two other objects related to long-time behavior are inertial manifold and exponential attractors [17, 40]. It is easy to check that the infinite-Prandtl-number model for convection possesses an exponential attractor that is positively invariant and attracts all orbits with exponential rate. We can ask if the Boussinesq system possesses an exponential attractor at large Prandtl number and whether the exponential attractors, if they exist, converge in some sense to one of the exponential attractors of the infinite-Prandtl-number model. See [19] for such a convergence result for a singularly perturbed wave equation, [21] for a general result, and [17] for more on exponential attractors. Indeed, one may even be hopeful for continuity of exponential attractors since these objects are more stable [17, 34]. The question regarding inertial manifold (a finite-dimensional manifold that is positively invariant under the dynamics and attracts all orbits with exponential rate) is much harder. Even the existence of an inertial manifold for the simplified infinite-Prandtl-number model is unknown.

A related result is a connection to the Landau-Lifshits theory on degrees of freedom for turbulent flows. With fixed Rayleigh and Prandtl numbers, the intensity of the turbulence is fixed, and thus it is expected that the degrees of freedom of the system scale linearly in each of the horizontal lengths (L_x and L_y) according to the Landau-Lifshits theory (see, for instance, [13, 20]). Utilizing techniques that we developed earlier for shear flows in elongated channels [15], we can obtain upper bounds on the dimension of attractors for the Boussinesq system and the infinite-Prandtl-number model that scale linearly in each of the horizontal lengths (L_x and L_y). We leave the details to the interested reader. Such a bound is optimal in terms of dependence on L_x and L_y since one can show that at a fixed value of Ra the number of linearly unstable modes around the conduction state is proportional to the "density of states" that is proportional to $L_x * L_y$ (C. R. Doering, private communication). Similar optimal bounds in the case of the free-slip boundary condition on top and bottom and with restriction to the two-dimensional case or certain functional invariant sets in the three-dimensional case are known [33, 35].

As we discussed earlier, statistical behavior is probably more important and realistic for systems like the Boussinesq system, where we expect turbulent/chaotic behavior. Thus a more important criterion for the validity of the infinite-Prandtl-number model for convection is if the statistical properties for the Boussinesq system are close to the corresponding statistical properties of the infinite-Prandtl-number model. A prominent statistical quantity in convection is the averaged heat transfer in the vertical direction that can be characterized via the time-averaged Nusselt number [4, 8, 43]. In the case of the infinite-Prandtl-number model, an upper bound on the Nusselt number that agrees with physical scaling (modulo a logarithmic term) has been derived by Constantin and Doering [8]. It is then interesting to see if we can derive an upper bound on the Nusselt number for the Boussinesq system that agrees with the Constantin-Doering result in the sense that the upper bound should be the Constantin-Doering bound for infinite-Prandtl-number convection plus a correction term that vanishes as the Prandtl number approaches infinity. This and a few other issues are currently under investigation [16].

In terms of the second question, we consider the Rayleigh-Bénard convection problem at large Prandtl number as a special case of more general physical systems with two explicitly separated time scales. In general, we should not expect such kind results to be true all the time. For instance, problems with fast oscillation on the fast-time scale cannot be expected to converge in the strong sense as is clear from the following example

$$\varepsilon \frac{dx}{dt} = -y - z, \quad \varepsilon \frac{dy}{dt} = x - z, \quad \frac{dz}{dt} = -z.$$

We see that the limit problem ($\varepsilon = 0$) has trivial dynamics (converge to the origin) while we have persistent oscillation for positive ε. We do not have convergence in terms of individual trajectory or in terms of long-time behavior. This example can easily be modified into a dissipative one via applying a filter in space. Although strong convergence is not possible in this oscillatory situation, it may still be possible to discuss convergence in the weak sense (see, for instance, [31]). Oscillation could occur spontaneously via Hopf bifurcation. For instance, the two-dimensional Navier-Stokes equation under generalized Kolmogorov forcing may experience Hopf bifurcation [7]. Therefore it may be relatively easier to deal with the case with explicit fast oscillation; it may still be very hard to handle the general situation with oscillation generated by the nonlinear mechanism.

If the fast dynamics is not oscillatory and the limit system is regular enough, i.e., the two-time-scale problem relaxes to the slow-time-scale problem as we have encountered here for convection at large Prandtl number, we expect similar results. For instance, we expect to have similar results (convergence of trajectory modulo a transitional layer and convergence of the global attractor) for convection in a porous medium at small Darcy-Prandtl number [42]. Convection in a porous medium is more regular than standard convection since the nonlinear advection term in the velocity field is missing and the well-posedness of the governing system is known [18, 29]. We remark that there are models for convection in porous media that retain the nonlinear advection term [5, 11]. In this case, the mathematical difficulty in terms of well-posedness is the same as that for the three-dimensional incompressible Euler systems. We also expect to have similar results for certain reaction-diffusion systems with one fast reaction/diffusion time. In this case, the limit system may not be well posed since the limit elliptic equation in the fast variable may not have a unique solution. In this case we employ the notion of generalized dynamical systems [2], and it seems that similar results may follow as well.

Acknowledgment. This work was supported in part by a grant from the National Science Foundation, a COFRS award from Florida State University, and a senior visiting scholar fellowship from Fudan University. The author thanks C. R. Doering, A. Majda, and A. Miranville for helpful suggestions and comments.

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Received September 2005.