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Abstract

This paper develops a new theory of independent fuzzy probabilities, that addresses limitations of fuzzy fault trees both and Zadeh's fuzzy extension of probability. In contrast to the fuzzy fault tree approach, the new theory is complete since it assigns a fuzzy probability to every event. In the case of a probability theory built from independent events, Zadeh's extension is not consistent with fuzzy fault trees. Our new extension is also consistent. The new theory is demonstrated with an example.

Introduction

Many safety assessment models require, as input, the probabilities of a number of independent events. Often these probabilities can be estimated from data or theory, but sometimes choosing probabilities for input is difficult. This work is part of an ongoing study in high-consequence surety analysis. Many of the factors of interest come from traditionally non-mathematical areas of research, such as estimating the probability of a terrorist attack, compliance with safety practices, or a flawed design of a safety system. Other factors are too expensive or dangerous to measure experimentally. Instead, expert opinion is used to provide these probabilities, but these estimates are rarely precise. Fuzzy sets and possibility theory provide a tool for describing and analyzing these uncertain quantities.

Fuzzy fault trees provide a powerful and computationally efficient technique for developing fuzzy probabilities based on independent inputs. The probability of any event that can be described in terms of a sequence of independent unions, intersections, and complements may be calculated by a fuzzy fault tree. Unfortunately, fuzzy fault trees do not provide a complete theory: Events of substantial practical interest for calculating safety margins cannot be described only by independent operations. Thus the standard fuzzy extension (based on fuzzy fault trees) is not complete, since not all events are assigned a fuzzy probability. Zadeh and others have proposed other complete extensions. Unfortunately, the calculations of these models are not consistent with the underlying fuzzy probabilities of the independent inputs.

In this paper, we discuss a new extension of crisp probability theory. Our model is based on *n* independent inputs, each with a fuzzy probability. The elements of our sample space describe exactly which of the *n* input events did and did not occur. Our extension is complete, since a fuzzy probability is assigned to every subset of the sample space. Our extension is also consistent with all calculations that can be arranged as a fault tree [1].

Our approach allows the reliability analyst to develop complete and consistent fuzzy reliability models from existing crisp models. This allows a comprehensive analysis of the system. Computational algorithms are provided both to extend existing models and develop new models. The technique is demonstrated with an example.

An uncertain parameter $F \in \Re$ may be assigned a fuzzy membership function $\underline{F}(y)$ mapping $\Re$ into [0,1], which is the membership function of a fuzzy set $\underline{F}$. Then the possibility that $\underline{F}$ is in a set S is designated by $\Pi_F(S)$, and

$$\Pi_F(S)=\sup_{y \in S} \underline{F}(y).$$

This is the sense in which we describe uncertainty in the probability of an event A. Note the inherent conservative nature of possibility theory: the possibility of a set is high if a single point in the set has high possibility. This may be viewed as a worst-case calculation and is appropriate for the study of rare but high- consequence events. An uncertainty model based on probability theory, on the other hand, better models the average risk over repeated trials.

In this paper, $\underline{P}_A$ is a fuzzy set describing uncertainty in the crisp number P(A). Fuzzy fault trees provide a method for developing fuzzy probabilities based on independent fuzzy inputs $\underline{P}_A$ [2]. The probability of any event that can be described in terms of a sequence of independent unions, intersections, and complements may be calculated by a fuzzy fault tree. Unfortunately, we show below that some events of substantial practical interest cannot be described only by independent operations; fuzzy fault trees do not provide a complete theory. Thus the standard fuzzy extension (based on fuzzy fault trees) is not complete, since not all events are assigned a fuzzy probability. Zadeh proposed another extension that is complete [3], but his extension is shown (in our context) to be inconsistent with the calculations from fuzzy fault trees.

Here we develop a new extension of crisp probability theory, based on *n* independent inputs, each with a fuzzy probability. The elements of our sample space describe exactly which of the *n* input events did and did not occur. This extension will be shown to be both complete and consistent. These results are discussed in more detail in [1].

Independent Calculations and Fuzzy Fault Trees

Throughout this paper, we use the bar notation $\underline{P}_A$ to indicate a fuzzy set representing probability of A, the notation $\underline{P}_A(y)$ to indicate the corresponding membership function, and $\underline{P}_A^{\alpha}=\{y: \underline{P}_A(y)\geq\alpha\}$ to indicate the corresponding α-cuts. A convex fuzzy set $\underline{P}_A$ has special structure; each α-cut is a closed and convex subset of $\Re$. We see for a convex fuzzy probability that each α-cut can be written as a closed interval with $\underline{P}_A^{\alpha} = [P_{A1}^{\alpha}, P_{A2}^{\alpha}]$. This assumption of convexity is equivalent to assuming that the membership function has a single mode. Earlier work with independent fuzzy probabilities relied on this (often quite reasonable) assumption of convexity, but our work will be more general. Following the lead of most fuzzy models, all fuzzy sets here are required to have nonempty $\alpha=1$ cut. This property is called normality.

Consider independent events $A_1, A_2, \ldots, A_n$ with estimated fuzzy probabilities $\underline{P}_{A1}$ $\underline{P}_{A2}, \ldots, \underline{P}_{An}$, which will be used in a reliability model. Our goal is to build a fuzzy probability theory to describe the probabilities of various unions, intersections, and complements of these sets. To this end, we follow the standard approach of Tanaka et. al. [2] and first build fuzzy intersections of independent events.

If events A_i are independent, then for crisp probabilities we have

$$P(A_i \cup A_j)=P(A_i)+P(A_j)-P(A_i)P(A_j)$$

and

$$P(A_i \cap A_j)=P(A_i)P(A_j).$$

Using the usual extension principle, we define the fuzzy independent union and intersection as

$$\underline{P}_{Ai\cup Aj}(y)= \quad \sup_{y=pi+pj-pi\,pj} \quad \min[\,\underline{P}_{Ai}(p_i), \underline{P}_{Aj}(p_j)] \qquad \text{(Eq. 1)}$$

and

$$\underline{P}_{Ai\cap Aj}(y)= \quad \sup_{y=pi\,pj} \quad \min[\,\underline{P}_{Ai}(p_i), \underline{P}_{Aj}(p_j)]. \qquad \text{(Eq. 2)}$$

Complements of fuzzy probabilities are similarly defined by

$$\underline{P}_{Ai}(y)= \quad \sup_{y=1-pi} \quad \underline{P}_{Ai}(p_i) \; = \; \underline{P}_{Ai}(1-y). \qquad \text{(Eq. 3)}$$

We then have the following familiar properties:

$$\underline{P}_{Ai\cup Aj} = \underline{P}_{Aj\cup Ai} \qquad \underline{P}_{Ai\cap Aj} = \underline{P}_{Aj\cap Ai}$$

$$\underline{P}_{(Ai\cup Aj)\cup Ak} = \underline{P}_{Ai\cup(Aj\cup Ak)} \qquad \underline{P}_{(Ai\cap Aj)\cap Ak} = \underline{P}_{Ai\cap(Aj\cap Ak)}$$

$$\underline{P}_{(Ai\cup Aj)'} = \underline{P}_{Aj'\cap Ai'} \qquad \underline{P}_{(Ai\cap Aj)'} = \underline{P}_{Aj'\cup Ai'}\,. \qquad \text{(Eq. 4)}$$

This third formula is DeMorgan's law and extends in the obvious way to

$$\underline{P}_{(A1\cup A2\cup\ldots\cup Ak)'} = \underline{P}_{A1'\cap A2'\cap\ldots\cap Ak'} \qquad \underline{P}_{(A1\cap A2\cap\ldots\cap Ak)'} = \underline{P}_{A1'\cup A2'\cup\ldots\cup Ak'} \qquad \text{(Eq. 5)}$$

If the fuzzy probabilities are convex, we have the relationships between endpoints of the α-cut intervals

$$[P_{Ai\cup Aj\ 1}^{\alpha}, P_{Ai\cup Aj\ 2}^{\alpha}] =$$

$$[P_{Ai\ 1}^{\alpha} + P_{Aj\ 1}^{\alpha} - P_{Ai\ 1}^{\alpha} P_{Aj\ 1}^{\alpha},\quad P_{Ai\ 2}^{\alpha} + P_{Aj\ 2}^{\alpha} - P_{Ai\ 2}^{\alpha} P_{Aj\ 2}^{\alpha}] \qquad \text{(Eq. 6)}$$

and

$$[P_{Ai\cap Aj\ 1}^{\alpha}, P_{Ai\cap Aj\ 2}^{\alpha}] = [P_{Ai\ 1}^{\alpha} P_{Aj\ 1}^{\alpha}, P_{Ai\ 2}^{\alpha} P_{Aj\ 2}^{\alpha}]. \qquad \text{(Eq. 7)}$$

Unfortunately, the distributive laws fail. Straightforward application of the above formulas shows

$$\underline{P}_{Ai\cup(Aj\cap Ak)} \neq \underline{P}_{(Ai\cup Aj)\cap(Ai\cup Ak)} \qquad \underline{P}_{Ai\cap(Aj\cup Ak)} \neq \underline{P}_{(Ai\cap Aj)\cup(Ai\cap Ak)}\,. \qquad \text{(Eq. 8)}$$

This formula fails because of the violation of independence.

As we see in Equation 8, care must be used in organizing calculations to maintain independence. This is usually done by describing calculations as a tree structure. This viewpoint was naturally assumed in several papers on fuzzy fault trees [2,4,5,6,7,8]. To illustrate this concept, consider the example tree diagram in Figure 1. This diagram contains three varieties of nodes: unions, intersections, and complements. At the nodes, fuzzy input probabilities are combined according to the formulas in equations (1-3). As long as the tree only feeds upward and each node has only one output, independence is maintained. Because of DeMorgan's laws in Equation 5, we can develop fault trees using only unions and intersections (but no complements) or only intersections and complements (but no unions). Thus several somewhat different approaches to fault trees are in fact equivalent when the standard extensions in Equations 1 through 3 are used.

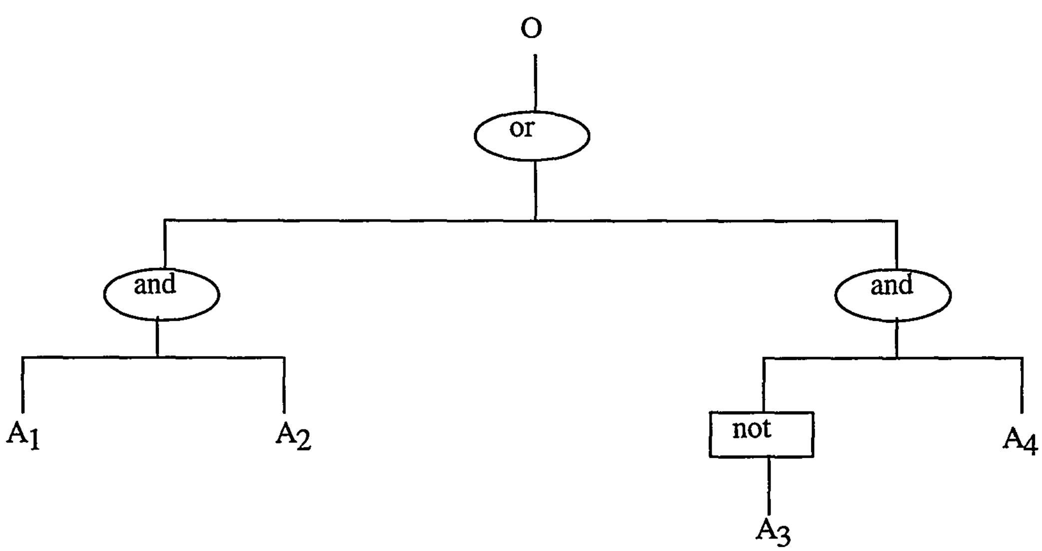


Figure 1. A fuzzy fault tree which maintains independence.

Unfortunately, many problems do not easily fit into a straightforward tree structure, with each node having only one output. In our investigations, certain factors (such as terrorism risk) influence many different events, so that construction of independent trees is problematic. As we will see in the next section, other problems also occur.

Completeness

The representation of some sets can be rearranged to allow use of Equations 1 through 3. For example, in Equation 8, since $A_i \cup A_j$ is not (necessarily) independent of $A_i \cup A_k$, we could simply define

$$\underline{P}_{(Ai \cup Aj) \cap (Ai \cup Ak)} = \underline{P}_{Ai \cup (Aj \cap Ak)} \cdot \qquad \text{(Eq. 9)}$$

Now A_j and A_k are independent so we can correctly calculate $\underline{P}_{Aj \cap Ak}$ using equation 2. Since A_i is independent of $A_j \cap A_k$, we can apply Equation 1 to calculate $\underline{P}_{Ai \cup (Aj \cap Ak)}$. Unfortunately, unraveling such relationships can be very difficult in complex models. Of greater concern is the fact that not all possible fuzzy probabilities can be calculated by rearranging them into a calculation that maintains independence.

For example, a listing of all possible independent calculations easily shows that $(A_i' \cap A_j) \cup (A_i \cap A_j')$ may not be rearranged to allow calculation by independence formulas. Consider two independent system components numbered i and j. If event A_i indicates that i is operational and A_j indicates that j is operational, then $\underline{P}_{(Ai' \cap Aj) \cup (Ai \cap Aj')}$ is the fuzzy probability that exactly one of the two components is operational. The inability of Equations 1 through 3 to calculate such probabilities is a serious limitation in reliability applications.

This limitation is illustrated by the example we use in this paper. Consider the three-stage manufacturing process shown in Figure 2. This diagram shows the flow of an industrial process through three stages. Stage 1 may be performed by two redundant units, each with a throughput capacity of 0.5 items per second. If both units 1 and 2 are operational, stage 1 has a throughput capacity of 1 item per second. If only one of the two units is operational, the stage 1 throughput is 0.5 items per second. If neither unit 1 nor unit 2 is operational, the throughput capacity of stage 1 is 0. This viewpoint may be used to build the throughput capacity of the entire process, with the capacity of stage 1 limiting the possible flow through stage 2, and so on. Let A_i be the event that unit i is operational. Assume the process has repairable (or replaceable) independent units, and that the process has been in operation long enough to approximately reach stationarity. Then $p_i = P(A_i)$ is the stationary readiness coefficient of unit i [9]. Letting T be the process throughput capacity, we can calculate the steady state distribution of T as

$$P(T=1) = P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6)$$

$$P(T=0.8) = P(A_1 \cap A_2 \cap ((A_3 \cap A_4 \cap A_5') \cup (A_3 \cap A_4' \cap A_5) \cup (A_3' \cap A_4 \cap A_5) \cap A_6), \qquad \text{(Eq. 10)}$$

and so on. Possible values of T are {0, 0.4, 0.5, 0.8, 1.0}. Calculation of the distribution of T follows easily when the stationary readiness coefficients are crisp; our goal is to study this process with fuzzy readiness coefficients. To calculate the fuzzy probability $\underline{P}_{T=0.5}$, we must calculate the fuzzy probability that exactly one of units 1 and 2 is functional. Unfortunately, as discussed in the proceeding paragraph, this fuzzy probability cannot be modeled using Equations 1 through 3. Several other "gaps" occur in the fuzzy reliability model of the system.

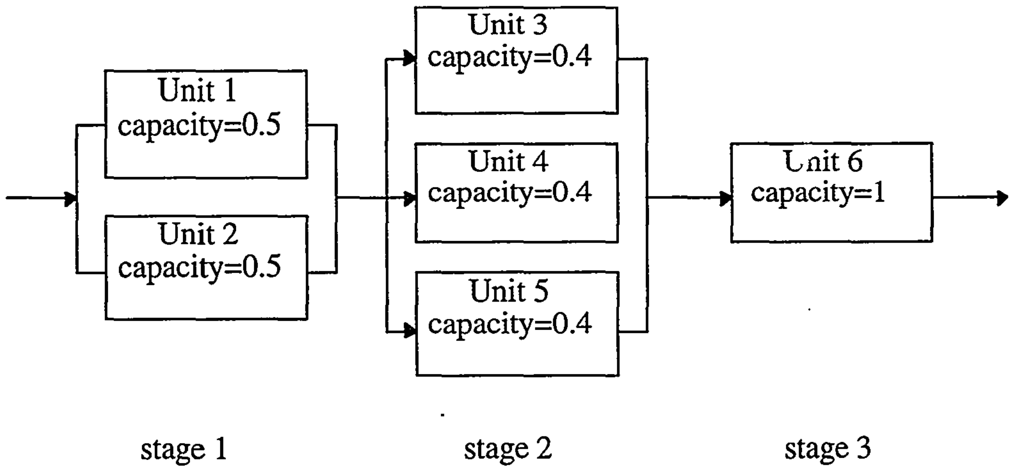


Figure 2. A three-stage industrial process.

Clearly, many important fuzzy probabilities cannot be reached by the standard independence formulas in equations 1-3. To understand what sets are missing, we should more carefully specify the probability space of interest in our reliability problem.

Definition: The sample space S_n based on n independent events $A_1, A_2, \ldots, A_n$ the set of 2^n distinct elements

$$S_n=\{s_1,s_2, \ldots, s_{2^n}\}$$

of the form

$$s=\cap_{i=1,n}B_i \quad \text{with} \quad B_i=A_i \quad \text{or} \quad B_i=A_i'.$$

For the remainder of this paper, the notation A_i will be used to indicate the independent events from which S_n is defined.

Note that S_n has a finite number of elements, so our sample space is discrete. A fuzzy probability theory, in keeping with both our needs and the structure of crisp probability theory for discrete sample spaces, should assign a probability for every subset of S_n.

Definition: A fuzzy probability theory is called complete if it assigns a fuzzy probability to every subset of S_n.

Consider a set subset B of S_n, which can be constructed through independent operations. For an event B, which can be organized as an independent calculation, we define $\underline{B}$ as the fuzzy probability theory resulting from repeated application of Equations 1 through 3.

Zadeh's Linguistic Probabilities and Consistency

Now we must build the definition of fuzzy probability for subsets of S_n from the given fuzzy probabilities $\underline{P}_{A1}$ $\underline{P}_{A2}$, ..., $\underline{P}_{An}$. Following Zadeh [3], we can define an extension. Consider a proper subset B of a sample space $S_n=\{s_1,s_2,...,s_{2^n}\}$, with $B=\{t_1, t_2, ..., t_k\}$ where t_i are the elements in S_n which are in B. We define, using a superscript Z to indicate Zadeh's extension,

$$\underline{P}_B{}^Z(y)=\sup_{y=x1+x2+...+xk;\ x1+x2+...+xk\leq 1} \min[\underline{P}_{\{t1\}}(x_1),\underline{P}_{\{t2\}}(x_2),...,\underline{P}_{\{tk\}}(x_k)]\ . \qquad \text{(Eq. 11)}$$

The inequality in the sup is a result of the interactivity of crisp probabilities, since

$$\Sigma_{i=1,2^n} P(\{s_i\}) = 1.$$

Each $\underline{P}_{si}(.)$ is calculated from $\underline{P}_{A1}$ $\underline{P}_{A2}$, ..., $\underline{P}_{An}$ using independence and Equations 1 through 3. This formulation does provide a fuzzy probability for every subset of S_n. Unfortunately, Equations 11 and 12 are not consistent with the calculations in Equations 1 through 3 [1].

A Complete and Consistent Formulation of Independent Fuzzy Probabilities

As an alternative to Zadeh's approach, we consider a different extension. Consider a reliability model built in terms of the independent fuzzy probabilities $\underline{P}_{Ai}$, i=1,2, ..., n, for sample space S_n. Using, for crisp probabilities, the definition $p_i=P(A_i)$, we see, for subset B of S_n, that

$$P(B)= P(\cup_{si\in B} \{s_i\})= \Sigma_{si\in B} P(\{s_i\})= f_B(p_1,p_2,....,p_n) \qquad \text{(Eq. 12)}$$

for a function $f_B(.)$. Thus the crisp probability of every B can be written uniquely as a function $f_B(.)$ in terms of p_1, p_2, ..., p_n. For the empty set ϕ we have $f_\phi(p_1,p_2,....,p_n)=0$ and for the sample space we have $f_{Sn}(p_1,p_2,....,p_n)=1$. We use these functions to build our extension of Equations 1 through 3. We can now define our extension for B.

Definition: For subset B of S_n, the extension of independent fuzzy probabilities is

$$\underline{P}_B{}^E(y)=\sup_{y= fB(p1,p2,....,pn)} \min(\underline{P}_{A1}(p_1),\underline{P}_{A2}(p_2),...,\underline{P}_{An}(p_n)]$$

with $P(B)=f_B(p_1,p_2,...,p_n)$ when $P(A_i)= p_i$. If, for a fixed y, the set $\{(p_1,p_2,...,p_n):y=f_B(p_1,p_2,...,p_n)\}$ is empty, we take $\underline{P}_B{}^E(y)=0$. The function $f_B(.)$ is defined in Equation 12.

The extension $\underline{P}_B{}^E$, when derived from independent fuzzy probabilities $\underline{P}_{Ai}$, is both consistent and complete. See [1] for a complete proof.

An Example

To demonstrate the technique, we consider the three-stage process discussed above and illustrated in Figure 2. To demonstrate the calculations, the event T=0.8 will be discussed. To simplify the illustration, all six independent units are assumed to have the fuzzy readiness coefficient shown in Figure 3. Note that

$$P(T=0.8)= f_{T=0.8}(p_1,...,p_6)=p_1p_2(p_3p_4(1-p_5)+p_3(1-p_4)p_5+(1-p_3)p_4p_5)p_6$$

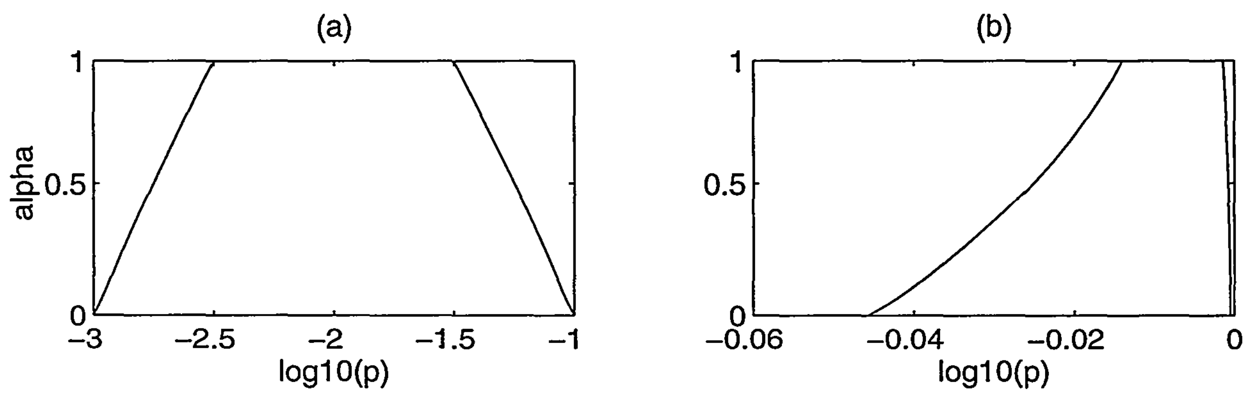
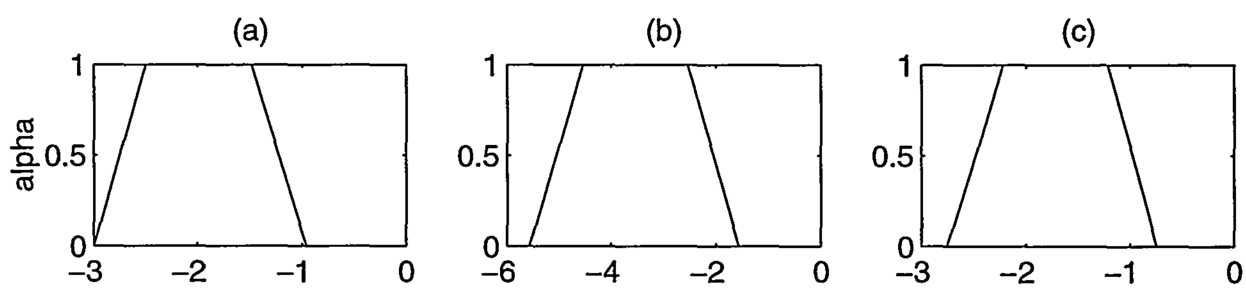


Figure 3. The fuzzy idleness coefficient (a) and readiness coefficient (b) for a single unit.

Figure 4 shows the resulting fuzzy probabilities for T=0, T=0.4, T=0.5, T=0.8, and T=1.0. These fuzzy probabilities describe the long-term performance of the industrial process.



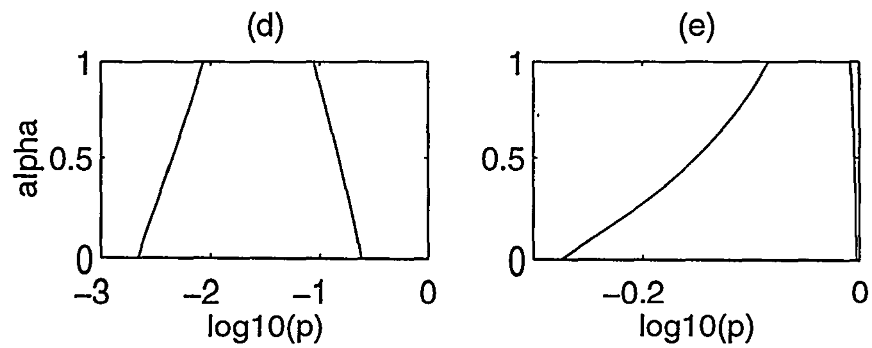


Figure 4. The resulting fuzzy probabilities for the process throughput T=0 (a), T=0.4 (b), T=0.5 (c), T=0.8 (d), and T=1 (e).

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Biography

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James Dunyak received a BS and MS in Engineering Mechanics from Virginia Tech in 1982 and 1987 respectively. From 1984 through 1992, he worked in systems engineering variously for Locus Inc, the Naval Research Laboratory, and Mitre Corporation. After completing his PhD in Applied Mathematics from the University of Maryland in 1994, he took a position in the Department of Mathematics at Texas Tech University. His research interests include random processes and their application to a wide variety of engineering and physics problems, fuzzy set theory as an alternate model of uncertainty, and neural networks.

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