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Generalization of the Schwartz-Soffer inequality for correlated random fields

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We investigate the influence of spatial correlations between the values of the random field on the critical behavior of random-field lattice models and derive a generalized version of the Schwartz-Soffer inequality for the averages of the susceptibility and its disconnected part. At the critical point this leads to a modification of the Schwartz-Soffer exponent inequality for the critical exponents η and $\bar{\eta}$ describing the divergences of the susceptibility and its disconnected part, respectively. It now reads $\bar{\eta}\leqslant 2\eta-2y$ where $2y$ describes the divergence of the random-field correlation function in Fourier space. As an example we exactly calculate the susceptibility and its disconnected part for the random-field spherical model. We find that in this case the inequalities actually occur as equalities.

Systems where the field conjugate to the order parameter is a random field are widely used in physics as prototypes of disordered systems. Typical examples are the random-field Ising model[1–3] of disordered magnetic systems and the Coulomb glass model[4–7] of disordered localized electrons. The critical behavior of such systems has been intensively studied since the pioneering work of Imry and Ma.[1] The theoretical investigation of critical properties is, however, very difficult and new results have been obtained only slowly. From perturbation theory Parisi and Sourlas[8] derived the "dimensional reduction," i.e., the critical exponents of the random-field Ising model in D dimensions are equal to that of the pure Ising model in $D-2$ dimensions. Later this was shown to be incorrect due to the very complicated energy landscape in the random-field Ising model (see, e.g., Refs. 2 and 3). Recently this behavior has been attributed to replica symmetry breaking.[9,10] While today there is a general agreement about the upper and lower critical dimensions of the different models, the critical behavior of random-field systems is still poorly understood. In particular the critical exponents and their relations are not known.

In 1985 Schwartz and Soffer[11] obtained an inequality relating the critical behavior of susceptibilities and correlation functions in a general random-field system. While the so-called disconnected part of the susceptibility [see Eq. (4) below] vanishes in systems without field, it does not vanish in random-field systems. Precisely at the critical point the disconnected part is expected to diverge even stronger than the susceptibility itself. A corresponding exponent inequality $\bar{\eta}\leqslant 2\eta$ relating the critical exponents η and $\bar{\eta}$ of the susceptibility and its disconnected part, respectively, was also derived by Schwartz and Soffer.[11] It has further been suggested[12] that both inequalities should actually occur as equalities. This is supported by numerical calculations for the random-field Ising model[13–16] and a rigorous result for the random-field spherical model.[17]

However, in almost all of these investigations the values of the random field are considered to be uncorrelated. This may be a reasonable approximation in the case of disordered magnetic systems where the random fields are generated by short-range exchange interactions. In other physical situations the spatial correlations between the random-field values are expected to play an important role such as in disordered electronic systems with localized states, where the random fields are generated by the long-range Coulomb interaction between the electrons and charged donor and acceptor atoms.

In this paper we therefore generalize the Schwartz-Soffer inequality to systems with spatially correlated random fields. We consider a system described by the Hamiltonian

$$H=H_0+\sum_i \varphi_i S_i . \tag{1}$$

Here H_0 is the nonrandom part of the Hamiltonian and S_i is the spin variable at site i. (For simplicity of notation we restrict the derivation to a single spin component.) The values φ_i of the field are random variables with a Gaussian probability distribution

$$W(\{\varphi_i\})=(2\pi)^{-N/2}(\det\mathbf{A})^{-1/2}\exp\left(-\frac{1}{2}\varphi^T\mathbf{A}^{-1}\varphi\right), \tag{2}$$

where φ is the vector of the random-field values, N is the number of sites in the system, and $\mathbf{A}$ is the symmetric covariance matrix with the elements A_{ij}, which we assume to be isotropic and translationally invariant, i.e., $A_{ij}=A(|r_i-r_j|)$.

We now derive the generalized Schwartz-Soffer inequality for the disorder averages of the susceptibility

$$\chi(\mathbf{q})=\langle S_\mathbf{q} S_{-\mathbf{q}}\rangle-\langle S_\mathbf{q}\rangle\langle S_{-\mathbf{q}}\rangle \tag{3}$$

and its disconnected part

$$\chi^{\text{dis}}(\mathbf{q})=\langle S_\mathbf{q}\rangle\langle S_{-\mathbf{q}}\rangle, \tag{4}$$

where $\mathbf{q}$ is the wave vector and $S_{\mathbf{q}}$ is the Fourier transform of the S_i. The symbol $\langle\ \rangle$ denotes the thermodynamic average. The disorder-averaged susceptibility is given by

$$[\chi(\mathbf{q})]_\varphi = -\frac{1}{\beta}\left[\frac{\partial\langle S_{\mathbf{q}}\rangle}{\partial\varphi_{\mathbf{q}}}\right]_\varphi = -\frac{1}{\beta}\int\prod_{\mathbf{k}} d\varphi_{\mathbf{k}} P(\{\varphi_{\mathbf{k}}\})\frac{\partial\langle S_{\mathbf{q}}\rangle}{\partial\varphi_{\mathbf{q}}}. \qquad (5)$$

Here $\varphi_{\mathbf{q}}$ is the Fourier transform of the random field, β is the inverse temperature, and $[\ ]_\varphi$ denotes the disorder average. After a partial integration we get

$$[\chi(\mathbf{q})]_\varphi = -\frac{1}{\beta A_{\mathbf{q}}}\int\prod_{\mathbf{k}} d\varphi_{\mathbf{k}} P(\{\varphi_{\mathbf{k}}\})\langle S_{\mathbf{q}}\rangle\varphi_{-\mathbf{q}}, \qquad (6)$$

where $A_{\mathbf{q}}$ is the Fourier transform of the covariance matrix. The integral on the right-hand side has the form of a scalar product and consequently the general Schwartz inequality can be applied to it:

$$[\chi(\mathbf{q})]_\varphi \leq \frac{1}{\beta A_{\mathbf{q}}}([\langle S_{\mathbf{q}}\rangle\langle S_{-\mathbf{q}}\rangle]_\varphi[\varphi_{\mathbf{q}}\varphi_{-\mathbf{q}}]_\varphi)^{1/2}. \qquad (7)$$

Since $[\varphi_{\mathbf{q}}\varphi_{-\mathbf{q}}]_\varphi = A_{\mathbf{q}}$ this yields the generalized Schwartz-Soffer inequality

$$\beta^2 A_{\mathbf{q}}[\chi(\mathbf{q})]_\varphi^2 \leq [\chi^{\mathrm{dis}}(\mathbf{q})]_\varphi. \qquad (8)$$

It differs from the original inequality[11] by the occurrence of the $\mathbf{q}$-dependent random field correlation function $A_{\mathbf{q}}$ on the left-hand side. We note that the inequality (8) actually occurs as equality, if $\langle S_{\mathbf{q}}\rangle \propto \varphi_{-\mathbf{q}}$, since in this case the Schwartz inequality for the scalar product is an equality.

Precisely at the critical point of the ferromagnetic transition the susceptibility and its disconnected part are expected to diverge for $\mathbf{q}\to 0$ as

$$[\chi(\mathbf{q})]_\varphi \sim |\mathbf{q}|^{-2+\eta} \qquad (9)$$

and

$$[\chi^{\mathrm{dis}}(\mathbf{q})]_\varphi \sim |\mathbf{q}|^{-4+\bar{\eta}}, \qquad (10)$$

which defines the critical exponents η and $\bar{\eta}$. We further assume that $A_{\mathbf{q}}$ diverges as

$$A_{\mathbf{q}} \sim |\mathbf{q}|^{-2y} \qquad (11)$$

for $\mathbf{q}\to 0$. Here $y=0$ corresponds to uncorrelated random fields, $y>0$ describes long-range correlations in real space, whereas $y<0$ corresponds to short-range oscillatory correlations in real space. Applied at the critical point the generalized Schwartz-Soffer inequality (8) implies an exponent inequality

$$\bar{\eta} \leq 2\eta - 2y. \qquad (12)$$

Consequently, when analyzing experimental results for the exponents η and $\bar{\eta}$ random field correlations have to be taken into account. We note in passing that (12) also influences the hyperscaling relations, since in a random field system the spatial dimension D is expected to be replaced by $D-(2-\bar{\eta}+\eta)$ (see, e.g., Ref. 2).

In the second part of this paper we will calculate the susceptibility and its disconnected part for the random-field spherical model with correlated random fields and compare the results to that of the generalized Schwartz-Soffer inequality (8). The spherical model was first proposed[18] as an approximation to the nearest-neighbor Ising model (for a review see Ref. 19). The effect of uncorrelated[17,20–23] and correlated[24,25] random fields on the critical behavior of the spherical model was studied by several authors. The advantage of the spherical model is that the evaluation of the free energy is much easier than in the corresponding Ising model since the trace over the dynamic variables becomes a multiple Gaussian integral instead of a multiple sum in the case of Ising variables. This renders the model exactly solvable even in the case of long-range correlated random fields.

We consider a model if $N=L^D$ dynamic variables S_i located on the sites $\mathbf{r}_i$ of a regular hypercubic D-dimensional lattice. The variables S_i interact via a translationally invariant pair interaction $U_{ij}=U(|\mathbf{r}_i-\mathbf{r}_j|)$. The Hamiltonian is given by

$$H=\sum_i \varphi_i S_i + \tfrac{1}{2}\sum_{i,j} U_{ij}S_iS_j. \qquad (13)$$

(We define $U_{ii}=0$ so that the terms with $i=j$ do not have to be explicitly excluded.) The dynamic variables are continuous real variables ranging from $-\infty$ to ∞. To make the model well defined and to ensure a finite ground-state energy, the spherical constraint

$$\sum_i S_i^2 = N/4 \qquad (14)$$

is imposed on the values of the variables. Our choice of the numerical value $N/4$ of this constant is influenced by the analogy of the model to the Coulomb glass model[4] and the respective analogy of the dynamic variables S_i and the occupation numbers $n_i=\pm\frac{1}{2}$. The distribution of the random field φ_i is given by (2).

In order to investigate the susceptibility and its disconnected part, we first have to calculate the partition function Z for a fixed realization of the random field which reads

$$Z=\prod_i\left(\int dS_i\right)\delta\left(\frac{N}{4}-\sum_i S_i^2\right) \times\exp\left(-\frac{\beta}{2}\sum_{i,j} U_{ij}S_iS_j - \beta\sum_i \varphi_i S_i\right). \qquad (15)$$

This can be done by means of the saddle-point method (see Ref. 18 or 19 for details). We obtain

$$Z=\pi^{N/2}[2\pi N|\Psi''(z_0)|]^{-1/2}e^{N\Psi(z_0)}, \qquad (16)$$

where Ψ'' denotes the second derivative of

$$\Psi(z)=\frac{z}{4}-\frac{1}{2N}\mathrm{Tr}\ \ln\mathbf{V}+\frac{\beta^2}{4N}\varphi\mathbf{V}^{-1}\varphi \qquad (17)$$

with respect to z. The matrix $\mathbf{V}$ comprises the matrix elements $V_{ij}=z\delta_{ij}+\beta U_{ij}/2$. The saddle-point equation

$$0=\frac{1}{4}-\frac{1}{2N}\mathrm{Tr}\,\mathbf{V}^{-1}-\frac{\beta^2}{4N}\,\varphi\mathbf{V}^{-2}\varphi \tag{18}$$

determines z_0. The thermodynamic averages $\langle S_{\mathbf{q}}\rangle$ and $\langle S_{\mathbf{q}}^2\rangle$ are given by

$$\langle S_{\mathbf{q}}^m\rangle=\frac{1}{Z}\prod_i\left(\int dS_i\right)\delta\left(\frac{N}{4}-\sum_i S_i^2\right)S_{\mathbf{q}}^m \times\exp\left(-\frac{\beta}{2}\sum_{i,j}U_{ij}S_iS_j-\beta\sum_i\varphi_iS_i\right). \tag{19}$$

We note that we use a real version of the Fourier transformation which is appropriate for the spherical model[18] so that $S_{\mathbf{q}}$ and $\varphi_{\mathbf{q}}$ are real variables. The calculation of $\langle S_{\mathbf{q}}\rangle$ and $\langle S_{\mathbf{q}}^2\rangle$ is completely analogous to the calculation of the partition function. After having carried out the saddle-point integration almost all terms in the numerator and denominator cancel and we obtain

$$\langle S_{\mathbf{q}}\rangle=-\frac{\beta\varphi_{\mathbf{q}}}{2\lambda_{\mathbf{q}}} \tag{20}$$

and

$$\langle S_{\mathbf{q}}^2\rangle=\frac{1}{2\lambda_{\mathbf{q}}}+\frac{\beta^2\varphi_{\mathbf{q}}^2}{4\lambda_{\mathbf{q}}^2}, \tag{21}$$

with

$$\lambda_{\mathbf{q}}=z_0+\beta U_{\mathbf{q}}/2, \tag{22}$$

where $U_{\mathbf{q}}$ is the Fourier transform of the interaction. Equation (20) establishes a linear relation between the thermal averages of the variables $S_{\mathbf{q}}$ and the random field. Thus according to the discussion after (8) we expect the generalized Schwartz-Soffer inequality to be fulfilled as an equality. Indeed, if we insert (20) and (21) into (8), we obtain

$$\beta^2A_{\mathbf{q}}[\chi(\mathbf{q})]_\varphi^2=[\chi^{\mathrm{dis}}(\mathbf{q})]_\varphi. \tag{23}$$

By comparing (20) and (21) with the definitions (9) and (10) one can directly calculate the critical exponents η and $\bar{\eta}$. If we assume $\lambda_{\mathbf{q}}$ to behave as $\lambda_{\mathbf{q}}\sim|\mathbf{q}|^x$ for $|\mathbf{q}|\rightarrow 0$ ($x=2$ corresponds to short-range interactions), η and $\bar{\eta}$ are given by

$$\eta=2-x, \qquad \bar{\eta}=4-2x-2y. \tag{24}$$

Consequently the exponent inequality also occurs as equality. Of course, this follows directly from (23) as well.

In conclusion, we have studied the behavior of random-field lattice models with spatially correlated random fields. We have derived a generalized Schwartz-Soffer inequality for the susceptibility and its disconnected part. The correlations between the random-field values yield a modification of the exponent inequality between the critical exponents η and $\bar{\eta}$. It now reads $\bar{\eta}\leq 2\eta-2y$, where $2y$ describes the divergence of the random field correlations in Fourier space. We have exactly calculated the susceptibility and its disconnected part for the random field spherical model with correlated fields and found that for this model the generalized Schwartz-Soffer inequality and the corresponding exponent inequality actually occur as equalities. In the light of this result and that of the calculations for uncorrelated fields[13–17] we expect both inequalities to hold as equalities in general. As for uncorrelated random fields a rigorous proof remains, however, a task for the future.

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