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Local Versus Nonlocal Order-Parameter Field Theories for Quantum Phase Transitions

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General conditions are formulated that allow us to determine which quantum phase transitions in itinerant electron systems can be described by a local Landau-Ginzburg-Wilson (LGW) theory solely in terms of the order parameter. A crucial question is the degree to which the order parameter fluctuations couple to other soft modes. Three general classes of zero-wave-number order parameters, in the particle-hole spin-singlet and spin-triplet channels and in the particle-particle channel, respectively, are considered. It is shown that the particle-hole spin-singlet class does allow for a local LGW theory, while the other two classes do not. The implications of this result for the critical behavior at various quantum phase transitions are discussed, as is the connection with nonanalyticities in the wave-number dependence of order-parameter susceptibilities in the disordered phase.

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I. INTRODUCTION

Much interest in the field of quantum many-body physics has recently focused on quantum phase transitions.[1–3] These are phase transitions that occur at zero temperature as a function of some nonthermal control parameter, often pressure or composition, and are driven by quantum fluctuations as opposed to thermal ones. In addition to being of fundamental interest, quantum phase transitions are important because they are believed to underlie a number of interesting low-temperature phenomena, in particular various forms of exotic superconductivity.[4,5]

Hertz, building on earlier work, has given a general scheme for the theoretical treatment of quantum phase transitions.[1] After identifying the order-parameter of interest, one performs a Hubbard-Stratonovich decoupling of the interaction term responsible for the ordering, with the order parameter field as the Hubbard-Stratonovich field. One then integrates out the fermions to obtain a field theory entirely in terms of the order parameter. This Landau-Ginzburg-Wilson (LGW) theory can then be analyzed by means of renormalization-group techniques. To the extent that the LGW theory is well behaved, the resulting critical behavior in three dimensions (3D) is often mean-field-like, since the quantum phase transition is related to the corresponding classical transition in a higher dimension. Until recently, it therefore was believed that most quantum phase transitions are not interesting from a critical phenomena point of view.

In recent years it has become clear that in general there are problems with Hertz's scheme. In particular, for one of the most obvious examples of a quantum phase transition, viz., the zero-temperature transition in itinerant ferromagnets, it was shown that Hertz's method does not lead to a local quantum field theory.[6,7] This is because there are soft modes other than the order-parameter fluctuations, specifically, soft particle-hole excitations other than the spin-density fluctuations, that couple to the order parameter. Since these "additional" soft modes are integrated out in deriving the LGW functional, the resulting field theory has vertices that are not finite in the limit of vanishing wave numbers and frequencies. Such nonlocal field theories are hard to analyze, and unsuitable for explicit calculations. A better, albeit more involved, strategy in such cases is to integrate out the massive modes only, keep all of the soft modes on equal footing, and derive a coupled field theory for the latter. Such a procedure for the quantum ferromagnetic transition in the presence of quenched disorder has recently revealed that the critical behavior is not mean-field-like as suggested by Hertz and not even given by a simple Gaussian fixed point as proposed on the basis of the nonlocal LGW theory[6] but rather given by the power laws of the simple Gaussian fixed point with complicated multiplicative logarithmic corrections to scaling.[8] In clean itinerant ferromagnets, Hertz theory also breaks down[7] and the quantum phase transition is generically of first order.[9]

This example of a breakdown of the order-parameter field theory method casts serious doubt on the very concept of LGW theory for quantum phase transitions.[10] However, an equally prominent example for which the LGW concept works, at least in 3D,[11] is given by the quantum antiferromagnetic transition, which was also discussed by Hertz.[1] This raises the following question: Which quantum phase transitions in itinerant electron systems can be described by a local order-parameter field theory, and which require a more complicated analysis in terms of a coupled field theory? The quantum ferromagnetic and antiferromagnetic transitions, respectively, in 3D provide examples for each of these two categories.

It is the purpose of the present paper to provide a partial answer to this question. Specifically, for quantum phase transitions with zero-wave number order parameters we will provide a classification scheme that shows for which transitions Hertz's scheme works and for which it does not. The paper is organized as follows. In Sec. II we consider some very general arguments to develop a criterion for the breakdown of LGW theory. In particular, we discuss a relation between the analytic properties of the order parameter susceptibility in the disordered phase and the applicability of LGW theory. In Sec. III we first formulate a general field theory for interacting fermions to which this criterion can be applied. We then consider three classes of zero-wave-number order-parameter fields, one each in the particle-hole spin-singlet and spin-triplet channels, and one in the particle-particle channel. We show that for the first class, LGW theory works, while for the other two it does not. In the last part of this section we show that these results are consistent with explicit calculations. In Sec. IV we discuss these results. We discuss the generality of the model we consider, the coupling between the order-parameter fluctuations and other soft modes, and questions regarding the coupling of statics and dynamics in quantum statistical mechanics. We also discuss the effect of disorder on our results. The paper closes with a series of final remarks. Perturbative results for some susceptibilities relevant to our dicussion are summarized the in Appendix.

II. A CRITERION FOR THE BREAKDOWN OF LOCAL LGW THEORY

Of the two examples mentioned in the Introduction, the quantum ferromagnetic and antiferromagnetic transitions of itinerant electrons, the ferromagnetic one has a zero-wave-number, or homogeneous, order parameter, while the antiferromagnetic one has a nonhomogeneous order parameter. The impossibility of constructing a local order-parameter field theory for the former is due to the coupling of fermionic soft modes to the order-parameter fluctuations, all of which are soft at the same (zero) wave number. For clarity we stress that these fermionic modes are soft modes in addition to the critical order-parameter fluctuations and to any Goldstone modes in the ordered phase. The coupling of these additional soft modes to the antiferromagnetic order parameter, which is an object at nonzero wave number, is much weaker and hence does not spoil the LGW concept in this case.

The above considerations imply that the existence of a local LGW theory will generically be much more questionable for quantum phase transitions with homogeneous, or zero-wave-number, order parameters, than for nonhomogeneous ones. In this paper we will therefore investigate the case of quantum phase transitions with homogeneous order parameters. The crucial question is then whether or not the coupling of the additional, fermionic, soft modes to the order-parameter fluctuations is strong enough to destroy the local nature of the LGW theory.

To sharpen this question, and put it in mathematical terms, we note that in Hertz's scheme the vertices of the LGW theory are given by order-parameter correlation functions in a reference ensemble that consist of the full action minus an interaction term that has been decoupled by a Hubbard-Stratonovich transformation (see Ref. 1 and Sec. III B below). Since the interaction term one chooses for the decoupling is the one that causes the phase transition, the reference system is always in the disordered phase. The two-point, or ordinary, order-parameter susceptibility $\chi^{(2)}\equiv\chi$ in the reference ensemble determines the Gaussian vertex, and the higher-order correlation functions $\chi^{(n)}$ $(n>2)$ give the higher vertices. The locality, or otherwise, of the LGW theory then depends on the properties of the $\chi^{(n)}$. More precisely, since the square-gradient term in the LGW functional comes from the wave number expansion of χ, and the coefficients of the important (i.e., quadratic and cubic) terms in the equation of state are determined by the zero-wave-number and zero-frequency limits of $\chi^{(3)}$ and $\chi^{(4)}$, it follows that in order for the local LGW approach to break down, the wave-vector-dependent susceptibility $\chi(\mathbf{q})$ must not be an analytic function of $|\mathbf{q}|$ at $\mathbf{q}=\mathbf{0}$, or $\chi^{(n)}$ $(n=3,4)$ must diverge as $\mathbf{q}\rightarrow\mathbf{0}$. Such nonanalyticities can only arise from infrared singularities, i.e., from fermionic soft modes. As has been discussed in Ref. 28, these soft modes exist for various symmetry reasons, and if the symmetry responsible for a particular soft mode is broken, the mode will acquire a mass that depends on the symmetry-breaking parameter. For reasons that will become clear below, let us suppose that the field H conjugate to the order parameter for the quantum phase transition, i.e., a source for the order-parameter field, breaks the above symmetry. Then the most general form of the the singular part of χ that is consistent with power-law scaling is form[12,13]

$$\chi_{\mathrm{sing}}(\mathbf{q},H)=(|\mathbf{q}|^{x}+|H|)^{y}. \tag{2.1}$$

Here $y,x>0$ are exponents that determine the nature of the singularity and the scaling of H with the wave number, respectively.[14] For example, for clean and disordered itinerant quantum ferromagnets in d spatial dimensions, one has $x=1$, $y=d-1$, and $x=2$, $y=(d-2)/2$, respectively. [In the clean case in 3D, $y=2$ should be interpreted as $(|\mathbf{q}|+|H|)^{2}\ln(|\mathbf{q}|+|H|)$; see Eq. (3.5) and the discussion below it.] The higher susceptibilities $\chi^{(n)}$ can be obtained from χ by differentiating $(n-2)$ times with respect to H. This leads to infinite zero-field, zero wavenumber $\chi^{(n)}$ for all $n>y+2$, and hence to a nonlocal field theory. With these considerations we can now state and justify a criterion for quantum phase transitions in itinerant electron systems that we will proceed to discuss and apply in the following sections of this paper.

Criterion. Hertz theory (a local LGW theory) breaks down if a quantum phase transition has a homogeneous, or zero-wave-number, order-parameter, and if a source term H for the order-parameter field changes the soft-mode spectrum of the fermionic or reference ensemble part of the action.

The validity of this criterion follows from our above discussion. We first note that interacting electronic systems are intrinsically nonlinear, and therefore generically all soft modes couple to all the physical quantities. Therefore, if some soft modes are given a mass by $H>0$, i.e., are of the form $1/(|\mathbf{q}|^x+H)$, then the free energy and thus all of the order-parameter susceptibilities will be nonanalytic functions of H. This is only possible if the two-point order-parameter susceptibility in the reference ensemble has the form given in Eq. (2.1). It then follows that at $H=0$ all order-parameter susceptibilities are singular functions of the wave number. That is, a local LGW theory will not be possible.

Two remarks might be helpful at this point: (1) The second condition in the above criterion ensures that there is a sufficiently strong coupling between the fermionic soft modes and the order-parameter fluctuations to invalidate the local LGW approach. (2) Instead of referring to the soft-mode spectrum of the reference ensemble, one could also demand that a nonzero H changes the soft-mode spectrum of the full action in the disordered phase away from the critical point. Since the order-parameter fluctuations, which are taken out of the reference ensemble, are massive in this region, these two requirements are equivalent.

To conclude this section we note that the analytical properties of various susceptibilities in interacting electronic systems have been examined in perturbation theory at $H=0$. The spin susceptibility is known to be a nonanalytic function of the wave number at second order in the screened interaction,[15] consistent with the known breakdown of Hertz theory for the itinerant quantum ferromagnet.[6,7] However, the number-density, number-density current, and number-density stress susceptibilities have all been shown to not have a nonanalytic wave-number dependence to that order; see the Appendix. This suggests that for quantum phase transitions with these observables as order parameters, Hertz theory might work. While in principle it is possible that nonanalyticities would appear in these correlation functions at higher order in the perturbation theory, at least in the case of the number-density current susceptibility this is very unlikely, for reasons explained in Sec. III D. Also, in order to ruin the local LGW theory, any nonanalyticity would have to be cut off by the appropriate external field. In the case of the number density, where the conjugate field is just the chemical potential, this is hard to imagine. In any case, the complexity of perturbation theory makes it impractical to go beyond second order, and a more powerful approach is needed to determine which quantum phase transitions can be described by local order-parameter field theories.

In the remainder of this paper we develop a classification scheme for quantum phase transitions with homogeneous order parameters that is based on the above criterion. We will show that there are classes of quantum phase transitions for which a local LGW functional exist, i.e., for which Hertz theory is valid, and classes for which it breaks down.

III. CLASSIFICATION SCHEME FOR QUANTUM PHASE TRANSITIONS WITH HOMOGENEOUS ORDER PARAMETERS

A. Fermionic field theory

Our starting point is a general action for itinerant, interacting electrons,[16]

$$S=-\int dx\,\bar{\psi}(x)[\partial_\tau+\epsilon(\partial_{\mathbf{x}})-\mu]\psi(x)+S_{\rm int}. \tag{3.1a}$$

Here $\bar{\psi}(x)\equiv(\bar{\psi}_\uparrow(x),\bar{\psi}_\downarrow(x))$ and $\psi(x)\equiv(\psi_\uparrow(x),\psi_\downarrow(x))$ are fermionic (i.e., Grassmann-valued) two-component spinor fields, and the index $x\equiv(\mathbf{x},\tau)$ comprises the real-space position $\mathbf{x}$ and the imaginary time τ. $\int dx\equiv\int d\mathbf{x}\int_0^\beta d\tau$ with $\beta=1/k_BT$, and the product of $\bar{\psi}$ and ψ is understood as a scalar product in spinor space that accomplishes the summation over the two spin projections. μ is the chemical potential, and ϵ denotes the dispersion relation. For instance, for free electrons one has $\epsilon(\partial_{\mathbf{x}})=-\partial_{\mathbf{x}}^2/2m$ with m the free electron mass.

$S_{\rm int}$ describes the electron-electron interaction, which we will keep general. At the most basic level, $S_{\rm int}$ is the Coulomb interaction, but often one starts at the level of an effective theory, where some degrees of freedom have already been integrated out to create short-ranged, effective interactions between various modes. For our purposes, we only assume that $S_{\rm int}$ contains an interaction between the order parameter modes. If we denote the order parameter, in terms of the fermionic fields by $n(\bar{\psi},\psi)$, this part of $S_{\rm int}$ reads schematically

$$S_{\rm int}^{\rm OP}=J\int dx\,n^2(x), \tag{3.1b}$$

with J an appropriate coupling constant. Notice that in general n will be a tensor, so the notation n^2 in Eq. (3.1b) is symbolic. For conceptual simplicity's sake, we also assume that the interacting system in the disordered phase has a Fermi liquid ground state, i.e., that the interactions are not sufficiently singular to destroy the Fermi liquid. We will come back to this assumption in the Discussion, Sec. IV.

Finally, we write

$$S=S_0+S_{\rm int}^{\rm OP}, \tag{3.1c}$$

with S_0 containing all pieces of the action other than $S_{\rm int}^{\rm OP}$. S_0 describes the reference ensemble that was alluded to in Sec. II. We note that, although the order-parameter interaction term is missing from the bare reference ensemble action S_0, such an interaction is in general generated in perturbation theory, albeit with a coupling constant that is smaller than the critical value necessary for a nonvanishing expectation value of the order parameter. The correlation functions of the reference ensemble thus are those of the full system, with action S, in the disordered phase.

B. Order-parameter field theory

Here we briefly review the derivation of the order parameter or LGW theory.[1] Let us decouple the order-parameter interaction $S_{\rm int}^{\rm OP}$, Eq. (3.1b), by means of a Hubbard-Stratonovich field M. That is, we write the partition function

$$Z=\int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]}$$
$$=\text{const}\int D[M]\exp\left(-J\int dx M^2(x)\right)$$
$$\times\left\langle \exp\left(-2J\int dx M(x)n(x)\right)\right\rangle_0 ,$$
$$\equiv\text{const}\times\int D[M]e^{-\Phi[M]}, \tag{3.2}$$

where $\langle\cdots\rangle_0$ denotes an average with the reference ensemble action S_0, and $\Phi[M]$ is the LGW functional. The latter reads explicitly

$$\Phi[M]=J\int dx M^2(x)-\ln\left\langle \exp\left(-2J\int dx M(x)n(x)\right)\right\rangle_0 \tag{3.3}$$

and can be expanded in powers of M,

$$\Phi[M]=\frac{1}{2}\int dx_1 dx_2 M(x_1)\left[\frac{1}{J}\delta(x_1-x_2)-\chi^{(2)}(x_1-x_2)\right]$$
$$\times M(x_2)+\frac{1}{3!}\int dx_1 dx_2 dx_3 \chi^{(3)}(x_1,x_2,x_3)$$
$$\times M(x_1)M(x_2)\,M(x_3)+O(M^4), \tag{3.4a}$$

where we have scaled M with $1/\sqrt{2}J$. The coefficients $\chi^{(l)}$ in the Landau expansion, Eq. (3.4a), are connected l-point correlation functions of $n(x)$ in the reference ensemble,

$$\chi^{(l)}(x_1,\ldots,x_l)=\langle n(x_1)\cdots n(x_l)\rangle_0^c . \tag{3.4b}$$

A crucial question now arises concerning the behavior of these correlation functions in the limit of long distances and times or small frequencies and wave numbers. If their Fourier transforms are finite in that limit, then the LGW functional Φ is local, and Hertz's analysis of the quantum phase transition applies. However, this is not the case if the fermionic soft modes that have been integrated out in the above procedure couple sufficiently strongly to the order-parameter field. A prominent example is the case of a ferromagnetic order-parameter, where $\chi^{(2)}$ is a nonanalytic function of the wave number,[15]

$$\chi^{(2)}(\mathbf{q}\to\mathbf{0},\omega=0)\propto\text{const}+|\mathbf{q}|^{d-1}+O(\mathbf{q}^2). \tag{3.5}$$

The integer exponents in $d=1$ and $d=3$ are to be interpreted as $\ln(1/|\mathbf{q}|)$ and $\mathbf{q}^2\ln(1/|\mathbf{q}|)$, respectively. Higher-order correlation functions, starting with $\chi^{(4)}$, diverge in this limit as $\chi^{(n)}\sim|\mathbf{q}|^{-2(n-1)+d}$.[7] A nonzero expectation value $\langle M\rangle\neq0$ in the ordered phase, or an external field H conjugate to M in the disordered phase, cuts off these nonanalyticities by giving a mass to some soft modes (to spin-triplet particle-hole excitations in the ferromagnetic example). This leads to the free energy being a nonanalytic function of M or H.[7,9] The above discussion is a more technical rerendering of the criterion given in Sec. II. In what follows, we will use this criterion as a diagnostic tool.

C. Sources and symmetry considerations

We now add source or external field terms to our action, and ask whether they change the soft-mode structure of the system. We will consider three explicit examples, corresponding to two classes of homogeneous spin-singlet and spin-triplet order parameters in the particle-hole channel, respectively, and a third class of particle-particle channel order parameters.

1. Particle-hole channel spin-singlet order parameters

Consider a source term for a class of homogeneous, spin-singlet particle-hole order parameter fields,

$$S_H=H\int dx\,\bar{\psi}(x)f(\partial_{\mathbf{x}})\psi(x), \tag{3.6}$$

with f an arbitrary polynomial function of the gradient operator. Obviously, this source term has the same structure as the most general dispersion term in the band electron part of the action S_0, the first term on the right-hand side of Eq. (3.1a). Depending on the actual structure of $\epsilon(\partial_{\mathbf{x}})$, S_H may or may not break a spatial symmetry of S. However, it does not change the soft-mode spectrum since any electronic action with this structure describes a Fermi liquid, and the soft modes of all Fermi liquids are in a one-to-one correspondence with one another. The free energy is therefore an analytic function of H. It follows, from the criterion in Sec. II, that for any quantum phase transition with an order parameter of the form

$$n(x)=\bar{\psi}(x)f(\partial_{\mathbf{x}})\psi(x), \tag{3.7}$$

Hertz theory works, and the quantum critical behavior in 3D is in general mean-field-like. From the discussion in connection with Eq. (2.1) it further follows that the order-parameter susceptibility in the disordered phase cannot have a nonanalytic wave-number dependence that is cut off by H.[17]

An example of an order parameter in this class is the one for the isotropic-to-nematic phase transition that has been proposed to occur in quantum Hall systems by Oganesyan *et al.*[18] For this phase transition Hertz theory works, and the mean-field critical behavior determined in Ref. 18 is the exact quantum critical behavior. Consistent with this, a perturbative calculation of the stress susceptibility χ_{xy} defined in the Appendix, which plays the role of the reference ensemble susceptibility for the isotropic-to-nematic transition, found no nonanalytic wave-number dependence.

2. Particle-hole channel spin-triplet order parameters

Now consider a class of source terms analogous to Eq. (3.6), but in the spin-triplet channel,

$$S_{\vec{H}}=\vec{H}\cdot\int dx\,\bar{\psi}(x)f(\partial_{\mathbf{x}})\vec{\sigma}\psi(x), \tag{3.8}$$

with $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ the Pauli matrices. Such terms break the invariance of S under the SU(2) spin rotation group, so the three components of the spin triplet are no longer equivalent, and transverse spin-triplet particle-hole excitations acquire a mass. For instance, take $\vec{H}$ to point in z direction, and consider the temporal Fourier transform of $\psi(x)$, $\psi_n(\mathbf{x}) \equiv \psi(\mathbf{x},\omega_n)$, with $\omega_n = 2\pi T(n+1/2)$ a fermionic Matsubara frequency. For $\vec{H}=\vec{0}$, the homogeneous transverse particle-hole susceptibility

$$\int d\mathbf{x}d\mathbf{y}\langle[\bar{\psi}_n(\mathbf{x})\sigma_x\psi_m(\mathbf{x})][\bar{\psi}_n(\mathbf{y})\sigma_x\psi_m(\mathbf{y})]\rangle, \quad nm<0$$

diverges for $\Omega_{n-m}\equiv\omega_n-\omega_m\to 0$ as $1/\Omega_{n-m}$. For a nonvanishing source field, this soft mode acquires a mass proportional to the magnitude of $\vec{H}$, i.e., the above susceptibility in the zero-frequency limit has a finite value proportional to $1/|\vec{H}|$.

With an order parameter of the general form

$$\vec{n}(x)=\bar{\psi}(x)f(\partial_{\mathbf{x}})\vec{\sigma}\psi(x), \tag{3.9}$$

Hertz's LGW approach will therefore break down: the free energy is a nonanalytic function of the average order parameter, the quantum critical behavior is in general not mean-field-like, and the order-parameter susceptibility in the disordered phase is a nonanalytic function of the wave number.

The primary example for this class is the quantum itinerant ferromagnetic transition that we have mentioned several times before. Other examples include the spin-triplet analog of the isotropic-to-nematic transition discussed in Ref. 18, for which the Hertz theory will not work, in contrast to the spin-singlet version.

3. Particle-particle channel order parameters

We finally consider a class of source terms for order parameters in the particle-particle channel,

$$S_H = H\int dx\,\psi_\sigma(x)f(\partial_{\mathbf{x}})\psi_{\sigma'}(x), \tag{3.10}$$

which are relevant for superconductivity. These sources give a mass to two-particle excitations in the particle-particle channel. For order parameters of the form

$$n(x)=\psi_\sigma(x)f(\partial_{\mathbf{x}})\psi_{\sigma'}(x), \tag{3.11}$$

Hertz theory will therefore break down. The prime example for this class is the zero-temperature metal-to-superconductor transition.[19]

D. Connection with perturbative results

The above considerations imply that for the class of order parameters given by Eq. (3.6) one need not worry about a breakdown of Hertz theory, and the exact quantum critical behavior is easy to determine. In addition, they also explain a number of results concerning the presence or otherwise of nonanalytic wave-number dependences in various susceptibilities, which show a pattern that was not understood before.

The topic of a possible nonanalytic wave number and/or temperature dependence of static correlation functions in a Fermi liquid has a long history, which has been reviewed in Ref. 15. In this reference it was also established that the wave-number-dependent spin susceptibility χ_s at zero temperature has the form given in Eq. (3.5), a result that was confirmed by an explicit calculation in 2D by Chitov and Millis.[20] This was done by means of perturbation theory to second order in the electron-electron interaction. Reference 15 also gave a physical argument, based on the coupling of zero-sound modes, that links the $\mathbf{q}^2\ln|\mathbf{q}|$ dependence in 3D, and the $|\mathbf{q}|^{d-1}$ dependence for general d, to the $\ln|\mathbf{q}|$ dependence in $d=1$.[21] This means that, unless a prefactor accidentally vanishes in some dimension, a correlation function that shows the former nonanalyticity in 3D will necessarily have the latter in 1D, as is the case for χ_s. Analogous calculations for various spin-singlet susceptibilities found no nonanalyticity to second order in the interaction.[15,22] All of these results are summarized in the Appendix. Chitov and Millis[20] speculated that, at least in the case of the number density susceptibility, this null result is an artifact of low-order perturbation theory, and that there actually is a nonanalytic wave-number dependence with the same strength as in the spin density susceptibility, with the prefactor being of cubic or higher order in the interaction. Given the computational effort of the perturbation theory, this would be very hard to check explicitly. However, in the case of the density current susceptibility there is a strong argument against such a hypothesis: The f-sum rule, which reflects particle number conservation, requires that the homogeneous density current susceptibility is equal to n_e/m, with n_e the electron density and m the electron mass. This means that for very fundamental reasons this susceptibility cannot have a $\ln|\mathbf{q}|$ singularity in $d=1$, and from the mode-mode coupling argument mentioned above it follows that therefore there cannot be a nonanalyticity stronger than $|\mathbf{q}|^x$ with $x>d-1$ in higher dimensions either.[23] Since the density and density current susceptibilities are closely related by the same conservation law, this casts serious doubt on a nonanalyticity in the former as well.

Our general arguments based on the soft-mode structure of the system provide an explanation for all of these results. In particular, they show that neither the density susceptibility, nor any other spin-singlet susceptibility, to any order in perturbation theory, has a nonanalytic wave-number dependence that is cut off by the appropriate conjugate field. While in principle this leaves open the possibility of a nonanalyticity of a different nature,[17] it makes it likely that these susceptibilities are analytic at zero wave-number, and the perturbative results summarized in the Appendix are consistent with this.

In the particle-particle channel, the susceptibility of the spin-singlet anomalous density has a nonanalytic wave-number dependence that is cut off by a superconducting gap, and accordingly the metal-superconductor transition at zero temperature is not described by Hertz theory.[19] This is again in agreement with the general arguments given in Sec. III C 3. These calculations for the particle-particle channel were for systems with quenched disorder, but that does not affect our arguments; see Sec. IV C below.

IV. DISCUSSION

A. Generalizations

In this paper we have provided a general scheme to answer the question of whether it is possible to construct a local LGW theory, i.e. a field theory solely in terms of the order parameter, for a given quantum phase transition. We have applied this general philosophy to the one-band model defined in Sec. III A, but more general models can easily be analyzed in the same way.

As an example we consider a toy model with two completely degenerate bands

$$S=-\int dx \sum_{a=1,2} \overline{\psi}^{(a)}(x)[\partial_\tau+\epsilon(\partial_{\mathbf{x}})-\mu]\psi^{(a)}(x) + \sum_{a=1,2} S_{\rm int}[\overline{\psi}^{(a)},\psi^{(a)}]+S_{12}[\overline{\psi}^{(1)},\psi^{(1)},\overline{\psi}^{(2)},\psi^{(2)}]. \tag{4.1}$$

Here $a=1,2$ denotes the band index. In the simplest case, the interband interaction S_{12} could be an interaction between the number or charge densities in the two bands

$$S_{12}=J_{12}\int dx n_c^{(1)}(x) n_c^{(2)}(x), \tag{4.2}$$

where $n_c^{(a)}(x)$ is the charge density in band a. With increasing J_{12} the system will undergo a quantum phase transition from a symmetric state, in which the number densities in both bands are identical, to an asymmetric state with different densities in the two bands. The order parameter for the transition is the density difference $n_c^{(1)}(x)-n_c^{(2)}(x)$.

Let us now apply the soft-mode considerations developed in Sec. III to this model. A source term for the order-parameter field,

$$S_{H_{12}}=H_{12}\int dx[n_c^{(1)}(x)-n_c^{(2)}(x)], \tag{4.3}$$

breaks the symmetry between the two bands manifest in Eq. (4.1) and changes the soft-mode structure. Therefore, the band symmetry breaking quantum phase transition will not be described by a local order-parameter field theory.

We also note that this transition is very similar to the magnetic transitions discussed above. If one neglects the spin degrees of freedom in the two-band model, it can be mapped onto a quantum ferromagnetic transition with Ising symmetry by identifying the band indices 1 and 2 with spin up and spin down, respectively.

B. Coupling between soft modes and order-parameter fluctuations

The mechanism for a breakdown of the LGW approach to quantum phase transitions that we have studied is soft modes that are integrated out in deriving a LGW theory. It is important to realize that such soft modes *always* exist in itinerant electron systems, and in general they always couple to the order-parameter fluctuations via mode-mode coupling effects, except for special cases where such a coupling is forbidden by some symmetry. The crucial question is whether this coupling is strong enough to lead to the free energy being a nonanalytic function of the order parameter. The criterion for this is whether a nonzero average value of the order parameter, or, equivalently, a nonzero external field conjugate to the order parameter, gives a mass to the soft modes in question. This is more easily accomplished if both the order parameter and the additional soft modes are massless at the same wave number, as is the case for the itinerant quantum ferromagnet. A counterexample is the quantum antiferromagnet, where the order parameter is the staggered magnetization, a finite-wave-number quantity that couples only weakly to the soft particle-hole excitations that are soft at zero wave number.

C. Effects of disorder

So far we have discussed clean systems, but all of our methods remain valid in the presence of quenched disorder, and so does our discussion. The only modification is that in the presence of quenched disorder the Fermi liquid ground state is destroyed for dimensions $d \leqslant 2$ rather than $d \leqslant 1$. Quenched disorder is described by a term in the action

$$S_{\rm dis}=\int dx u(\mathbf{x})\overline{\psi}(x)\psi(x), \tag{4.4}$$

with $u(\mathbf{x})$ a random potential. The source term for a homogeneous spin-singlet order parameter, Eq. (3.6), still does not change the soft-mode structure of the disordered system (although we stress that the latter is different from that of a clean system), and consequently the quantum phase transitions with such order parameters in disordered systems can be described by local LGW theories. Furthermore, the corresponding susceptibilities are expected to be analytic functions of the wave number. This is consistent with the fact that the static density susceptibility has no $|\mathbf{q}|^{d-2}$ nonanalyticity in perturbation theory, and its homogeneous limit, $\partial n/\partial\mu$, is finite in 2D.[24–26] Similarly, the spin-triplet source, Eq. (3.8), still breaks spin rotation invariance and changes the soft-mode structure by giving the transverse particle-hole excitations in the spin-triplet channel a mass. Consequently, LGW theory breaks down for the disordered itinerant quantum ferromagnetic transition, and the quantum critical behavior is not mean-field-like. Consistent with this, in perturbation theory the spin susceptibility has a $|\mathbf{q}|^{d-2}$ nonanalyticity for $d>2$, and a $\ln|\mathbf{q}|$ behavior in 2D.[24,6]

D. Statics versus dynamics

It is well known that in quantum statistical mechanics, statics and dynamics are coupled. Naively, this might lead to the expectation that correlation functions that do not show a nonanalytic wave-number dependence at zero frequency have no nonanalytic frequency dependence either. This, however, is not true. In quenched disordered systems, the real part of the electrical conductivity, which is the imaginary part of the dynamical density current susceptibility divided by the frequency, has an $\omega^{(d-2)/2}$ frequency dependence, even though the static current susceptibility, as pointed out above, has no analogous wave-number nonanalyticity. Similary, the dynamical counterpart of the stress susceptibility χ_{xy} describes the sound attenuation coefficient, and it also is a nonanalytic function of the frequency.[27] It is therefore *not* correct to conclude from the presence of a nonanalytic frequency dependence in a time correlation function that the corresponding static susceptibility will be a nonanalytic function of the wave number, and that hence for the phase transition with the corresponding order parameter LGW theory will break down.

In order to understand this asymmetry between wave-number and frequency dependences it is important to remember that, even in quantum statistical mechanics, statics and dynamics are not equivalent. It is known that a finite frequency in the fermionic field theory breaks the symmetry between retarded and advanced Green functions, and gives a mass to soft modes.[28] Therefore, considering a *dynamical* source, as opposed to the static sources in Eqs. (3.6) and (3.8), changes the situation. If one expands a dynamical spin-singlet source in powers of ω, then the zeroth-order term does not change the soft-mode spectrum, but the term linear in ω does, and hence the dynamical piece of a spin-singlet susceptibility has in general nonanalyticities, even though the static part does not. The electrical conductivity and the sound attenuation coefficient mentioned above are examples of this effect.

E. Final remarks

We conclude with a few final remarks. First, we have restricted ourselves to systems where the ground state in the disordered phase is a Fermi liquid, but we have not explicitly used this property. We expect our general method to still work in more general cases. As an example, consider the case of a metamagnetic transition, i.e., a magnetic transition inside a ferromagnetic phase. In this case the spin rotation symmetry is already broken on either side of the transition, and adding the source term, Eq. (3.8), does not further change the soft-mode spectrum. We therefore conclude that in this case LGW theory will work, in agreement with the recent treatment of this situation by Millis *et al.*[29]

Second, we have restricted our explicit discussion to homogeneous order parameters. As we have mentioned in the context of the antiferromagnetic phase transition, inhomogeneous order parameters are in general expected to couple less strongly to the fermionic soft modes than homogeneous ones. However, the consequences of a spontaneous breaking of the translational invariance warrant a more thorough investigation. This is of interest, for instance, for the nature of the transition to the stripe phases that have been predicted and observed to occur in high-temperature superconductors.[30] These questions can also be analyzed within the framework set up in this paper.

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APPENDIX: PERTURBATIVE RESULTS FOR ORDER-PARAMETER SUSCEPTIBILITIES

In this appendix we list perturbative results for various susceptibilities. In Ref. 15 it was shown that the static spin density susceptibility χ_s has a nonanalytic wave-number dependence in 3D,

$$\chi_s(\mathbf{q}) = 2N_F[1 + c_s(\mathbf{q}/2k_F)^2 \ln(2k_F/|\mathbf{q}|) + O(\mathbf{q}^2)], \tag{A1}$$

with N_F the density of states per spin at the Fermi surface, k_F the Fermi wave number, and c_s a positive constant that, for weak interactions, is quadratic in the interaction amplitude. This result was confirmed in Ref. 20, which also showed that in 2D there is a corresponding $|\mathbf{q}|$ nonanalyticity. All of these results are consistent with a mode-mode coupling argument that lets one expect a singularity of the the form $|\mathbf{q}|^{d-1}$, with the integer exponents in $d=1$ and $d=3$ corresponding to logarithms; see Eq. (3.5). In particular, this argument links the perturbative $\ln|\mathbf{q}|$ dependence in $d=1$ (see Ref. 21) to the more general nonanalyticity in any dimension.

The perturbation theory developed in Ref. 15 is readily generalized to calculate other susceptibilities. For the number density susceptibility χ_n one finds that, to quadratic order in the interaction, the terms of order $\mathbf{q}^2 \ln|\mathbf{q}|$ cancel,[15] leaving one with the behavior

$$\chi_n(\mathbf{q}) = 2N_F[1 + O(\mathbf{q}^2)]. \tag{A2}$$

We have also calculated the number density current susceptibility χ_j and the stress susceptibility χ_{xy} that is the static correlation function of the electronic stress operator

$$(2/k_F^2)\sum_{\mathbf{k}} k_x k_y c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}.$$

For both of these we have found that the same cancellations hold as in the case of the number density susceptibility. That is, to second order in the interaction,

$$\chi_j(\mathbf{q}) = (n_e/m)[1 + O(\mathbf{q}^2)], \tag{A3}$$

$$\chi_{xy}(\mathbf{q}) = (8N_F/15)[1 + O(\mathbf{q}^2)], \tag{A4}$$

with n_e the electron density and m the electron mass. The null results expressed by Eqs. (A2)–(A4) are valid for interaction amplitudes with arbitrary wave-number and frequency dependences.

In the particle-particle channel, the susceptibility of the anomalous density $\psi_{\uparrow}(x)\psi_{\downarrow}(x)$ is known to have a leading wave-number dependence proportional to $1/\ln|\mathbf{q}|$,[19] but no exact perturbative results are available.

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