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Applicability of the position-dependent diffusion approach to localized transport through disordered waveguides

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In this work we show analytically and numerically that the localized regime of wave transport can be modeled as position-dependent diffusion with a diffusion coefficient that retains the memory of the source location. The dependence on the source diminishes when absorption is introduced.

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I. INTRODUCTION

The diffusive description of wave transport in random media has a long history [1]. This macroscopic approach describes the ensemble-averaged intensity of the wave on scales longer than the transport mean free path ℓ. As such, the diffusive description has a practical advantage compared to the direct solution of the wave equation for each statistical realization of disorder and subsequent averaging over the ensemble of solutions.

The diffusion coefficient can become renormalized [2] due to the wave localization phenomenon [3]. In three dimensions, for sufficiently strong disorder, diffusion vanishes for an infinitely large system [4]. In practice, however, one deals with transport through finite systems. Both the self-consistent theory (SCT) of localization [2,5] and supersymmetric (SUSY) theory [6] have predicted [7,8] that the diffusive-like description can also be applied to the *finite* systems that exhibit localized transport, in particular, to low-dimensional systems. In such a description, the diffusion coefficient becomes dependent on position, system size [9–11], and geometry [12]. In quasi-one-dimensional (quasi-1D) or 1D lossless media both SCT and SUSY lead to the following equation for the ensemble-averaged intensity $\langle I(z,z')\rangle$ in the presence of a point source J_0 at z':

$$-\frac{\partial}{\partial z}\left[D(z)\frac{\partial}{\partial z}\langle I(z,z')\rangle\right] = J_0\delta(z-z'). \qquad (1)$$

Diffusion of this kind leads to highly unusual macroscopic transport [13]. We stress that the medium itself, i.e., the density of scatterers, is statistically uniform and that the position-dependent diffusion is brought about by nonlocal wave-interference effects.

To date, the studies of position-dependent diffusion have concentrated on the geometry in which a wave is incident upon the random medium from an outside (free-space) region [7–12,14–17]. The position-dependent diffusion description was successful in describing light intensity under various measurement conditions (see Ref. [13] for a review). The ensemble-averaged intensity, however, is only the first step in characterizing the wave transport in random media. Indeed, the second-order statistical quantities, such as fluctuations or correlations, become important at the onset of Anderson localization [18,19]; they require the knowledge of the Green's function of the diffusion equation, e.g., $\langle I(z,z')\rangle$ with an arbitrary z' [20–22].

In this work, we test the applicability of Eq. (1) with an arbitrary position of the source. We show analytically that for $z'\neq 0$, the diffusion equation is applicable; however, the position-dependent diffusion coefficient (PDDC) $D(z)$ acquires a dependence on the position of the source z'. We derive a closed-form analytic expression for $D(z,z')$ and verify it with *ab initio* numerical simulations. We show that $D(z,z')$ is reduced to the known result [11] for $z'\to 0$, i.e., for the wave incident from the outside region. We demonstrate that when an absorption, unavoidable in experiment, is present in the system, the dependence of $D(z,z')$ on the position of the source is diminished so that PDDC can be adequately determined using the self-consistent theory [9,10]. Although in this work we make references to the transport of light (i.e., the electromagnetic waves) in random media, our results are also applicable to other types of waves such as acoustic waves and matter waves.

This paper is organized as follows. In Sec. II, we obtain an analytic expression for PDDC describing wave transport in a single-mode waveguide with an external source. We verify the applicability of the result with numerical simulations. In Sec. III, we demonstrate analytically and confirm numerically that PDDC depends on the position of the source inside the random medium. The effect of absorption on the position dependence of PDDC is studied in Sec. IV.

II. POSITION-DEPENDENT DIFFUSION IN PASSIVE RANDOM MEDIA WITH AN EXTERNAL SOURCE

We consider a one-dimensional random medium occupying the $0\leqslant z\leqslant L$ region. Propagation of a scalar wave $E(z)$ is described by

$$\frac{d^2E(z)}{dz^2}+k^2[1+\epsilon(z)]E(z)=0, \qquad (2)$$

where $k=2\pi/\lambda$ is the wave number and $\epsilon(z)$ is a random process. For a wave with unit amplitude incident from the left, the boundary conditions can be expressed in terms of reflection r and transmission t coefficients as

$$E(z)=e^{ikz}+r\,e^{-ikz},\quad z<0,$$
$$E(z)=te^{ikz},\quad z>L. \qquad (3)$$

We are interested in obtaining a closed-form expression for the intensity $\langle I(z)\rangle\equiv\langle|E(z)|^2\rangle$ averaged over an ensemble

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of random processes $\epsilon(z)$. Indeed, the position-dependent diffusion coefficient $D(z)$ can be found with Fick's law

$$\langle J(z)\rangle = -D(z)d\langle I(z)\rangle/dz, \tag{4}$$

where $\langle J(z)\rangle$ is the flux. In a passive random medium the flux is conserved during propagation and thus can be found from the boundary conditions. Indeed, the fraction of the flux propagating in the positive (+) or negative (−) direction can be expressed as [1]

$$\langle J^{(\pm)}(z)\rangle = (v/2)\langle I(z)\rangle \mp [D(z)/2]\, d\langle I(z)\rangle/dz, \tag{5}$$

where v is the wave speed. The right boundary $\langle J^{(-)}(z)\rangle$ vanishes, so $\langle J(z)\rangle = \langle J^{(+)}(L)\rangle = v\langle I(L)\rangle$. Substituting this expression into Eq. (4), we obtain

$$D(z) = -v\langle I(L)\rangle/[d\langle I(z)\rangle/dz]. \tag{6}$$

Therefore, finding PDDC requires the knowledge of (only) $\langle I(z)\rangle$ for the problem defined by Eqs. (2) and (3). Such a solution has been obtained in Refs. [23–25]. The common theme in these studies is to relate the statistical property of the wave field inside the medium to those at the boundary [see Eq. (3)], where it can be obtained using the limiting theorems (see Ref. [26] for a review). Such an approach is in the spirit of the well-known self-embedding method [27]. We will assume that $\epsilon(z)$ is a δ-correlated Gaussian process with $\langle\epsilon(z)\rangle$ and $\langle\epsilon(z)\epsilon(z')\rangle = a\delta(z-z')$. Under these conditions, the solution for the ensemble-averaged intensity is obtained in the form [23–25]

$$\langle I(z)\rangle = 1 - \sqrt{\frac{\xi}{\pi L}}\int_{-\infty}^{\infty}\exp\left[-\frac{[\zeta-(z-L/2)/\xi]^2}{L/\xi}\right] \times\left(\tanh(\zeta)+\frac{\zeta}{\cosh(\zeta)^2}\right)d\zeta, \tag{7}$$

where we introduced the localization length as $\xi^{-1} = ak^2/2$. Substitution of Eq. (7) into Eq. (6) gives us the analytical expression for PDDC.

A compact expressions for both $\langle I(z)\rangle$ and $D(z)$ can be obtained when $L \gg \xi$. In this limit, the expression in parentheses in the integrand of Eq. (7) can be approximated with the step function $h(\zeta)$, and the integral can be computed in terms of the error function $\mathrm{erf}(x)$:

$$\langle I(z)\rangle \simeq 1 - \mathrm{erf}[\tilde{z}_c], \tag{8}$$

with a scaling parameter $\tilde{z}_c = (z-L/2)/\sqrt{L\xi}$ as the argument. In Fig. 1(a), $\langle I(z)\rangle$ is plotted for $L/\xi = 5$, 10, 20, 50, and 100 with and without the scaling z coordinate. We confirm that Eq. (8) approximates the exact expression (7) well. We note that such a distribution has been observed in the numerical simulations of energy deposition in wave-front shaping in a random medium [28,29].

The asymptotic expression for PDDC in the limit $L \gg \xi$ is obtained by substituting Eq. (8) into Eq. (6):

$$D(z) \simeq D_0 \exp\left[\tilde{z}_c^2 - L/4\xi\right]. \tag{9}$$

Here $D_0 = v\ell$ is the unrenormalized value of the diffusion coefficient in terms of the transport mean free path $\ell = \xi$. Figure 1(b) confirms the universality of PDDC inside a 1D passive random medium in terms of the scaling parameter $\tilde{z}_c$. Equation (9) agrees with the one derived in the framework of the supersymmetric theory of Ref. [11] for quasi-1D geometry (a multimode waveguide), where it also applies in the crossover regime between diffusion and localization.

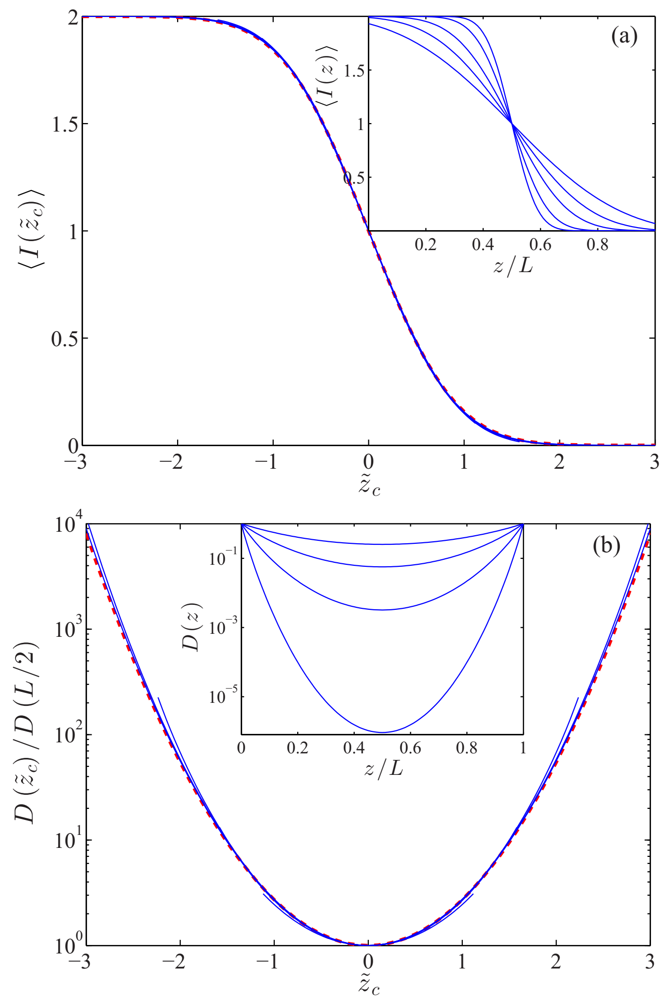

FIG. 1. (Color online) (a) Spatial distribution of the ensemble-averaged intensity $\langle I(z)\rangle$ from Eq. (7) (blue lines) without scaling (inset) for systems with $L/\xi = 5,10,20,50,100$ (left to right). The main panel presents the same data in terms of the scaled coordinate $\tilde{z}_c = (z-L/2)/\sqrt{L\xi}$; the approximate expression given by Eq. (8) is shown as a thick dashed line. (b) The position-dependent diffusion coefficient found from Eq. (6) for systems with $L/\xi = 5,10,20,50$ without (inset) and with (main panel) scaling. The thick dashed line is found with the asymptotic $L \gg \xi$ expression in Eq. (9).

III. POSITION-DEPENDENT DIFFUSION IN PASSIVE RANDOM MEDIA WITH AN INTERNAL SOURCE

For the source located inside the random medium, Eq. (2) is modified to include a point source at z':

$$\frac{d^2E(z,z')}{dz^2} + k^2[1+\epsilon(z)]E(z,z') = \delta(z-z'), \tag{10}$$

whereas the boundary conditions in Eq. (3) are replaced with the outgoing wave conditions at both ends of the waveguide,

$$E(z) = t_1\, e^{-ikz}, \quad z<0,$$
$$E(z) = t_2\, e^{ikz}, \quad z>L. \tag{11}$$

Under these conditions, the flux inside the medium is a piecewise constant function with a jump at the position of the source z'. The values of $\langle J(z,z')\rangle$ for $z<z'$ and $z>z'$ can be determined by applying Eq. (5) at $z=0$ and $z=L$, respectively. We find $\langle J(z<z')\rangle = -v\langle I(0,z')\rangle$ and $\langle J(z>z')\rangle = v\langle I(L,z')\rangle$, where $\langle I(z,z')\rangle = \langle |E(z,z')|^2\rangle$ and $E(z,z')$ is the solution of Eq. (10) with boundary conditions in Eq. (11). Therefore, PDDC can be written based on Fick's law Eq. (4) as

$$D(z,z') = -v[d\langle I(z)\rangle/dz]^{-1}\begin{cases}-\langle I(0,z')\rangle, & z<z';\\ \langle I(L,z')\rangle, & z>z'.\end{cases} \quad (12)$$

As in Sec. II, the above expression for PDDC requires knowledge of the ensemble-averaged intensity $\langle I(z,z')\rangle$. The latter has been obtained in Refs. [25,30] in the form

$$\begin{aligned}\langle I(z,z')\rangle &= \pi \exp\left[\frac{3L}{4\xi} - \frac{|z-z'|}{\xi}\right]\int_{-\infty}^{\infty} d\mu\\ &\times \exp\left[-\frac{\mu^2 L}{\xi}\right]\frac{\sinh(\pi\mu)}{\mu\cosh(\pi\mu)^2}\\ &\times\left[\left(\mu^2+\frac{1}{4}\right)\cos\left(2\mu\frac{(z+z'-L)}{\xi}\right)\right.\\ &+\left(\mu^2-\frac{1}{4}\right)\cos\left(2\mu\frac{(L-|z-z'|)}{\xi}\right)\\ &\left.+\mu\sin\left(2\mu\frac{(L-|z-z'|)}{\xi}\right)\right]. \end{aligned} \quad (13)$$

Substituting this expression into Eq. (12) gives us the final result.

We make the following observations. First of all, in the limit of $z'=0$, we recover the result for an external source found in the previous section. Indeed, Eq. (13) with $z'=0$ can be shown [26] to reduce to Eq. (7). Second, unlike Eq. (1), PDDC $D(z,z')$ depends on the source position z'. In Fig. 2 we evaluated Eq. (12) for $L/\xi = 7.6$ and four values of z': 0 (outside source), $L/4$, $L/2$, and $3L/4$. Indeed, PDDC shows strong dependence on the position of the source. We note that $D(z,0)$ is always greater than $D(z,z'>0)$, with the minimum value at the middle of the sample for $z'=L/2$.

We verified the above results with the numerical simulations for the wave with $k=1.45$ propagating normally through a stack of alternating dielectric slabs with dielectric constants $\epsilon_1 = 1$ and $\epsilon_2 = 1.2$. The width of the stacks of the first kind is distributed uniformly in the interval $d_1 \in (0.9,1.1)$, while the width of the other slabs is kept constant at $d_2 = 1$. The wave propagation in the system consists of the free propagation inside the slabs and scattering at the interfaces, where the proper boundary conditions should be satisfied. It can be described using the transfer-matrix formalism (cf. Refs. [31,32]). We computed $J(z) = -kc\text{Im}[E(z)dE(z)/dz]$ and $I(z) = (k^2/2)|E(z)|^2 + (1/2)|dE(z)/dz|^2$ in a system with $N = 8\times10^3$ layers numerically and then found the average over 10^8 disorder realizations. PDDC was found from Fick's law (4). The results reported in Fig. 2

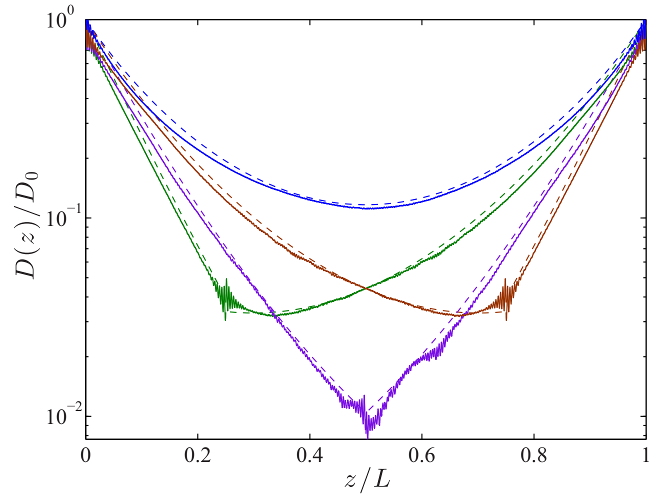

FIG. 2. (Color online) The position-dependent diffusion coefficient inside 1D passive random media with $L = 7.6\xi$, computed for four source positions: $z'=0$ (upper curves), $L/4$, $L/2$, and $3L/4$. The latter three curves have cusps at the position of the source. Dashed lines were found by substituting the analytical result (13) into Eq. (12). The solid lines were obtained numerically.

(solid lines) agree with the analytical expression (dashed lines).

IV. EFFECT OF ABSORPTION

Absorption is inevitable in optical experiments. The effect of absorption is to suppress resonant tunneling of the wave [11], thereby increasing PDDC [33,34]. In this section we perform numerical analysis of the effect of absorption on PDDC for an internal source.

Modeling an absorbing random medium is accomplished by adding a constant imaginary part to $\epsilon(z)$ in Eq. (10) as $\epsilon(z) \to \epsilon(z) + i\gamma$. The addition of the loss results in an extra term, $\xi_a^{-2}\langle I(z,z')\rangle$, in the inhomogeneous diffusion equation. The absorption length ξ_a can be obtained for a given value of γ from the continuity condition $d\langle J(z,z')\rangle/dz = \langle I(z,z')\rangle/\tau_a$, where $\tau_a = \xi_a^2/D_0$ [33].

Figure 3 shows PDDC obtained numerically for the model in Sec. III. We choose the number of layers in a stack $N = 1.6\times10^4$ and two values of γ, $\gamma = 10^{-5}$ and 10^{-4}. These parameters give $L/\xi_{a0} = 3.2$ and 10, respectively. Similar to the passive (nonabsorbing) case in Sec. III, PDDC clearly shows a dependence on the position of the source inside the medium ($z' = 0$, $L/4$, $L/2$, and $3L/4$ are shown); it has a cusp feature at z'. However, closer inspection of Fig. 3 shows that the dependence on the source position is strongly suppressed at large L/ξ_a; in this limit, ξ_a becomes comparable to the localization length ξ.

Performing computationally expensive numerical simulations is not practical, particularly in higher-dimensional systems. Self-consistent theory [7,9,10] has been successful in providing a good prediction for PDDC for systems with L/ξ not too large [11], and it was shown [33] to be accurate in the absorbing systems. In all previous works, an external source has been considered. Here we computed the prediction

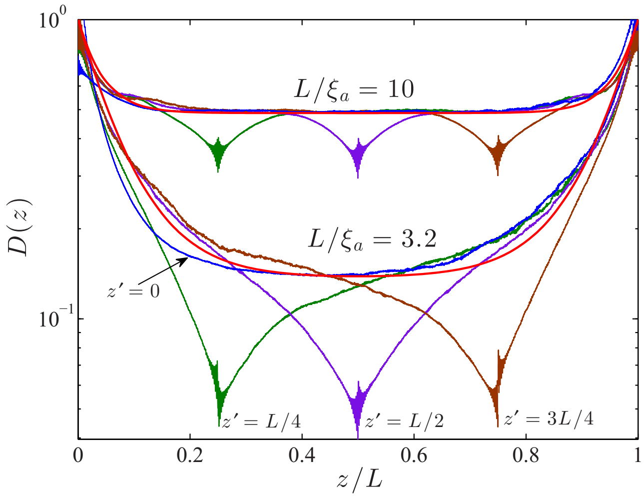

FIG. 3. (Color online) The position-dependent diffusion coefficient in an absorbing random medium with $L/\xi = 15.2$. The lower set of curves corresponds to the absorption $L/\xi_a = 3.2$ and four positions of the source. The thick red line is the prediction of SCT from Eqs. (14) and (15). For a stronger absorption $L/\xi_a = 10$ (upper set of curves), SCT provides an adequate prediction for PDDC.

of the self-consistent theory for systems with different amounts of absorption. The self-consistency condition relates PDDC to the return probability $\langle I(z,z)\rangle$ as

$$-\frac{d}{dz}D^{(SCT)}(z)\frac{d\langle I(z,z')\rangle}{dz} + \frac{\langle I(z,z')\rangle}{\xi_a^2} = J_0\delta(z-z'), \quad (14)$$

$$D_0/D^{(SCT)}(z) = 1 + v\langle I(z,z)\rangle. \quad (15)$$

These equations form a closed set, sufficient for finding $D^{(SCT)}(z)$. The solution of Eqs. (14) and (15) is shown with a thick solid line in Fig. 3. We find that in the presence of sufficiently strong absorption, SCT makes an adequate prediction for PDDC even for the internal source.

V. CONCLUSION

In this work, we investigated the applicability of the position-dependent diffusion approach to describing the localized wave transport in random media. We have shown analytically and numerically that even for $L > \xi$, the position-dependent diffusion coefficient can be defined through Fick's law. The benefit of such an approach is that it allows one to obtain the ensemble-averaged value of intensity without the need to perform statistical averaging.

Our analysis shows that the position-dependent diffusion coefficient exhibits significant dependence on the source position z'. Such a dependence has not been discussed before. That is because previous studies have concentrated on the common experimental arrangement: the incident wave impinging on the sample from the outside. Our study of PDDC with an internal source is of practical interest for a number of reasons. First of all, the solution $\langle I(z,z')\rangle$ of the diffusion equation with PDDC and an internal source z' is the Green's function, which can be used to define the second-order statistics (e.g., fluctuation, correlations) of wave transport. Second, Fick's law with PDDC in the form of $D(z,z')$ points to a highly unconventional type of diffusion in the localized systems. The spatial dependence of PDDC has been shown [13] to exhibit an unusual macroscopic transport behavior. The additional dependence on the source position found in our work may necessitate a completely new *nonlocal approach* to transport. An accurate description of the wave transport through random media would inform studies of the limitations of wave-front shaping [35] with applications, in particular, in the field of biological imaging [36].

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