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U. D. Jentschura and J. H. Noble, "Nonrelativistic Limit of the Dirac-Schwarzschild Hamiltonian: Gravitational Zitterbewegung and Gravitational Spin-Orbit Coupling," *Physical Review A - Atomic, Molecular, and Optical Physics*, vol. 88, no. 2, pp. 022121-1-022121-6, American Physical Society (APS), Aug 2013.

The definitive version is available at https://doi.org/10.1103/PhysRevA.88.022121

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Nonrelativistic limit of the Dirac-Schwarzschild Hamiltonian: Gravitational *Zitterbewegung* and gravitational spin-orbit coupling

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(Received 17 May 2013; published 23 August 2013)

We investigate the nonrelativistic limit of the gravitationally coupled Dirac equation via a Foldy-Wouthuysen transformation. The relativistic correction terms have immediate and obvious physical interpretations in terms of a gravitational *Zitterbewegung* and a gravitational spin-orbit coupling. We find no direct coupling of the spin vector to the gravitational force, which would otherwise violate parity. The particle-antiparticle symmetry described recently by one of us [Jentschura, Phys. Rev. A **87**, 032101 (2013)] is verified on the level of the perturbative corrections accessed by the Foldy-Wouthuysen transformation. The gravitational corrections to the electromagnetic transition current are calculated.

DOI: 10.1103/PhysRevA.88.022121 PACS number(s): 03.65.Pm, 11.10.−z, 03.70.+k

I. INTRODUCTION

The ever-increasing precision of spectroscopic experiments necessitates the consideration of gravitational effects in relativistic quantum mechanics. General relativity effects already have to be taken into account in the synchronization of clocks in the global positioning system (GPS) satellites, which implies that the satellite-based clocks get ahead of ground-based clocks by about 45 μs/d. For nonrelativistic neutron waves, quantum-mechanical phase shifts due to the gravitational and inertial forces have been measured in Refs. [1–3]. The derivation of the gravitationally coupled Dirac equation has been discussed in textbooks [4–10] as well as original research literature [11–17]. Recently, it has been argued that a symmetry exists [16] for the gravitationally coupled Dirac equation, which implies that particles and antiparticles are both attracted to the gravitational field and that this symmetry holds to all orders in the velocity of the particles, i.e., including all relativistic quantum corrections of motion. This symmetry was obtained [16] for a class of static space-time metrics which give rise to a (generalized) Dirac-Schwarzschild equation. We also refer to the corresponding Hamiltonian as the Dirac-Schwarzschild Hamiltonian.

However, the result obtained in [16] was not reconciled with other articles from the literature, which investigate the nonrelativistic limit of the quantum dynamics and, in particular, discuss the conceivable presence [18] of a spin-gravity coupling of the form $\vec{\Sigma}\cdot\vec{g}$, where $\vec{\Sigma}$ is the (4×4)-spin matrix and $\vec{g}$ is the acceleration due to gravity. Specifically, in Ref. [18], a conceivable spin-gravity interaction and the pertinent experimental detection have been investigated. In Ref. [19], via a Foldy-Wouthuysen transformation [20] of the Dirac-Schwarzschild Hamiltonian, a term proportional to $\vec{\Sigma}\cdot\vec{g}$ is obtained in the leading nonrelativistic approximation. References [21,22] discuss whether such a term would violate parity. Indeed, it is well known that orbital as well as spin angular momenta constitute pseudovectors. Notably, the spin operator $\vec{\Sigma} = \gamma^5\,\gamma^0\,\vec{\gamma}$ transforms under parity as $\vec{\Sigma} \to \gamma^0\,\vec{\Sigma}\,\gamma^0 = \vec{\Sigma}$ and therefore as a pseudovector. By contrast, the gravitational force $\vec{F}_g = m\,\vec{g}$ with $mg = m\,|\vec{g}| = GmM/r^2 = m\,r_s/(2r)$ is a vector. (Here, m and M are the masses of the test particle and planet, respectively, and r_s is the Schwarzschild radius; we use natural units with $\hbar = c = \epsilon_0 = 1$.) One might thus conclude that any term in the Hamiltonian proportional to $\vec{\Sigma}\cdot\vec{g}$ would indeed violate parity symmetry. Apparently, the question of how to physically interpret the leading relativistic corrections in a curved space-time akin to the Schwarzschild geometry still constitutes an open problem [23–26].

The gravitational Dirac Hamiltonian is similar in its mathematical structure to the Dirac-Coulomb Hamiltonian [27–29], and we would *a priori* expect that the Foldy-Wouthuysen transformation should yield similar terms but respect the particle-antiparticle symmetry from [16]. The nature of the Foldy-Wouthuysen program is inherently perturbative. In the following, we expand to first order in the gravitational coupling constant; i.e., we only keep terms of first order in the Schwarzschild radius r_s. The corresponding dimensionless expansion parameter is $\chi = r_s/r$, where r measures the distance from the center of the black hole (in the sense of a space-time coordinate). Regarding the momenta and distances, we assume that the Cartesian components p^i are of order ξm and that $r^i \sim 1/(\xi m)$ and expand to order $\chi\,\xi^3$ or to order ξ^4 (for the gravitationally uncoupled terms). After a rederivation of the Hamiltonian form of the Dirac-Schwarzschild equation in Sec. II, the Foldy-Wouthuysen transformation is carried out explicitly in Sec. III, while conclusions are reserved for Sec. IV. We again reemphasize the use of natural units throughout the article ($\hbar = c = \epsilon_0 = 1$).

II. FORMALISM

We use the same conventions as in Ref. [16] and assume that the curved-space Dirac γ (overline) and flat-space (tilde) Dirac matrices fulfill the algebraic relations

$$\{\overline{\gamma}^\mu(x),\overline{\gamma}^\nu(x)\} = 2\,\overline{g}^{\mu\nu}(x)\,,\quad \{\widetilde{\gamma}^\mu,\widetilde{\gamma}^\nu\} = 2\,\widetilde{g}^{\mu\nu}. \tag{1}$$

Here, $\{\cdot,\cdot\}$ is the anticommutator. The curved-space metric is $\overline{g}^{\mu\nu}$, with $\mu,\nu = 0,1,2,3$, while the "West-Coast" flat-space metric is $\widetilde{g}^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Overlining the curved-space Dirac matrices and using the tilde for the flat-space variants avoids a conceivable confusion with the particle physics literature [30–34], where one denotes the flat-space matrices as γ^μ and the flat-space metric as $g^{\mu\nu}$, whereas in the literature on general relativity, one usually denotes the

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curved-space Dirac matrices without a tilde and uses $g^{\mu\nu}$ for both flat-space and curved-space metrics [35–39]. Explicit use of the overlining and the tilde avoids any possible confusion.

The formulation of the gravitationally coupled Dirac equation [4–16] suggests the use of the vierbein formalism, which is particularly straightforward to formulate if the "square root of the metric" can be taken with ease (see Sec. 6 of Ref. [40]). The Dirac action in curved space-time is given by the invariant integral

$$S = \int d^4x \, \sqrt{-\det \overline{g}} \;\; \overline{\psi}(x) \left(\frac{i}{2} \overline{\gamma}^{\rho}(x) \overleftrightarrow{\nabla}_{\rho} - m \right) \psi(x), \tag{2}$$

where the covariant derivative is $\nabla_\rho = \partial_\rho - \Gamma_\rho$ and

$$\Gamma_\rho = -\frac{i}{4} \, \overline{g}_{\mu\alpha} \left(\frac{\partial b_\nu{}^\beta}{\partial x^\rho} \, a^\alpha{}_\beta - \Gamma^\alpha{}_{\nu\rho} \right) \overline{\sigma}^{\mu\nu} \tag{3}$$

is the affine spin-connection matrix. Here, the curved-space spin matrices are $\overline{\sigma}^{\mu\nu} = \frac{i}{2} [\overline{\gamma}^\mu, \overline{\gamma}^\nu]$. We represent the $\overline{\gamma}^\nu$ matrices in terms of the flat-space $\widetilde{\gamma}^\mu$,

$$\overline{\gamma}_\rho = b_\rho{}^\alpha \, \widetilde{\gamma}_\alpha \, , \quad \widetilde{\gamma}_\rho = a^\alpha{}_\rho \, \overline{\gamma}_\alpha , \tag{4a}$$
$$\overline{\gamma}^\alpha = a^\alpha{}_\rho \, \widetilde{\gamma}^\rho \, , \quad \widetilde{\gamma}^\alpha = b_\rho{}^\alpha \, \overline{\gamma}^\rho . \tag{4b}$$

We use the flat-space Dirac matrices in the Dirac representation,

$$\widetilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \widetilde{\gamma}^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \tag{5a}$$
$$\widetilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \widetilde{\gamma}^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \tag{5b}$$
$$\widetilde{\gamma}^5 = i \, \widetilde{\gamma}^0 \, \widetilde{\gamma}^1 \, \widetilde{\gamma}^2 \, \widetilde{\gamma}^3 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}. \tag{5c}$$

The metric is recovered as

$$\{\overline{\gamma}_\rho, \overline{\gamma}_\sigma\} = b_\rho{}^\alpha \, b_\sigma{}^\beta \, \{\widetilde{\gamma}_\alpha, \widetilde{\gamma}_\beta\} = 2 \, \widetilde{g}_{\alpha\beta} \, b_\rho{}^\alpha \, b_\sigma{}^\beta = 2 \overline{g}_{\rho\sigma}, \tag{6a}$$
$$\{\overline{\gamma}^\rho, \overline{\gamma}^\sigma\} = a^\rho{}_\alpha \, a^\sigma{}_\beta \, \{\widetilde{\gamma}^\alpha, \widetilde{\gamma}^\beta\} = 2 \, \widetilde{g}^{\alpha\beta} \, a^\rho{}_\alpha \, a^\sigma{}_\beta = 2 \overline{g}^{\rho\sigma}. \tag{6b}$$

The gravitationally coupled Dirac equation, obtained by variation of Eq. (2), is

$$(i \, \overline{\gamma}^\mu \, \nabla_\mu - m) \psi(x) = 0. \tag{7}$$

We now specialize to a static space-time metric [41] of a generalized Schwarzschild type,

$$\overline{g}_{\mu\nu} = \mathrm{diag}[w^2(r), -v^2(r), -v^2(r), -v^2(r)], \tag{8}$$

where the vierbein coefficients are given as $b_0{}^\beta = \delta_0{}^\beta \, w(r)$, $b_i{}^j = \delta_i{}^j \, v(r)$, $a^\alpha{}_0 = \delta^0{}_\alpha / w(r)$, and $a_i{}^j = \delta_i{}^j / v(r)$. The a and b matrices are symmetric in this case, $a^\mu{}_\nu = a^\nu{}_\mu$ and $b_\mu{}^\nu = b_\nu{}^\mu$. With these coefficients, using computer algebra [42], it is easy to evaluate all Christoffel symbols and to establish that [16]

$$\overline{\gamma}^0 \, \overline{\gamma}^\mu \, \Gamma_\mu = -\frac{\widetilde{\gamma}^0 \, \vec{\widetilde{\gamma}} \cdot \vec{r}}{v(r) w(r)} \, G(r), \tag{9a}$$

$$G(r) = \frac{2 \, v'(r) \, w(r) + v(r) \, w'(r)}{2 \, v(r) \, w(r)}. \tag{9b}$$

The Hamiltonian form of the gravitationally coupled Dirac equation,

$$i (\overline{\gamma}^0)^2 \, \partial_t \psi = (\overline{\gamma}^0 \, \overline{\gamma}^j \, p^j + i \, \overline{\gamma}^0 \, \overline{\gamma}^\mu \, \Gamma_\mu + \overline{\gamma}^0 \, m) \psi, \tag{10}$$

translates into $i \, \partial_t \psi = H \, \psi$, where H is given by

$$H = \frac{w}{v} \, \vec{\alpha} \cdot \vec{p} - \frac{i}{2v} \, \vec{\alpha} \cdot \vec{\nabla} w - \frac{i w}{v^2} \, \vec{\alpha} \cdot \vec{\nabla} v + \beta m w. \tag{11}$$

Here, we use the notation $\vec{\alpha} = \gamma^0 \, \vec{\widetilde{\gamma}}$ and $\beta = \gamma^0$. We now stretch space according to the scaling

$$\psi' = v^{3/2} \, \psi \, , \quad H' = v^{3/2} \, H \, v^{-3/2}. \tag{12}$$

This leads to a Hermitian Hamiltonian, which acts on the Hilbert space of square-integrable functions with the scalar product $\langle \phi, \psi \rangle = \int d^3 \, \phi^*(\vec{r}) \, \psi(\vec{r})$ and in three-space reads as

$$H' = \frac{1}{2} \, \{\vec{\alpha} \cdot \vec{p}, \mathcal{F}\} + \beta m w, \quad \mathcal{F} = \frac{w}{v}. \tag{13}$$

We here confirm the result given in Eq. (14) of Ref. [19]. For the Schwarzschild metric in isotropic coordinates (see Sec. 43 of Chap. 3 of Ref. [41]), we have to first order in the Schwarzschild radius r_s,

$$w = \left(1 - \frac{r_s}{4r}\right) \left(1 + \frac{r_s}{4r}\right)^{-1} = \frac{4r - r_s}{4r + r_s} \approx 1 - \frac{r_s}{2r} \, ,$$
$$v = \left(1 + \frac{r_s}{4r}\right)^2 \approx 1 + \frac{r_s}{2r}, \tag{14}$$
$$\frac{w}{v} = \frac{16 \, r^2 \, (4r - r_s)}{(4r + r_s)^3} \approx 1 - \frac{r_s}{r} \, .$$

The Schwarzschild radius is given as $r_s = 2 \, G M$, where G is Newton's gravitational constant and M is the mass of the planet. So, to first order in r_s, we have to analyze the Dirac-Schwarzschild Hamiltonian $H_{\rm DS}$, which is given by

$$H_{\rm DS} \approx \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \left(1 - \frac{r_s}{r}\right) \right\} + \beta m \left(1 - \frac{r_s}{2r}\right). \tag{15}$$

We can now carry through the Foldy-Wouthuysen program as usual.

III. FOLDY-WOUTHUYSEN TRANSFORMATION

A. Transformation of the Hamiltonian

Contrary to widespread belief, the rationale of the Foldy-Wouthuysen transformation [20] actually is rather well defined [30] and in some sense tied to the Dirac representation of the Dirac matrices given in Eq. (5). (i) One has to identify the odd (in spinor space) part of the Hamiltonian H, which is denoted as $\mathcal{O}$. (ii) One then defines the Hermitian operator $S = -i \, \beta \, \mathcal{O}/(2m)$ and the unitary operator $U = \exp(i S)$. (iii) The calculation of multiple nested commutators of U and H proceeds until further nested commutators only yield higher-order terms when expressed in terms of defined operational parameters of the expansion. (iv) If there are odd terms left after the first transformation, to the desired order in the expansion, then one employs a second Foldy-Wouthuysen transformation by identifying the odd part of the new Hamiltonian, which resulted from the first Foldy-Wouthuysen step, as $\mathcal{O}'$. One does this recursively until all odd parts of the input Hamiltonian are eliminated. In the case of the free Dirac Hamiltonian, it is possible to perform a Foldy-Wouthuysen transformation to all orders in the momenta [30], but one may also choose a perturbative approach (see the Appendix). However, an exact Foldy-Wouthuysen transformation has not been described for more complicated Hamiltonians like the Dirac-Coulomb Hamiltonian [27–29]

$$H_{\rm DC} = \vec{\alpha}\cdot\vec{p} + \beta m - \frac{Z\alpha}{r}, \tag{16}$$

where Z is the nuclear charge number and α is the fine-structure constant, or for other, nontrivial extensions of the standard free Dirac equation [43]. The Dirac-Schwarzschild Hamiltonian (15) still is of an intricate nature. A nonperturbative Foldy-Wouthuysen transformation has not been described in the literature for the Dirac-Coulomb Hamiltonian (16), and we can thus conclude that a perturbative approach seems most promising for the Dirac-Schwarzschild Hamiltonian.

For the gravitational correction terms, the expansions below are carried through to first order in $\chi = r_s/r$ and to third order in $\xi = \vec{p}/m$ or $\xi = 1/(m\,r)$, and we assume the exchanged photons to be soft, i.e., $k \sim \xi^2 m$. This is the same expansion as for the Lamb shift [30,44] if we identify χ with $Z\alpha$, where Z is the nuclear charge number and α is the fine-structure constant. Terms are calculated up to order χ^4 if there is no gravitational interaction and up to order $\xi\,\chi^3$ if there is a gravitational interaction. Terms of order ξ^2 (second order in the gravitational interaction) are ignored.

To leading order in r_s, we have to analyze the Hamiltonian $H_1 = H_{\rm DS}$ as given in Eq. (15). For the first Foldy-Wouthuysen transformation, we therefore have

$$S_1 = -\frac{i}{2m}\,\beta\,\mathcal{O}_1, \quad \mathcal{O}_1 = \frac{1}{2}\left\{\vec{\alpha}\cdot\vec{p},\left(1-\frac{r_s}{r}\right)\right\}. \tag{17}$$

The transformation is calculated as

$$\begin{aligned} H_2 &= e^{iS_1}\,H_1\,e^{-iS_1} \\ &= H_1 + i\,[S_1,H_1] + \frac{i^2}{2!}\,[S_1,[S_1,H_1]] + \cdots . \end{aligned} \tag{18}$$

The result is

$$\begin{aligned} H_2 &= \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3}\right) - \beta\,\frac{m\,r_s}{2\,r} \\ &\quad + \beta\left(-\frac{3r_s}{8m}\left\{\vec{p}^{\,2},\frac{1}{r}\right\} + \frac{3\pi r_s}{4m}\delta^{(3)}(\vec{r}) + \frac{3r_s}{8m}\,\frac{\vec{\Sigma}\cdot\vec{L}}{r^3}\right) \\ &\quad - \frac{(\vec{\alpha}\cdot\vec{p})^3}{3m^2} + \frac{1}{4}\left\{\frac{r_s}{r},\vec{\alpha}\cdot\vec{p}\right\}. \end{aligned} \tag{19}$$

A central ingredient of the Foldy-Wouthuysen transformation is the presence of the term $\beta\,m$ in the initial Hamiltonian and the commutator relation $[\beta\mathcal{O},\beta m] = -2m\,\mathcal{O}$, which holds for any odd (in the spinor space) term $\mathcal{O}$ in the Hamiltonian. Indeed, the first commutator in Eq. (18) then eliminates the leading odd terms in the initial Hamiltonian H_1. One might think that this scheme is not applicable to the Hamiltonian (15) because the mass term is multiplied by a factor $[1 - r_s/(2r)]$, but that is not the case: The nature of the Foldy-Wouthuysen transformation is perturbative, and the term $-\beta\,m\,r_s/(2r)$ is a perturbative gravitational correction to the mass. The modification of the mass term therefore is of higher order in χ and does not inhibit the application of the perturbative approach to the Foldy-Wouthuysen transformation.

For the second Foldy-Wouthuysen transformation, we have

$$S_2 = -\frac{i}{2m}\,\beta\,\mathcal{O}_2, \quad \mathcal{O}_2 = -\frac{(\vec{\alpha}\cdot\vec{p})^3}{3m^2} + \frac{1}{4}\left\{\frac{r_s}{r},\vec{\alpha}\cdot\vec{p}\right\}. \tag{20}$$

The transformation is calculated as $H_{\rm FW} = e^{iS_2}\,H_2\,e^{-iS_2}$. Taking notice of the well-known identity $\vec{p}^{\,2}(\frac{1}{r}) = 4\,\pi\,\delta^{(3)}(\vec{r})$, the Foldy-Wouthuysen transformation gives the result

$$\begin{aligned} H_{\rm FW} &= \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3}\right) - \beta\,\frac{m\,r_s}{2\,r} \\ &\quad + \beta\left(-\frac{3r_s}{8m}\left\{\vec{p}^{\,2},\frac{1}{r}\right\} + \frac{3\pi r_s}{4m}\delta^{(3)}(\vec{r}) + \frac{3r_s}{8m}\,\frac{\vec{\Sigma}\cdot\vec{L}}{r^3}\right). \end{aligned} \tag{21}$$

In retrospect, this result is simple and also offers a straightforward physical interpretation as follows: First, we have the usual relativistic corrections to the kinetic energy. The second term on the right-hand side of Eq. (21) is the gravitational potential, duly decorated with a β prefactor, which ensures that antiparticles are attracted. The third term consists of a kinetic correction to the gravitational coupling and a Darwin (*Zitterbewegung*) term which is not experimentally relevant because it is located at the center of the planet, i.e., inside the Schwarzschild radius. However, for completeness, we should include this term. The very last term on the right-hand side of Eq. (21) is a spin-orbit coupling term, which is also decorated with a β prefactor. It commutes with the Dirac angular operator $K = \beta(\vec{\Sigma}\cdot\vec{L} + 1)$ and with the total angular momentum (orbital + spin), and the prefactor β ensures that the particle-antiparticle symmetry $E \leftrightarrow -E$ holds. The spin-orbit coupling term describes the gravitational spin-orbit coupling; it is in agreement with the classical result for the interaction of a spinning classical particle with the gravitational field, which is otherwise known as the geodesic precession or Fokker precession [45,46]. When comparing the spin-orbit term with Eq. (26) of Ref. [45], which has a prefactor of 3/2 instead of 3/8, one should take note that the spin operator carries a factor one half ($\vec{S} = \frac{1}{2}\,\vec{\Sigma}$), and there is an additional factor 2 in the definition of the Schwarzschild radius. The prefactor β in Eq. (21) describes the particle-antiparticle symmetry, which cannot be obtained based on classical considerations [45,46].

Let us conclude the discussion by pointing out a subtlety. One might think that, if the total angular momentum operator commutes with the Hamiltonian, then it should automatically commute with the Foldy-Wouthuysen transformed Hamiltonian. However, that is not the case. If A is an operator that commutes with the Hamiltonian, $[H_{\rm DS},A] = 0$, and $H_{\rm FW} = U\,H_{\rm DS}\,U^{-1}$, then the transformed operator $A_{\rm FW} = U\,A\,U^{-1}$ commutes with $H_{\rm FW}$. This can be seen as follows:

$$[H_{\rm FW},A_{\rm FW}] = U\,[H_{\rm DS},A]\,U^{-1} = 0\,. \tag{22}$$

So, if $\vec{J}$ commutes with $H_{\rm DS}$, then this does not automatically mean that $\vec{J}$ commutes with $H_{\rm FW}$. Let us recall that the total angular momentum $\vec{J}$ and the Dirac angular operator K are defined as follows:

$$\vec{J} = \vec{L} + \tfrac{1}{2}\vec{\Sigma}, \quad K = \beta(\vec{\Sigma}\cdot\vec{L} + 1). \tag{23}$$

We can establish the following commutator relations:

$$[H_{\rm DS}, K] = 0, \quad [H_{\rm DS}, \vec{J}] = \vec{0}, \tag{24a}$$

$$[H_{\rm FW}, K] = 0, \quad [H_{\rm FW}, \vec{J}] = \vec{0}. \tag{24b}$$

In view of (22), it is not a triviality to separately establish that both $\vec{J}$ and K commute with $H_{\rm DS}$ and $H_{\rm FW}$ individually, but the relations hold. Eigenfunctions of $H_{\rm DS}$ and of $H_{\rm FW}$ are eigenstates of $\vec{J}$ and K; i.e., the spinor components of the eigenfunction are $\chi_{\varkappa\mu}(\hat{r})$ (upper component) and $\chi_{-\varkappa\mu}(\hat{r})$ (lower component), and they have the same $j = |\varkappa| - 1/2$.

B. Transformation of the transition current

The main result derived in the current work concerns the Foldy-Wouthuysen Hamiltonian $H_{\rm FW}$ derived in Eq. (21). However, it is also interesting to investigate the gravitational corrections to the electromagnetic transition current. The transition current operator for the emission of transverse photons in flat space is given by the matrix-valued expression $j^i = \alpha^i \exp(i\vec{k}\cdot\vec{r})$; an illustrative discussion can be found in Refs. [30,44,47]. The Hermitian adjoint of this operator is obtained by the replacement $\exp(i\vec{k}\cdot\vec{r}) \to \exp(-i\vec{k}\cdot\vec{r})$, i.e., by the replacement of a photon emission by a photon absorption process.

The Dirac-Schwarzschild Hamiltonian (15) is coupled to an external electromagnetic field by the replacement $\vec{p} \to \vec{p} - e\vec{A}$, where $\vec{A}$ is the vector potential. The interaction Hamiltonian is $H_{\rm int} = -\vec{j}\cdot\vec{A}$. So with relativistic gravitational coupling included, the transition current reads

$$j^i = \frac{1}{2}\left\{1 - \frac{r_s}{r}, \alpha^i \exp(i\vec{k}\cdot r)\right\}. \tag{25}$$

We now employ the multipole expansion

$$\alpha^i \exp(i\vec{k}\cdot r) \approx \alpha^i + \alpha^i\,(i\vec{k}\cdot\vec{r}) - \tfrac{1}{2}\alpha^i(\vec{k}\cdot r)^2 \tag{26}$$

in Eq. (25). A subsequent calculation of the Foldy-Wouthuysen transformation $j^i_{\rm FW} = U\, j^i\, U^{-1}$ of the transition current [with $U = \exp(i S_2)\exp(i S_1)$] gives the result

$$\begin{aligned} j^i_{\rm FW} = {} & \frac{p^i}{m} - \frac{p^i\,\vec{p}^{\,2}}{2m} - \frac{i}{2m}(\vec{k}\times\vec{\sigma})^i + \frac{1}{2}\left\{\frac{p^i}{m}, (i\vec{k}\cdot\vec{r})\right\} \\ & - \frac{1}{4}\left\{(\vec{k}\cdot\vec{r})^2, \frac{p^i}{m}\right\} + \frac{1}{2m}(\vec{k}\cdot\vec{r})(\vec{k}\times\vec{\sigma})^i \\ & - \frac{3}{4}\left\{\frac{p^i}{m}, \frac{r_s}{r}\right\} + \frac{r_s}{2r}\frac{(\vec{\sigma}\times\vec{r})^i}{m\,r^2} \\ & - \frac{1}{2}\left\{(i\vec{k}\cdot\vec{r}), \left\{\frac{p^i}{m}, \frac{r_s}{r}\right\}\right\} + \frac{3 i r_s}{4r}\frac{(\vec{k}\times\vec{\sigma})^i}{m} \\ & + \frac{1}{4}\left\{\frac{r_s}{r}\,(i\vec{k}\cdot\vec{r}), \frac{p^i}{m}\right\}. \end{aligned} \tag{27}$$

In addition to the canonical corrections to the relativistic transition current [30,44,47] (kinetic corrections and magnetic coupling), this result contains a gravitational kinetic correction and gravitational corrections to the magnetic coupling.

IV. CONCLUSIONS

In this paper, we investigate the nonrelativistic limit of the gravitationally coupled Dirac Hamiltonian for a Dirac particle bound to the gravitational field of a planet. In order to calculate the relativistic corrections in the gravitational field, we carry out a Foldy-Wouthuysen transformation with relativistic corrections up to fourth combined third order in the momenta ($\vec{p}^{\,4}$) and first order in the gravitational constant G. Within our expansion, we have $|\vec{p}| \sim \xi\, m$, where m is the fermion mass. Our calculations include terms up to the order ξ^4 and to the combined fourth order $\chi\,\xi^3$, where $\chi = r_s/r$ is the gravitational expansion parameter. We verify that the equivalence principle holds for the gravitational interaction of particles and antiparticles based on an inspection of the functional form of the relativistic corrections, which are all proportional to the $\beta \equiv \gamma^0$ matrix [see Eqs. (5) and (21)]. The conceivable existence of a spin-gravity coupling, which would otherwise break parity and the equivalence principle, had been discussed in various studies in the literature; we find that the corresponding terms vanish.

It has recently been argued [48] that one might speculate about alternative forms for the relativistic spin operator, which might not be given by $\vec{\Sigma}$ but by a different operator which commutes with the Hamiltonian. However, the spin of the electron is not a constant of motion. Rather, the total angular momentum $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ is conserved (a particularly clear exposition can be found in Sec. 11.3 of Ref. [49]). Indeed, $\vec{J}$ (not $\vec{L}$ or $\vec{\Sigma}$) is an integral of the relativistic motion; if we could observe the spin of the electron in real time, then we would see it precess during the motion. Indeed, the spin of the electron couples to the orbital angular momentum, and this spin-orbit coupling term expresses that the spin is not conserved; i.e., it precesses while being coupled to the orbital angular momentum, and only the total angular momentum is conserved. This fact is precisely confirmed in our approach, and the gravitational analogs of the spin-orbit coupling and the *Zitterbewegung* term are found. We can thus uniquely identify the gravitational spin-orbit coupling term corresponding to the classical geodesic precession (Fokker precession) of a spinning object in the gravitational field [45], formulated within quantum mechanics and respecting the particle-antiparticle symmetry (prefactor β).

ACKNOWLEDGMENTS

The authors acknowledge helpful conversations with T. N. Moentmann, Professor M. I. Eides as well as support by the National Science Foundation (Grant No. PHY-1068547) and from the National Institute of Standards and Technology (precision measurement grant).

APPENDIX: FOLDY-WOUTHUYSEN TRANSFORMATION OF THE FREE DIRAC HAMILTONIAN

It is instructive to carry out the Foldy-Wouthuysen transformation of the free Dirac Hamiltonian in exactly the same two-step approach with two iterative transformations as described in Ref. [30] for the Dirac-Coulomb Hamiltonian (keeping in mind that the Foldy-Wouthuysen transformation of the free Dirac Hamiltonian can otherwise be carried out in a single step). The free Dirac Hamiltonian is given as

$$H_{\rm FD} = H_1 = \vec{\alpha}\cdot\vec{p} + \beta m. \tag{A1}$$

For the first Foldy-Wouthuysen transformation, we have

$$S_1 = -\frac{i}{2m}\,\beta\,\mathcal{O}_1\,, \quad \mathcal{O}_1 = \vec{\alpha}\cdot\vec{p}. \tag{A2}$$

The transformation is calculated as

$$H_2 = e^{iS_1}\,H_1\,e^{-iS_1}. \tag{A3}$$

This leads to the result

$$H_2 = \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3}\right) - \frac{(\vec{\alpha}\cdot\vec{p})^3}{3m^2}. \tag{A4}$$

For the second Foldy-Wouthuysen transformation, we have

$$S_2 = -\frac{i}{2m}\,\beta\,\mathcal{O}_2, \quad \mathcal{O}_2 = -\frac{(\vec{\alpha}\cdot\vec{p})^3}{3m^2}. \tag{A5}$$

The transformation is calculated as

$$H_{\rm FW} = e^{iS_2}\,H_1\,e^{-iS_2}. \tag{A6}$$

The transformed Hamiltonian is

$$H_{\rm FW} = \beta\left(m + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3}\right). \tag{A7}$$

This result corresponds to our Eq. (21) and is the limit of vanishing r_s, as it should be.

All unitary transformations discussed here respect the basic symmetries of the Dirac Hamiltonian such as parity. This is essential; as an extreme example, let us briefly supplement the discussion by considering the nonrelativistic free Schrödinger Hamiltonian $H_0 = \vec{p}^{\,2}/(2m)$ and the unitary transformation $U = \exp(i\vec{A}\cdot\vec{r})$, where $\vec{A}$ is a constant vector. Then, $H_0' = U\,H_0\,U^+ = (\vec{p}-\vec{A})^2/(2m)$. Upon binomial expansion, one obtains a term proportional to $\vec{A}\cdot\vec{p}$, which breaks parity. However, the parity-breaking term in H_0' is an artifact generated by the parity-breaking unitary transformation. As a further illustrative remark, let us consider the transformation S_1, given in Eq. (A2), under parity,

$$S_1 = -\frac{i}{2m}\,\beta\,\vec{\alpha}\cdot\vec{p}$$
$$\stackrel{\mathcal{P}}{\to} \beta\left[-\frac{i}{2m}\,\beta\,\vec{\alpha}\cdot(-\vec{p})\right]\quad \beta = \frac{i}{2m}\,\vec{\alpha}\cdot\vec{p}\;\beta = S_1. \tag{A8}$$

By construction, the iterative Foldy-Wouthuysen transformations discussed here respect parity symmetry, and so does the final result given in Eq. (21).

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