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Analyses of Thin-Walled Sections under Localised Loading for General End Boundary Conditions – Part I: Pre-Buckling

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Analyses of thin-walled sections under localised loading for general end boundary conditions – Part 1: Pre-buckling

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Abstract

The Semi-Analytical Finite Strip Method (SAFSM) for pre-buckling analysis of thin-walled sections under localised loading has been developed for general end boundary conditions. For different boundary conditions at supports and loading point, different displacement functions are required for both flexural and membrane displacements. As the stresses are not uniform along the member due to localised loading, the pre-buckling analysis also requires multiple series terms with orthogonal functions.

This paper briefly summaries the displacement functions used for different boundary conditions. In addition, the theory of the SAFSM for pre-buckling analysis of thin-walled sections under localised loading with general end boundary conditions is developed. The analysis is benchmarked against the Finite Element Method (FEM) using software package ABAQUS/Standard. The results from this pre-buckling analysis are deflections (pre-buckling modes) and membrane stresses which are used for the buckling analysis described in Part 2 - Buckling in the companion paper.

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1. **Introduction**

In order to carry out a buckling analysis of a thin-walled member under localised loading, it is necessary to compute the pre-buckling membrane stresses in the member. The Part 1 - Pre-buckling analysis described in this paper is an important step which provides stresses for conducting the buckling analysis described in Part 2 in the companion paper.

The analysis of structural members can be performed by a variety of methods. Two of the most popular numerical methods are the Finite Element Method (FEM) and Finite Strip Method (FSM). While the FEM allows the analysis of structural members with all kinds of geometry and general boundary conditions, the FSM provides analysis of structural members with complex geometry in their section, but simple along the length. For particular types of structures such as thin-walled sections, the FSM can be extremely competitive in terms of computational efficiency due to the simplicity of displacement functions and the decrease in number of degrees of freedom.

The first application of the SAFSM was presented by Cheung (1976). This method was first used for buckling analysis by Przemieniecki (1973) to study the initial local buckling stresses of plates and plate assemblies under biaxial compression. Bradford and Azhari (1995) used two sets of displacement functions in the buckling analysis of plates for different ends boundary conditions using the SAFSM. Their first basic functions were derived from the solution of the beam vibration differential equations employed by Cheung (1976) to study plate vibration. However, in static analyses of structural members under localised loading for some boundary conditions such as the Clamped-Clamped case, the shear stress at the ends of the structural member is equal to zero. It is an impossible situation in a beam as there is no reaction to resist the applied load at the supports. The second basic functions used by Bradford and Azhari are trigonometric functions, and satisfy the boundary conditions. However, in the Clamped-Clamped case, the displacement functions are fairly complex with the product of two sine functions which cause difficulty in solving the integrations in both pre-buckling and buckling analyses.

In this Part 1 - Pre-buckling, the paper summaries the displacement functions for different end boundary conditions of structural members. The theory of the SAFSM for pre-buckling analysis of thin walled sections under localised loading for general end boundary conditions is given as also built into the THIN-WALL-2 program developed by the authors (Nguyen, Hancock, & Pham, 2015). Numerical examples have been performed using the THIN-WALL-2 program and compared with the results from the analyses by the FEM using ABAQUS (ABAQUS/Standard Version 6.13, 2013) to validate the accuracy of the SAFSM against the FEM. The results from the pre-buckling analysis step are membrane stresses and deflections of the structural member which are used for the buckling analysis described in Part 2 – Buckling in the companion paper. A convergence study of deflections and stresses with the number of series terms is also provided in this paper.

2. Displacement functions

2.1. Choice of displacement functions

In the Finite Strip Method (FSM), it is seen that the choice of suitable displacement functions for a strip is the most important stage of the analysis, and great care must be exercised at such a stage. An incorrectly chosen displacement function may lead to results which converge to incorrect answers for successively refined meshes. The FSM can be considered as a special form of the FEM procedure using the displacement approach. Unlike the standard FEM which uses the polynomial displacement functions in all directions, the FSM calls the use of simple polynomials in the transverse direction and continuously differentiable smooth series in the longitudinal direction, with the stipulation that such series should satisfy the boundary conditions at the ends of the strips. The displacements of a strip are a combination of the flexural displacements perpendicular to the strip and membrane displacements in the plane of the strip. Generally, the form of the displacement function is given as a product of polynomials and smooth series.

2.2. The flexural displacement functions of a strip

An isometric view of flexural displacements of a strip is shown in Fig.1

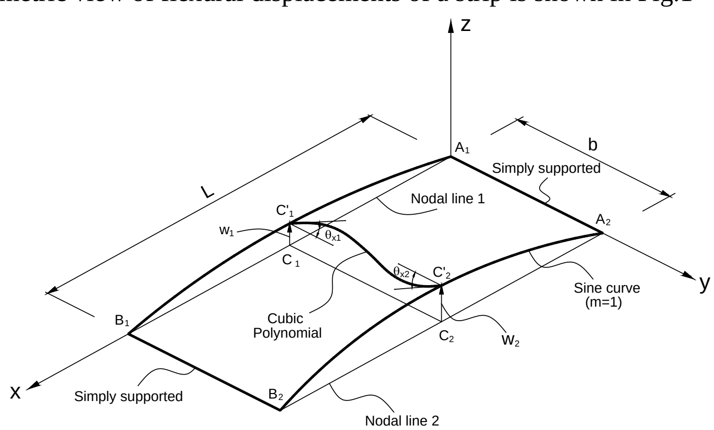


Figure 1: Flexural displacements of a strip with both ends simply supported

The flexural deformations w of a strip can be described by the summation over μ series terms as:

$$w = \sum_{m=1}^{\mu} f_{1m}(y) X_{1m}(x) \qquad (1)$$

where:

μ is the number of series terms of the harmonic longitudinal function

$X_{1m}(x)$ is the curve for longitudinal variation

$f_{1m}(y)$ is a polynomial for transverse variation. This function for the mth series term is given by:

$$f_{1m}(y) = \alpha_{1Fm} + \alpha_{2Fm}\left(\frac{y}{b}\right) + \alpha_{3Fm}\left(\frac{y}{b}\right)^2 + \alpha_{4Fm}\left(\frac{y}{b}\right)^3 \qquad (2)$$

$\{\alpha_{Fm}\}$ are the vector polynomial coefficients for the mth series term which depend on the nodal line flexural deformations of the strip

$$\{\alpha_{Fm}\} = [\alpha_{1Fm} \quad \alpha_{2Fm} \quad \alpha_{3Fm} \quad \alpha_{4Fm}]^T$$

b and *L* are the strip width and length respectively.

2.3. The membrane displacement functions of a strip

An isometric view of membrane displacements of a strip is shown in Fig.2

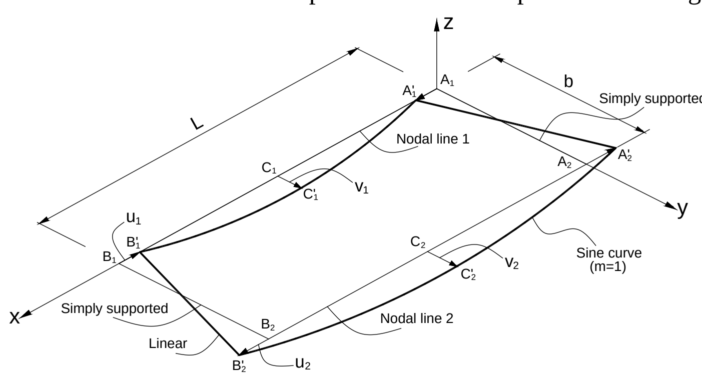


Figure 2: Membrane displacements of a strip with both ends simply supported

The membrane deformations in the longitudinal and transverse directions of a strip can be described by the summation over μ series terms as:

$$v = \sum_{m=1}^{\mu} f_{vm}(y) X_{1m}(x) \qquad (3)$$

$$u = \sum_{m=1}^{\mu} f_{um}(y) X_{2m}(x) \qquad (4)$$

where:

$X_{1m}(x)$ and $X_{2m}(x)$ are the longitudinal variation curves for the membrane transverse v and longitudinal u deformations respectively

$f_{vm}(y)$ and $f_{um}(y)$ are the transverse variations. These functions for the mth series term are given by:

$$f_{vm}(y) = \alpha_{1Mm} + \alpha_{2Mm}\left(\frac{y}{b}\right) \quad (5)$$

$$f_{um}(y) = \alpha_{3Mm} + \alpha_{4Mm}\left(\frac{y}{b}\right) \quad (6)$$

$\{\alpha_{Mm}\}$ is the vector of polynomial coefficients for the mth series term which depends on the nodal line membrane deformations of the strips

$$\{\alpha_{Mm}\} = \left[\alpha_{1Mm} \quad \alpha_{2Mm} \quad \alpha_{3Mm} \quad \alpha_{4Mm}\right]^T$$

2.4. Available displacement functions for different boundary conditions

2.4.1. Both ends simply supported (SS)

The displacement functions by Cheung (1976) are:

$$X_{1m}(x) = \sin\left(\frac{m\pi x}{L}\right) \quad (7)$$

$$X_{2m}(x) = \cos\left(\frac{m\pi x}{L}\right) \quad (8)$$

2.4.2. One end simply supported and the other end clamped (SC)

The displacement functions by Cheung (1976) are:

$$X_{1m}(x) = \sin\left(\frac{\mu_m x}{L}\right) - \alpha_m \sinh\left(\frac{\mu_m x}{L}\right) \quad (9)$$

$$X_{2m}(x) = \cos\left(\frac{\mu_m x}{L}\right) - \alpha_m \cosh\left(\frac{\mu_m x}{L}\right) \quad (10)$$

with $\mu_m = 3.9266, 7.0685, 10.2102, ..., \frac{4m+1}{4}\pi$

$m = 1, 2, 3, ..., \infty$ and $\alpha_m = \frac{\sin \mu_m}{\sinh \mu_m}$

2.4.3. One end simply supported and the other end free (SF)

The displacement functions by Cheung (1976) are:

Case 1: $m = 1$ and $\mu_1 = 1$

$$X_{11}(x) = \frac{x}{L} \text{ and } X_{21}(x) = 1 \tag{11}$$

Case 2:

$$m = 2,3,4,5,...,\infty \text{ and } \alpha_m = \frac{\sin \mu_m}{\sinh \mu_m}$$

$$\mu_m = 3.9266, 7.0685, 10.2102, 13.3520, ..., \frac{4m-3}{4}\pi$$

$$X_{1m}(x) = \sin\left(\frac{\mu_m x}{L}\right) + \alpha_m \sinh\left(\frac{\mu_m x}{L}\right) \tag{12}$$

$$X_{2m}(x) = \cos\left(\frac{\mu_m x}{L}\right) + \alpha_m \cosh\left(\frac{\mu_m x}{L}\right) \tag{13}$$

2.4.4. Both ends clamped (CC)

The displacement functions by Cheung (1976) are:

$$X_{1m}(x) = \sin\left(\frac{m\pi x}{L}\right) \tag{14}$$

$$X_{2m}(x) = \sin\left[\frac{(m+1)\pi x}{L}\right] \tag{15}$$

These functions were selected by Cheung (1976) in Chapter 3 to satisfy equilibrium at the ends.

2.4.5. One end clamped and the other end free (CF)

The displacement functions by Bradford and Azhari (1995) are:

$$X_{1m}(x) = 1 - \cos\left[\left(m - \frac{1}{2}\right)\frac{\pi x}{L}\right] \tag{16}$$

$$X_{2m}(x) = \left(\frac{2m-1}{2m}\right)\sin\left[\left(m - \frac{1}{2}\right)\frac{\pi x}{L}\right] \tag{17}$$

These functions have been chosen as they are simpler to implement in Part 2 - Buckling described later.

2.4.6. Both ends free (FF)

The new displacement functions which are used in this paper are:

Case 1: $m = 1$

$$X_{11}(x) = 1 \text{ and } X_{21}(x) = 0 \tag{18}$$

Case 2: $m = 2$

$$X_{12}(x) = 1 - \frac{2x}{L} \text{ and } X_{22}(x) = -\frac{1}{\pi} \tag{19}$$

Case 3: $m \geq 3$

$$X_{1m}(x) = 1 - 2\sin\left[\frac{(2m-5)\pi x}{L}\right] \tag{20}$$

$$X_{2m}(x) = -2\left(\frac{2m-5}{m}\right)\cos\left[\frac{(2m-5)\pi x}{L}\right] \tag{21}$$

These functions have been chosen as they are simpler to implement in Part 2 - Buckling described later.

3. Load vector

The localised load applied on the structural member is assumed to be line loads as shown in the Fig.3. The loads may be applied in different directions and at any position along the structural member.

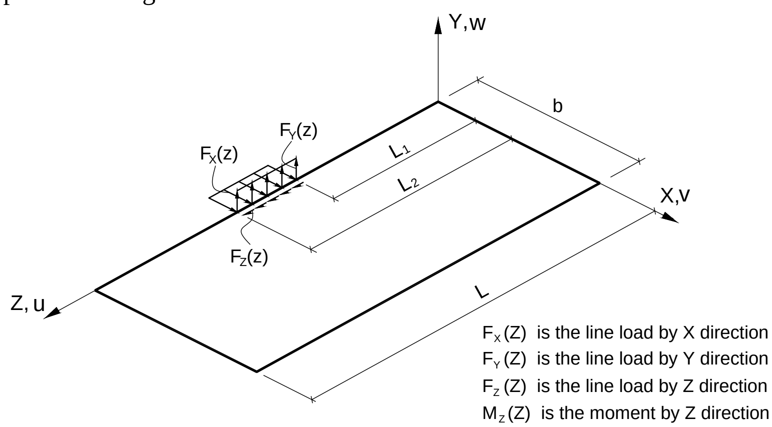


Figure 3: Localised loading applied on a strip

The deformation of the nodal line u,v,w in Z,X,Y directions is given by:

$$\begin{aligned} v &= V_{1m} X_{1m}(Z) \\ w &= W_{1m} X_{1m}(Z) \\ u &= U_{1m} X_{2m}(Z) \end{aligned} \tag{22}$$

where:

$X_{1m}(x)$ is the longitudinal variation curve for the membrane transverse deformation (v), also for the flexural deformation

$X_{2m}(x)$ is the longitudinal variation curve for the membrane longitudinal deformation (u)

U_{1m}, V_{1m}, W_{1m} are amplitude deformations of the loaded nodal line for the mth series term

The terms in the load vector can be derived from the potential energy of the external forces to be:

$$W_{Xm} = \int_{L_1}^{L_2} F_X(Z) X_{1m}(Z) dZ \; ; \; W_{Ym} = \int_{L_1}^{L_2} F_Y(Z) X_{1m}(Z) dZ$$
$$W_{Zm} = \int_{L_1}^{L_2} F_Z(Z) X_{2m}(Z) dZ \; ; \; W_{Mm} = \int_{L_1}^{L_2} M_Z(Z) X_{1m}(Z) dZ \tag{23}$$

where:

L_1 and L_2 are the starting and ending points of the line loads respectively as shown in Fig.3

$F_X(Z), F_Y(Z), F_Z(Z)$ and $M_Z(Z)$ are the distributed lines load in the X, Y, Z directions. These loads may be constant or vary with Z

W_{Xm}, W_{Ym}, W_{Zm} and W_{Mm} are the X,Y,Z and M components of the load vector for each nodal line for the mth series terms.

4. Strain energy and potential energy

In order to compute the stiffness matrix of a strip according to conventional finite strip theory (Cheung, 1976), it is necessary to define the strain energy in the strip under deformation and the potential energy of the external forces.

4.1. Strain energy of a strip

The flexural strain energy U_F is given by:

$$U_F = \frac{1}{2} \int_0^L \int_0^b \left(-M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) dy dx \tag{24}$$

$$\Rightarrow \; U_F = \frac{1}{2} \int_0^L \int_0^b \sum_{m=1}^{\mu} \sum_{n=1}^{\mu} \{\sigma_{Fm}\}^T \{\in_{Fn}\} dy dx \tag{25}$$

where $\{\sigma_{Fm}\}$ and $\{\in_{Fn}\}$ are the flexural stress and strain vectors respectively

The membrane strain energy U_M is given by:

$$U_M = \frac{1}{2}\int_0^L\int_0^b\left(\sigma_x \in_x + \sigma_y \in_y + \tau_{xy}\gamma_{xy}\right)tdydx \qquad (26)$$

$$\Rightarrow \quad U_M = \frac{1}{2}\int_0^L\int_0^b\sum_{m=1}^{\mu}\sum_{n=1}^{\mu}\{\sigma_{Mm}\}^T\{\in_{Mn}\}tdydx \qquad (27)$$

where $\{\sigma_{Mm}\}$ and $\{\in_{Mn}\}$ are the membrane stress and strain vectors respectively

4.2. Potential energy of the external forces

The potential energy of the external forces is given by:

$$V_W = -\int_0^L F(Z)X_m(Z)dZ \qquad (28)$$

where $F(Z)$ and $X_m(Z)$ are the line load and displacement functions respectively for different directions.

5. Stiffness matrix

The flexural strain energy U_F from equation (25) is rewritten as given:

$$U_F = \{\delta_{Fm}\}^T\left[k_{Fmn}\right]\{\delta_{Fn}\} \qquad (29)$$

where $\left[k_{Fmn}\right]$ is the flexural stiffness matrix corresponding to the mth and nth series terms and $\left[\delta_{Fn}\right]$ is the flexural displacement vector of a strip corresponding to the nth series term. The matrix $\left[k_{Fmn}\right]$ is given in the Research Report 958 (Nguyen, Hancock, & Pham, 2016). The coefficients I_{1F}, I_{2F}, I_{3F}, I_{4F}, I_{5F} in the report have been evaluated exactly for the displacement functions satisfying the different boundary conditions described in 2.4

The membrane strain energy U_M from equation (27) is rewritten as:

$$U_M = \{\delta_{Mn}\}^T\left[k_{Mmn}\right]\{\delta_{Mn}\} \qquad (30)$$

where $\left[k_{Mmn}\right]$ is the membrane stiffness matrix corresponding to the mth and nth series terms and $\left[\delta_{Mn}\right]$ is the membrane displacement vector of a strip corresponding to the nth series term. The matrix $\left[k_{Mmn}\right]$ is given in the Research Report 958 (Nguyen et al., 2016). The coefficients I_{1M}, I_{2M}, I_{3M}, I_{4M}, I_{5M}, I_{6M}, I_{7M}, I_{8M} in the report have been evaluated exactly for the displacement functions satisfying the different boundary conditions described in 2.4.

The stiffness matrix of a strip is assembled from both the flexural stiffness matrix and the membrane stiffness matrix in local coordinates. These matrices are transformed to global coordinates by a multiplication with transformation matrices. The stiffness matrix of the whole section for each series term is assembled from the stiffness matrices of individual strip. Finally, the complete stiffness matrix of the whole section is assembled from the stiffness matrices taken over the series terms, thus the size of this matrix is 4 times the node number and times the number of series terms.

6. Pre-buckling analysis

The total potential energy is the sum of the elastic strain energy stored in a strip and the potential energy of the external loads, thus:

$$\phi = U + V_W \tag{31}$$

The principle of minimum total potential energy requires that:

$$\left\{ \frac{\partial \phi}{\partial \{\delta_p\}} \right\} = \{0\} \tag{32}$$

Thus, we have:

$$[K]\{\delta_p\} = \{W\} \tag{33}$$

where $[K]$ is the system stiffness matrix based on a strip subdivision of a thin-walled section, $\{\delta_p\}$ are the nodal line displacements (pre-buckling modes) of strips in the global X,Y,Z axes, and $\{W\}$ are the nodal line forces (line loads) given by Eq.(23).

The amplitude of the pre-buckling displacements is obtained from Equation (33). These values are multiplied with the displacement functions to get the pre-buckling deformations for all sections along the structural member.
The membrane stresses of a strip are given by:

$$\{\sigma_{Mm}\} = [D_M]\{\in_{Mm}\} \tag{34}$$

where $\{\in_{Mm}\}$ is the membrane strain vector:

$$\{\in_{Mm}\} = [B_{Mm}]\{\alpha_{Mm}\} \tag{35}$$

The summation can be taken over the m series terms at any longitudinal position to get the membrane stresses for all sections:

$$\{\sigma_M\} = \sum_{m=1}^{\mu} \{\sigma_{Mm}\} \tag{36}$$

7. Numerical example

A pre-buckling analysis has been performed for a lipped channel section with rounded corners and lips under localised loading using the THIN-WALL-2 program. The geometry of the beam and the loading are shown in Fig.4. The beam is analysed with different boundary conditions for the web and the flanges of the end sections. In addition, lateral restraints are applied along the beam at Nodal Lines 11 and 35 to avoid twisting caused by eccentric loading. The results from the pre-buckling analysis of the beam under localised loading include deflections and stresses. The stress and deflection values are obtained from Nodal Line 23 in the middle of the section for all sections along the beam.

The beam has also been analysed using a pre-buckling analysis by ABAQUS with an equivalent loading and boundary conditions. It was meshed into 5mm x 5mm, except at the section's corners. The corners were modelled with 1mm x 5mm mesh to accurately represent the influence of corner radius. The stress and deflection values are obtained from a group of nodes at the same positions as the nodal lines from THIN-WALL-2.

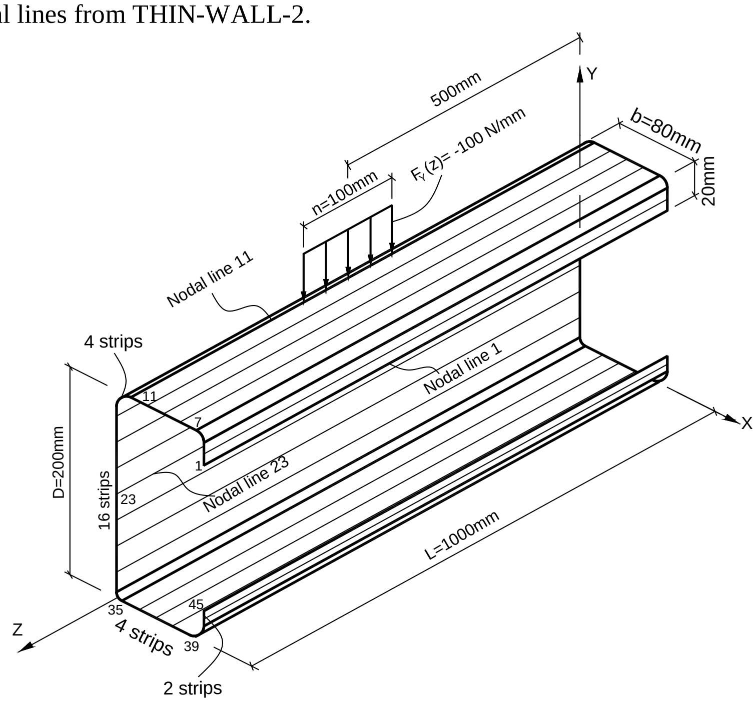
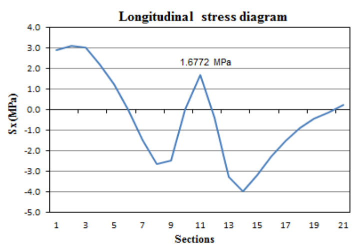


Figure 4: Lipped channel section under localised loading

The comparison between the stresses and deflections from the SAFSM and the FEM are shown in Table 1 and Table 2 for the Clamped - Free (CF) case which uses the Bradford and Azhari (1995) displacement functions. The results for other boundary conditions can be seen in the Research Report 958 (Nguyen et al, 2016). The comparison demonstrates the accuracy of the SAFSM when 15 series terms are used particularly for the transverse and shear stresses. There is a small difference in the local peak of the longitudinal stress at the centre but this is unlikely to have an effect on the buckling analysis in the companion paper – Part 2 - Buckling.

Table 1: Stress comparison for CF case (Nodal Line 23)

<table>
<tr><th></th><th>SAFSM (THIN-WALL-2)
(15 series terms)</th><th>FEM (Abaqus)</th></tr>
<tr><td>S_x (MPa)</td><td>


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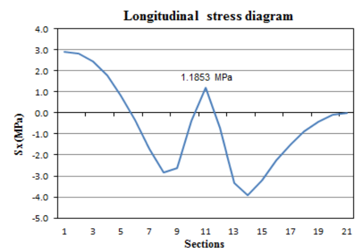

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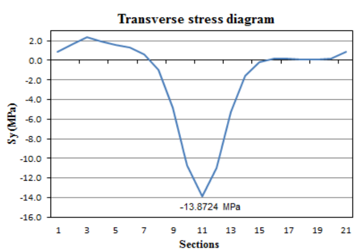
<tr><td>S_y (MPa)</td><td>


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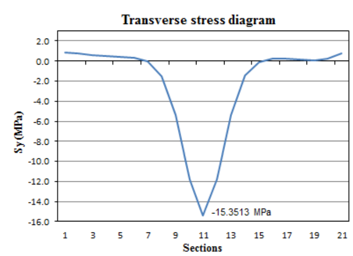

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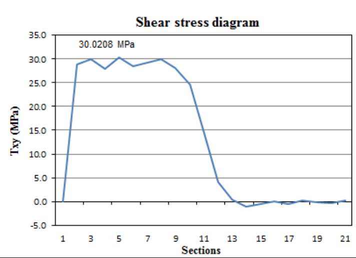
<tr><td>T_{xy} (MPa)</td><td>


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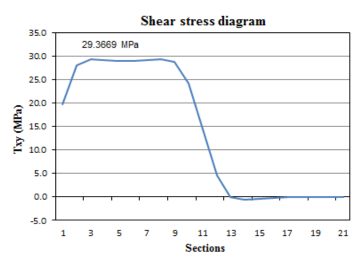

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Table 2: Deflection comparison for CF case (Nodal Line 23)

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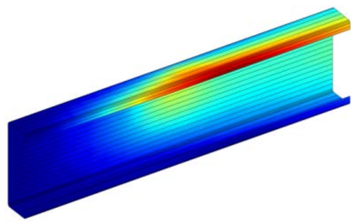
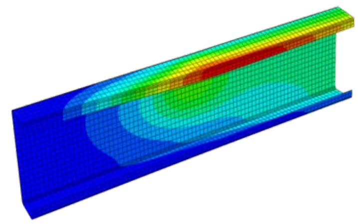
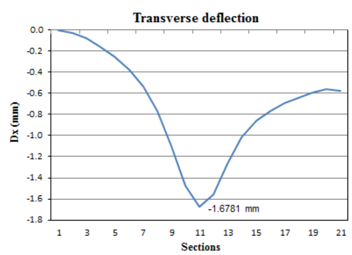
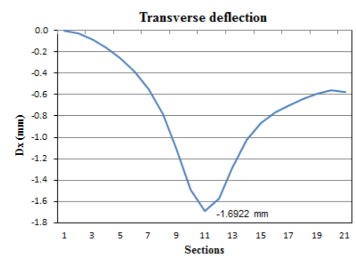
(15 series terms)</th><th>FEM (Abaqus)</th></tr>
<tr><td>Mode</td><td></td><td></td></tr>
<tr><td>D_x
(mm)</td><td>
</td><td>
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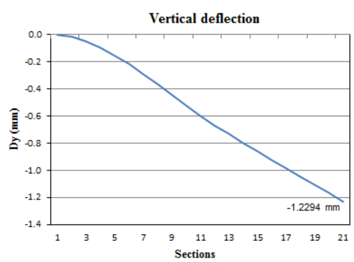
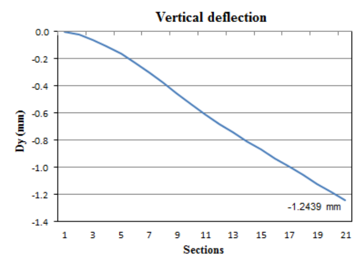
(mm)</td><td>
</td><td>
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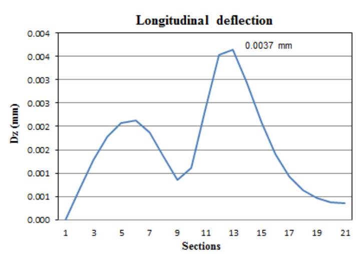
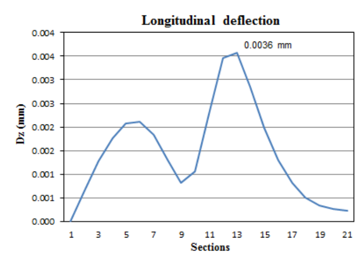
(mm)</td><td>
</td><td>
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8. Convergence study

A study has been performed for the lipped channel section in 7 with different boundary conditions and different numbers of series terms to find the acceptable number of series terms for the pre-buckling analysis. The relationships between the longitudinal stress at Nodal Line 23, Section 11 at the middle of the beam and the number of series terms are shown in Fig.5 for different boundary conditions. There is convergence of the longitudinal stress when the number of series terms reaches 25 in comparison with ABAQUS as shown in Table 1. It means that about 25 series terms are required to get the converged stresses as well as deflections in the pre-buckling analysis for a localised load one tenth the length of the member.

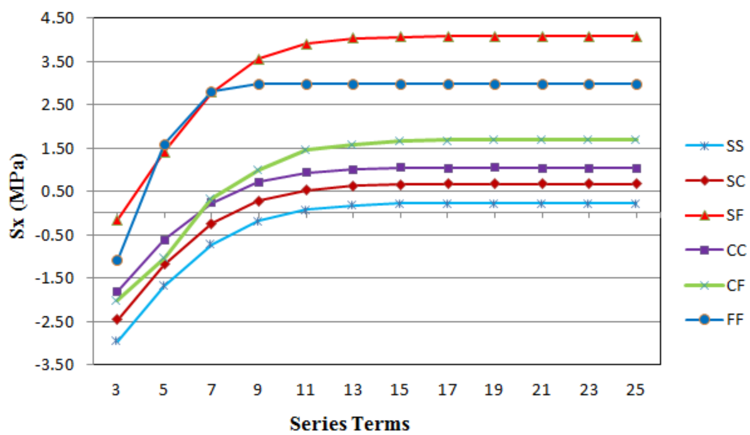

Figure 5: Convergences of longitudinal stress (S_x)

9. Conclusion

A Semi-Analytical Finite Strip Method of pre-buckling analysis of thin-walled section under localised loading has been developed for general end boundary conditions. This method has been benchmarked against the Finite Element Method.

Suitable displacement functions are used for different support and loading conditions for both flexural and membrane displacements. For a load over one-tenth of the span, about 25 series terms are required in the analysis process to get accurate pre-buckling results, particularly stress.

Acknowledgement

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