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Bound $\mu^+\mu^-$ system

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We consider the hyperfine structure, the atomic spectrum, and the decay channels of the bound $\mu^+\mu^-$ system (dimuonium). The annihilation lifetimes of low-lying atomic states of the system lie in the 10^{-12} s range. The decay rates could be measured by detection of the decay products (high-energy photons or electron-positron pairs). The hyperfine-structure splitting of the dimuonic system and its decay rate are influenced by electronic vacuum polarization effects in the far timelike asymptotic region. This constitutes a previously unexplored kinematic regime. We evaluate next-to-leading-order radiative corrections to the decay rate of low-lying atomic states. We also obtain order $\alpha^5 m_\mu$ corrections to the hyperfine splitting of the $1S$ and $2S$ levels. [S1050-2947(97)06511-6]

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I. INTRODUCTION

The bound system consisting of two muons (dimuonium) can be produced in heavy nuclei inelastic scattering at high energies and in particle decays. The decay of the neutral η^0 meson into dimuonium has been investigated theoretically by Nemenov [1] ($\eta^0 \rightarrow \text{dimuonium} + \gamma$). The formation of dimuonium in pion-proton collisions ($\pi^- + p \rightarrow \text{dimuonium} + n$) and by photons on nuclei ($\gamma + Z \rightarrow \text{dimuonium} + Z$) has been discussed in [2]. For the direct production of dimuonium in muon-antimuon collisions, considerable experimental difficulties associated with slow muon beams would have to be overcome. Another possible pathway for the production of the system, which we do not discuss in any further detail here, is the e^+e^- annihilation (near or above the μ threshold). Dimuonium, once produced, undergoes atomic decay (into energetically lower atomic states) and annihilation decay (into electrons and photons). Because the annihilation products are hard photons and relativistic electron-positron pairs, the decays could be investigated experimentally by established methods of particle physics.

In this work, we devote special attention to the annihilation decay rates of low-lying atomic states. The decay rate of the $S=1$ orthostates of dimuonium has been evaluated by Malenfant in pioneering investigations [3,4]. The name “dimuonium” was proposed in [4]. This work can therefore be regarded as a continuation of the earlier investigations on the system. We try to refine the analysis of the orthostates and we investigate in addition the $S=0$ parastates. We obtain corrections to the decay rate in next-to-leading order.

In this work we also investigate the hyperfine structure in

next-to-leading order. We further discuss briefly the atomic spectrum of dimuonium. The contribution of electronic vacuum polarization to the hyperfine structure and to the decay rate of orthostates lies in the far timelike asymptotic region. Its observation would constitute a test of QED in this previously unexplored kinematic regime. Due to the small length scale of the system, the hyperfine structure of the dimuonic atom is influenced by the effect of hadronic vacuum polarization (at the level of one part in 10^3).

Among the exotic atomic systems, some attention recently has been devoted to ponium (the $\pi^+\pi^-$ system). The spectrum and decay channels of ponium have been discussed extensively (see, e.g., [5–7]). The production of ponium has been observed recently [8]. The formation of bound atomic systems in particle decays has been observed earlier. Positronium has been detected in the decay $\pi^0 \rightarrow \text{Ps} + \gamma$ [9]. The bound system of a pion and a muon ($\pi\mu$) has been observed in the decay $K_L^0 \rightarrow (\pi\mu \text{ atom}) + \nu$ [10].

In contrast to the short-lived $\pi^+\pi^-$ system, where $c\tau$ is in the range of $1 \mu\text{m}$, $c\tau$ is in the range of 1 mm for dimuonium, so that the atom could leave the beam target after production. In any of the cases discussed the production rate is expected to be proportional to the atomic wave function at the origin $|\psi_{nl}(0)|^2 = \alpha^3 m_\mu^3 \delta_{l0} / 8\pi n^3$ (we use relativistic units in which $\hbar = c = 1$ and $\alpha = e^2$). Hence only low-lying states with a zero orbital moment (S states) are expected to be produced.

This paper is organized as follows. We first study the spectrum of the system including radiative corrections (Lamb shift) in Sec. II. We then proceed to an investigation of the hyperfine structure in Sec. III. As the highest accuracy can presumably be obtained in decay rate measurements, we devote special attention to the decay channels, which are investigated in detail in Secs. IV and V we conclude in Sec. VI.

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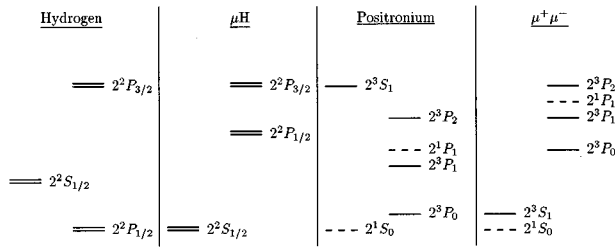


FIG. 1. Overview over the spectrum of $n=2$ levels in hydrogen, muonic hydrogen (μH), positronium, and dimuonium ($\mu^+\mu^-$). Double lines denote the hyperfine-structure splitting of the levels in hydrogen and muonic hydrogen, dashed lines denote $S=0$ states, and full lines denote $S=1$ states.

II. ATOMIC SPECTRUM OF DIMUONIUM

The dimuonic atom is an analog of the hydrogen, muonic hydrogen, and positronium atoms. The main feature of muonic hydrogen (μH) that differs from properties of hydrogen is the order of magnitude of the Lamb shift relative to the fine-structure (fs) splitting (see, e.g., [11]). The fs splitting in μH and H (also in positronium and dimuonium) is of the order of $\alpha^4 m_r$, where m_r denotes the reduced mass of the system, which is roughly equal to the mass of the lighter particle in a system with a heavy nucleus. In hydrogen the Lamb shift is of the order of $\alpha^5 m_r$, whereas in muonic systems the Lamb shift is of the order of $\alpha^3 m_r$ because of the large effect of vacuum polarization. The above considerations are parametrically true for all states, but numerical coefficients turn out to be largest for S states. Because the coefficients to the Lamb shift are negative (the Uehling potential is attractive), S states are significantly lower in energy in muonic systems. In Fig. 1 we present an overview over the spectrum of hydrogen, μH , positronium, and $\mu^+\mu^-$.

In both hydrogen and μH , the hyperfine structure is a smaller effect of order $\alpha^4(m_r^2/M)$ (M denotes the mass of the nucleus). This is different in positronium. Positronium has a more complicated spectrum than the hydrogen atom. Due to the larger magnetic moment of the “central” particle (and its smaller mass) and due to virtual pair-annihilation processes, the effects that are responsible for the fine and hyperfine structures and for the Lamb shift have the same order of magnitude: $\alpha^4 m_r$. For positronium as well as for hydrogen, the fine structure is defined as the separation of levels with the same principal and angular orbital momentum quantum numbers.

In positronium, different quantum numbers have to be used for the classification of levels from those in an atom with a heavy nucleus. For a system with a heavy nucleus, the total electron moment $j=l+s$ is conserved within the central field approximation. States are classified as $n l j F$. In positronium the total spin $S=s_1+s_2$ is an exactly conserved quantity. States are classified as $n l S F$ (denoted as $n^{2S+1} l_F$), where $S=1$ for an orthostate and $S=0$ for parastates. In all systems considered we denote the total angular atomic moment by F . Although the atomic spectrum of dimuonium is determined by a large Lamb shift of order $\alpha^3 m_r$ (as is the case for μH), the classification of levels is like in positronium (quantum numbers $n l S F$).

There exists no close analogy of dimuonium with pionium

because pionium contains only spinless particles and the spectrum includes neither a fine nor a hyperfine structure. Due to the strong interaction, the hadronic pionium system lives only a very short time and the atomic structure cannot be investigated (because the decay rate exceeds the rate of atomic transitions). By contrast, for P states in dimuonium, the atomic transition $nP \rightarrow n'S$ ($n' < n$) dominates over the annihilation decay rate. The lifetime of the atomic levels is considered in Sec. III.

The main contribution to the Lamb shift in dimuonium is caused by the one-loop electronic vacuum polarization. Due to the small spatial separation of the two muons in the $\mu^+\mu^-$ system, the charge of the two particles, which is larger than the effective charge to be observed at larger distance, is less screened and causes a deviation of the energy levels of the order $\alpha^3 m_r$. For hydrogen, the vacuum polarization enters at the order of $\alpha^5 m_r$, but for dimuonium, two powers of α are compensated for by the muon-electron mass ratio (or, by employing a coordinate space picture, by the smaller separation of the particles).

We now turn to a brief discussion of the atomic levels of dimuonium. The effect of electronic vacuum polarization on the spectrum is investigated. Any vacuum polarization insertion in the photon line can be represented as a substitution of the form

$$\frac{1}{q^2 + i\epsilon} \rightarrow \frac{\alpha}{\pi} \int_{s_0}^{\infty} ds \, \rho(s) \frac{1}{q^2 - s + i\epsilon} \quad (1)$$

in the photon propagator. The integration has to be performed from the threshold s_0 of pair production of the loop particle. For electronic vacuum polarization, the spectral function is given by (see, for instance, [12], p. 323)

$$\rho(s) = \frac{1}{3s} \sqrt{1 - \frac{4m_e^2}{s}} \left(1 + \frac{2m_e^2}{s} \right). \quad (2)$$

The substitution

$$v^2 = 1 - \frac{4m_e^2}{s} \quad (3)$$

brings the one-loop vacuum polarization integral into the form [13]

$$\frac{1}{q^2 + i\epsilon} \rightarrow \frac{\alpha}{\pi} \int_0^1 dv \, \frac{v^2(1-v^2/3)}{1-v^2} \frac{1}{q^2 - \lambda^2 + i\epsilon}. \quad (4)$$

For spacelike momentum transfer, this can be Fourier transformed into coordinate space and yields the Uehling potential

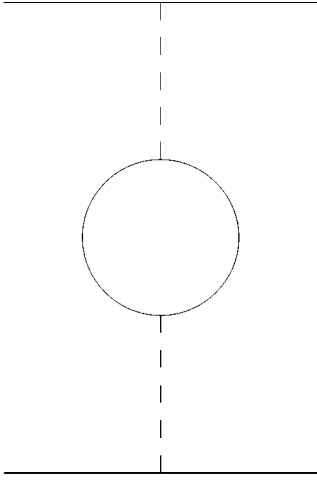


FIG. 2. Electronic vacuum polarization insertion in the Coulomb photon (main contribution to the Lamb shift in muonic systems). The dashed line denotes a Coulomb photon, bold fermionic lines denote muons, and thin lines denote electrons and positrons.

$$V_U(\mathbf{r}) = \frac{\alpha}{\pi} \int_0^1 dv \frac{v^2(1-v^2/3)}{1-v^2} \exp(-\lambda r) \left[\frac{-\alpha}{r} \right]. \quad (5)$$

We have introduced the notation

$$\lambda = \frac{2m_e}{\sqrt{1-v^2}}. \quad (6)$$

The correction to the energy due to the diagram in Fig. 2 is the main contribution to the Lamb shift. The evaluation of the matrix element of the Uehling potential on the nonrelativistic wave functions leads to the results listed in Table I. The Rydberg constant for the dimuonic atom is given by

$$E_0 = \frac{\alpha^2 m_r^2}{2} = \frac{\alpha^2 m_\mu}{4} = 1406.6133(5) \text{ eV}, \quad (7)$$

using the recommended values for $\alpha^{-1} = 137.0359895(61)$ and $m_\mu = 105.658389(91) \text{ MeV}$ [14,15]. The approximate Lamb shift values are (neglecting higher-order corrections corrections, which are estimated to be suppressed by an additional factor of α/π)

$$\begin{aligned} \mathcal{L}(1S) &= -0.49 \text{ eV}, & \mathcal{L}(2S) &= -0.058 \text{ eV}, \\ \mathcal{L}(2P) &= -0.0014 \text{ eV}. \end{aligned} \quad (8)$$

TABLE I. Contributions to the Lamb shift due to the electronic vacuum polarization. C is given in relative units, $\Delta E = (\alpha/\pi)CE_0$, where E_0 is the Rydberg constant for the dimuonic atom.

State	C	\mathcal{L} (eV)
1S	-0.15	-0.49
2S	-0.018	-0.058
2P	-0.00043	-0.0014

The Lamb shift values are in agreement with Malenfant's results [4]. In contrast to muonic hydrogen, dimuonium is a purely leptonic system. Therefore, its spectrum is not influenced by nuclear structure effects.

III. HYPERFINE STRUCTURE OF DIMUONIUM

The fine and hyperfine structures in dimuonium and positronium in lowest order $\alpha^4 m$ are given by the formula (for details see textbooks [11,12,16] and articles [17–19])

$$\begin{aligned} E_{\text{Ps}}(nlSF) &= -\frac{\alpha^2}{4n^2} m + \alpha^4 m \\ &\times \left[\frac{11}{64n^4} + \frac{\delta_{S,1}}{n^3} \left(\frac{7}{6} \delta_{l,0} + \frac{1-\delta_{l,0}}{4(2l+1)} B_{j,l} \right) \right. \\ &\quad \left. - \frac{1}{2n^3(2l+1)} \right], \end{aligned} \quad (9)$$

where

$$B_{j,l} = \begin{cases} \frac{3l+4}{(l+1)(2l+3)} & \text{for } F=l+1 \\ -\frac{1}{l(l+1)} & \text{for } F=l \\ -\frac{3l-1}{l(2l-1)} & \text{for } F=l-1. \end{cases} \quad (10)$$

Here m is the mass of the constituent particle, i.e., $m=m_e$ for positronium and $m=m_\mu$ for dimuonium. Note that formula (9) determines the position of energy levels in positronium up to the order of α^4 . Radiative corrections in Ps enter at the order of α^5 and in dimuonium at the order of α^3 . However, for the fine and hyperfine structures of dimuonium in lowest order, formula (9) remains valid. The results for the hyperfine-structure (hfs) order in dimuonium in leading order are

$$E_{\text{hfs}}^{(0)}(nS) = E_F/n^3$$

where

$$\begin{aligned} E_F &= \frac{7}{12} \alpha^4 m_\mu = 0.175 \text{ eV} \\ &= 4.23 \times 10^7 \text{ MHz}. \end{aligned} \quad (11)$$

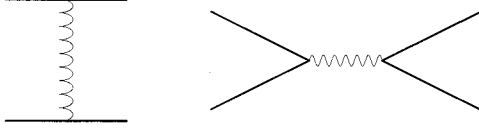


FIG. 3. Transverse photon exchange diagram and (timelike) photon annihilation diagram. Both of the diagrams contribute to the hyperfine structure of S states in leading order. The zigzag line denotes a transverse photon and the wavy line denotes the full photon propagator.

The hyperfine-structure interval $E_{\text{hfs}}^{(0)}(nS)$ arises from the exchange and annihilation diagrams (see Fig. 3), which contribute $(4/7)E_F$ and $(3/7)E_F$ to the Fermi energy E_F , respectively. The triplet n^3S_1 states are energetically higher than the singlet n^1S_0 states. For P states, the α^4 corrections imply the following order of the spectrum, in decreasing energy: n^3P_2 , n^1P_1 , n^3P_1 , and n^3P_0 (we follow here the usual spectroscopic notation $n^{2S+1}l_F$).

We now turn to the evaluation of radiative corrections to the hyperfine-structure splitting for S states of order $(\alpha/\pi)E_F$ [corresponding to $(\alpha/\pi)E_{\text{hfs}}^{(0)}$]. Some contributions are the same for positronium and dimuonium and some are specific to the dimuonic system. The contributions that are alike for both systems are depicted in Fig. 4. These contributions originate from the anomalous magnetic moment of the electron in the transverse photon diagram [denoted as “ $(g-2)-T$ ”], from recoil corrections (denoted by “Rec”),

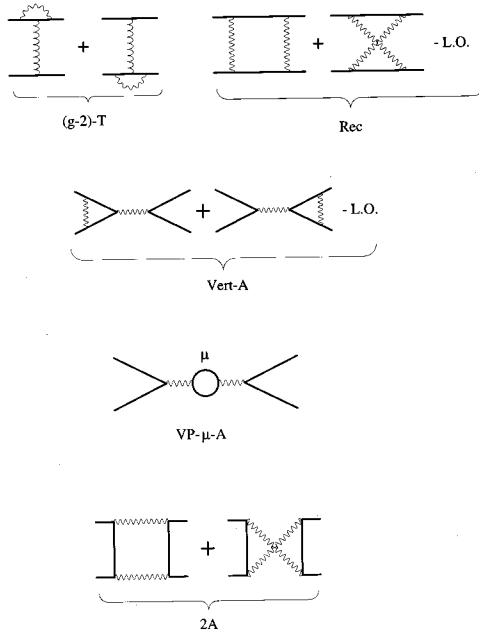


FIG. 4. Known corrections up to the order $(\alpha/\pi)E_F$ to the hyperfine splitting of dileptonic systems (positronium and dimuonium). Diagrams are explained in the text. For the dimuonic atom, the corrections depicted here contribute to the hfs, but there are additional terms specific to the dimuonic system that need to be taken into account. The subtraction of lower-order (LO) contributions is necessary for some of the diagrams in order to prevent double counting. The bold fermionic lines denote muons. The direction of time in all diagrams is from left to right.

TABLE II. Corrections of relative order α/π to the dimuonium hyperfine splitting. All contributions are explained in the text. The corrections $(g-2)-T$, Rec, Vert-A, VP- μ -A, and 2A contribute to the positronium hyperfine splitting just as in dimuonium. The remaining contributions (VPC-T, VPT, VPC-A, VP-e-A, and VP-h-A) are specific to the dimuonic system. Results are given in relative units $\Delta E = \alpha/\pi C E_F$.

Diagram	$C(1S)$	$8C(2S)$
$(g-2)-T$	0.571	0.571
Rec	-0.857	-0.857
Vert-A	-1.714	-1.714
VP- μ -A	-0.381	-0.381
2A	0.263	0.263
VPC-T	0.605	0.523
VPT	0.345	0.355
VPC-A	0.454	0.393
VP-e-A	1.483	1.483
VP-h-A	-0.080(9)	-0.080(9)
Sum	0.689(9)	0.556(9)

from the vertex correction to the annihilation diagram (denoted by “Vert-A”), from the muonic vacuum polarization (“VP- μ -A”), and from box diagrams corresponding to two-photon annihilation (denoted by “2A”). In this paper, due to the multitude of corrections considered, we prefer to denote the contributions by short abbreviations, which we hope are self-explanatory, rather than single letters, in order to enhance the clarity of presentation. The results for the above corrections are listed in Table II (first five contributions, above the separating line). The sum of corrections to the Fermi energy of order $(\alpha/\pi)E_F$ amounts to [12,20]

$$\Delta E_{\text{Ps}}(nS) = \frac{\alpha}{\pi} \left[-\frac{32}{21} - \frac{6}{7} \ln 2 + \frac{3}{7} \pi i \right] \frac{E_F}{n^3}, \quad (12)$$

where the imaginary part is entirely due to paradimuonium. It corresponds to the two-photon decay of the parasystem [18,21]

$$\frac{1}{\Gamma^{(0)}(n^1S_0)} = \tau^{(0)}(n^1S_0) = \frac{2n^3}{\alpha^5 m_\mu} = n^3 0.6021 \times 10^{-12} \text{ s}. \quad (13)$$

Now we turn to the evaluation of corrections specific to dimuonium (the relevant diagrams are depicted in Fig. 5). We first evaluate the correction due to vacuum polarization insertions in the Coulomb photon line. For the dimuonic atom, the correction to the wave function due to electronic vacuum polarization has relative order α/π and therefore modifies the hyperfine splitting in α/π relative order. The order of magnitude of this correction is specific to the dimuonic atom (small length scale of the system; see Sec. I). For “VPC-T” (see Fig. 5), the contribution is given by the matrix element

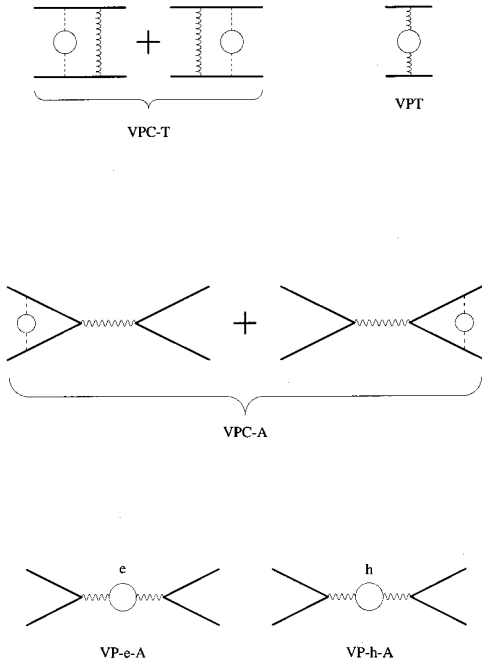


FIG. 5. Corrections of order $(\alpha/\pi)E_F$ to the hyperfine splitting specific to the dimuonic atom. Diagrams are further explained in the text.

$$\Delta E_{\text{VPC-T}} = 2 \times \frac{4}{3} \frac{\pi \alpha}{m_\mu^2} \langle \psi | \delta(\mathbf{r}) \bar{G}(E_\psi) V_U | \psi \rangle, \quad (14)$$

where

$$\bar{G}(E_\psi) = \sum_{\psi_n \neq \psi} \frac{|\psi_n\rangle \langle \psi_n|}{E_\psi - E_n} \quad (15)$$

is the reduced Coulomb Green's function (the pole of the reference state is excluded from the sum over all intermediate states). Because atomic momenta are of the order of αm_μ in the system under consideration, it is sufficient to carry out the calculations with nonrelativistic wave functions. The reduced Green's function is a function of two coordinates \mathbf{r}_1 and \mathbf{r}_2 . Due to the appearance of the δ -like potential, the Green's function is needed only for $\mathbf{r}_1 = \mathbf{0}$. It can be expressed in closed form [24]

$$\bar{G}_{1S}(E_{1S}; \mathbf{0}, \mathbf{r}) = \frac{\alpha m_r^2}{4\pi} \frac{2e^{-z_1/2}}{z_1} [2z_1(\ln z_1 + C) + z_1^2 - 5z_1 - 2] \quad (16)$$

and

$$\begin{aligned} \bar{G}_{2S}(E_{2S}; \mathbf{0}, \mathbf{r}) = & -\frac{\alpha m_r^2}{4\pi} \frac{e^{-z_2/2}}{2z_2} [4z_2(z_2 - 2)(\ln z_2 + C) + z_2^3 \\ & - 13z_2^2 + 6z_2 + 4], \end{aligned} \quad (17)$$

where $C = 0.5772\dots$ is the Euler constant, $z_n = 2\alpha m_r r/n$, and m_r denotes the reduced mass of the system ($m_r = m_\mu/2$ for the dimuonic atom). The radial integration can be per-

formed by standard formulas. We finally obtain a one-dimensional integral over the v parameter of the Uehling potential. For the 1S ground state, we obtain

$$\begin{aligned} \Delta E_{\text{VPC-T}}(1S) = & \frac{\alpha}{\pi} E_F \int_0^1 dv \frac{4x^2(1-v^2/3)}{(2+x\sqrt{1-v^2})^3} \\ & \times \left[2 + \frac{2}{7x} \frac{1}{\sqrt{1-v^2}} + \frac{3}{7} \sqrt{1-v^2} \right. \\ & \left. + \left(\frac{4}{7} + \frac{2}{7} x \sqrt{1-v^2} \right) \right. \\ & \left. \times \ln \left(1 + \frac{2}{x \sqrt{1-v^2}} \right) \right] \\ = & \frac{\alpha}{\pi} (0.605) E_F, \end{aligned} \quad (18)$$

where

$$x = \alpha \frac{m_\mu}{m_e} = 1.50886. \quad (19)$$

For 2S, an analogous evaluation yields

$$\Delta E_{\text{VPC-T}}(2S) = \frac{\alpha}{\pi} (0.523) \frac{E_F}{8}. \quad (20)$$

The analytical calculations were partially performed with the computer algebra system MATHEMATICA [22]. The contribution due to the VPC-A diagram can be evaluated by considering the relative contributions of these diagrams to the first-order hfs (4/7 and 3/7, respectively). An independent evaluation is done by utilization of a spectral decomposition of the reduced Coulomb Green's function. If that representation of the Green's function is chosen, then the calculation is closely related to [23]. In order to simplify the calculation, we observe that the perturbation can be expressed as a modification of the wave function at $r=0$. The discrete and continuous spectra give distinct contributions. The results of the respective terms (discrete and continuous spectrum) for the 1S and 2S states are

$$\begin{aligned} \Delta E_{\text{VPC-T}}(1S) = & 2 \frac{\alpha}{\pi} E_F \times \frac{4}{7} \left[\frac{\Delta \psi_{1S}(0)}{\psi_{1S}(0)} \right]_{\mu^+ \mu^-} \\ = & \frac{\alpha}{\pi} [0.04 + 0.56] E_F, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \Delta E_{\text{VPC-T}}(2S) = & 2 \frac{\alpha}{\pi} \frac{E_F}{8} \times \frac{4}{7} \left[\frac{\Delta \psi_{2S}(0)}{\psi_{2S}(0)} \right]_{\mu^+ \mu^-} \\ = & \frac{\alpha}{\pi} [-0.14 + 0.66] \frac{E_F}{8}. \end{aligned} \quad (22)$$

The continuous spectrum contribution [the second numerical term in Eqs. (21) and (22)] is very large. It can be understood in the following way. If one would try to omit the energy of the intermediate state in the denominator of the expression

$$\begin{aligned}\Delta\psi(\mathbf{0}) &= \int d^3r \bar{G}(E_\psi, \mathbf{0}, \mathbf{r}) V_U(\mathbf{r}) \psi(\mathbf{r}) \\ &= \int d^3r \sum_{n,l \neq \psi} \frac{\psi_{nl}(\mathbf{0}) \psi_{nl}^*(\mathbf{r})}{E_\psi - E_n} V_U(\mathbf{r}) \psi(\mathbf{r})\end{aligned}$$

in the sense of $E_\psi - E_n \rightarrow E_\psi$, then the result should include a divergence in the sum over states. In view of the equation $\sum_{n,l \neq \psi} \psi_{nl}(\mathbf{0}) \psi_{nl}^*(\mathbf{r}) = \delta(\mathbf{r}) - \psi(\mathbf{0}) \psi^*(\mathbf{r})$ (completeness of the spectrum), this is seen as

$$\begin{aligned}\int d^3r \sum_{n,l} \frac{1}{E_\psi} \psi_{nl}(\mathbf{0}) \psi_{nl}^*(\mathbf{r}) V_U(\mathbf{r}) \psi(\mathbf{r}) \\ = \frac{V_U(\mathbf{0}) - \langle \psi | V_U | \psi \rangle}{E_\psi} \psi(\mathbf{0})\end{aligned}$$

where $V_U(\mathbf{0})$ is a diverging quantity. Its appearance means that virtual intermediate states with large wave numbers $k > \alpha m_\mu$ are important for the convergence of the integral for $\Delta\psi(\mathbf{0})$ and they lead to a numerically large contribution. The final results for the VPC- T and VPC- A contributions are listed in Table II.

The correction due to the vacuum polarization insertion into the transverse photon line (VPT) is proportional to the matrix element

$$\begin{aligned}\mathcal{M} &= \langle \psi | \nabla^2 V_U | \psi \rangle = \frac{\alpha}{\pi} \int_0^1 dv \frac{v^2(1-v^2/3)}{1-v^2} \int d^3r |\psi(\mathbf{r})|^2 \\ &\quad \times \nabla^2 \left[\left(\frac{-\alpha}{r} \right) \exp \left(-\frac{2m_e r}{\sqrt{1-v^2}} \right) \right] \\ &= \frac{\alpha}{\pi} \int_0^1 dv \frac{v^2(1-v^2/3)}{1-v^2} \int \frac{d^3p}{(2\pi)^3} \\ &\quad \times \int \frac{d^3k}{(2\pi)^3} \psi^*(\mathbf{p}) \frac{-4\pi\alpha(\mathbf{p}-\mathbf{k})^2}{(\mathbf{p}-\mathbf{k})^2 + 4m_e^2/(1-v^2)} \psi(\mathbf{k}).\end{aligned}\tag{23}$$

The matrix element \mathcal{M} is evaluated both in momentum space and in coordinate space with the same result. We obtain for the 1S state

$$\mathcal{M}(1S) = \frac{\alpha^2(\alpha m_\mu)^4}{2\pi} \int_0^1 dv \frac{v^2(1-v^2/3)}{1-v^2} \frac{2\lambda + \alpha m_\mu}{(\lambda + \alpha m_\mu)^2},\tag{24}$$

where λ is defined in Eq. (6). The final results for the VPT contribution (after numerical v integration) are

$$\Delta E_{\text{VPT}}(1S) = \frac{\alpha}{\pi} 0.345 E_F, \quad \Delta E_{\text{VPT}}(2S) = \frac{\alpha}{\pi} 0.355 E_F.\tag{25}$$

The results are included in Table II.

The term due to the muonic vacuum polarization in the photon line in the annihilation diagram (denoted ‘‘VP- μ -A’’) is known, since it has the same relative magnitude as that of electronic vacuum polarization in the positronium system. The electronic part (denoted as VP- e -A) is found from the well-known asymptotic behavior of the vacuum polarization [11,12]

$$\Delta E_{\text{VP-}e\text{A}}(nS) = \frac{\alpha}{\pi} \frac{3}{7} \left[\frac{1}{3} \ln \frac{q^2}{m_e^2} - \frac{5}{9} - \frac{\pi}{3} i \right] \frac{E_F}{n^3}\tag{26}$$

for $q^2 = (2m_\mu)^2$. The contribution due to the electronic vacuum polarization in the annihilation is the second largest correction to the dimuonium hyperfine splitting in α/π relative units (see Table II, the result for VP- e -A). The timelike vector $q = (2m_\mu, \mathbf{0})$ is lying in the far timelike asymptotic region for the electronic vacuum polarization. The logarithmic term that appeared originally by the replacement $\ln(q^2/m^2) \rightarrow \ln(-q^2/m^2)$ (going from spacelike to timelike virtual photons) leads to the imaginary part. It can be ascribed to the decay of the orthostate into free electrons ($\mu^+ \mu^- \rightarrow \gamma \rightarrow e^+ e^-$)

$$\frac{1}{\Gamma^{(0)}(n^3 S_1)} = \tau^{(0)}(n^3 S_1) = \frac{6n^3}{\alpha^5 m_\mu} = n^3 1.806 \times 10^{-12} \text{ s}.\tag{27}$$

In contrast to positronium, the parastate and orthostate lifetimes have the same order of magnitude in dimuonium ($\alpha^5 m_\mu$).

The hadronic contribution (denoted by VP- h -A) to the vacuum polarization is found as the sum of four terms. The main contribution results from a pionic loop. We follow here the approach in [25]. The spectral function is of the form

$$\rho(s) = \frac{(s - 4m_\pi^2)^{3/2}}{12s^{5/2}} |F_\pi(s)|^2,\tag{28}$$

where the pionic form factor is used in the form given by Gounaris and Sakurai [26]

$$F_\pi(s) = \frac{N}{D_1 + D_2 - iD_3}.\tag{29}$$

The quantities N , D_1 , D_2 , and D_3 , are given by

$$N = m_\rho^2 + dm_\rho \Gamma_\rho,\tag{30}$$

$$d = \frac{3}{\pi} \frac{m_\pi^2}{k_\rho^2} \ln \frac{m_\rho + 2k_\rho}{2m_\pi} + \frac{m_\rho}{2\pi k_\rho} - \frac{m_\pi^2 m_\rho}{\pi k_\rho^3} \approx 0.48,\tag{31}$$

$$D_1 = m_\rho^2 - s, \quad D_3 = m_\rho \Gamma_\rho [k(s)/k_\rho]^3 m_\rho / \sqrt{s}, \quad (32)$$

$$D_2 = \Gamma_\rho \frac{m_\rho^2}{k_\rho^3} \{k(s)^2 [h(s) - h_\rho] + k_\rho^2 h' (m_\rho^2) (m_\rho^2 - s)\}, \quad (33)$$

where h' denotes the derivative of h and the functions k and h are defined as

$$k(s) = \frac{1}{2} \sqrt{s - 4m_\pi^2}, \quad h(s) = \frac{2}{\pi} \frac{k(s)}{\sqrt{s}} \ln \left(\frac{\sqrt{s} + 2k(s)}{2m_\pi} \right). \quad (34)$$

with the special values $k_\rho \equiv k(m_\rho^2)$, $h_\rho \equiv h(m_\rho^2)$, $\Gamma_\rho = 150.7(1.2)$ MeV, and $m_\rho = 768.5(6)$ MeV [15]. We give results for the $1S$ state only in the following. The $1/n^3$ scaling is easily restored in the final result. The contribution from the pionic vacuum polarization is given by

$$\begin{aligned} \Delta E_{\pi^+\pi^-}(1S) &= \frac{\alpha}{\pi} \frac{3}{7} \left[4m_\mu^2 \int_{4m_\pi^2}^\infty ds \frac{\rho(s)}{4m_\mu^2 - s} \right] E_F \\ &= \frac{\alpha}{\pi} (-0.055) E_F. \end{aligned} \quad (35)$$

In the simple ρ -meson pole approximation, where $\rho(s) = 4\pi^2/f_\rho^2 \delta(s - m_\rho^2)$ with $f_\rho^2/4\pi = 2.2$ [27], we have

$$\Delta E_{\pi^+\pi^-}(1S) \approx \Delta E_\rho(1S) = \frac{\alpha}{\pi} (-0.050) E_F. \quad (36)$$

This result agrees with the full pion form factor of Eq. (29) to about 10%, so it is justified to consider mesonic resonances of higher energy in the pole approximation only. The higher energy mesonic contributions are due to the ω and ϕ resonances. These resonances are not included in the Gounaris-Sakurai form factor and are treated separately. Estimating the coupling constants as $f_\omega^2/4\pi = 18(2)$ and $f_\phi^2/4\pi = 11(2)$ [25,27] and given the meson masses of $m_\omega = 782$ MeV and $m_\phi = 1019$ MeV [15], the results are

$$\Delta E_\omega(1S) = \frac{\alpha}{\pi} (-0.006) E_F, \quad \Delta E_\phi(1S) = \frac{\alpha}{\pi} (-0.005) E_F. \quad (37)$$

The background above 1 GeV is estimated by assuming a form of

$$\rho(s) = \frac{R}{3s} \quad \text{where} \quad R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}, \quad (38)$$

for the spectral function with a branching ratio $R \approx 2$ (constant) below $\sqrt{s} = 4$ GeV and $R \approx 4$ above $\sqrt{s} = 4$ GeV (see [15], p. 190). Integrating from an estimated lower threshold of $s_{\text{th}} \approx (1 \text{ GeV})^2$ we obtain

$$\Delta E_{>}(1S) \approx \frac{\alpha}{\pi} (-0.014) E_F. \quad (39)$$

Summing all contributions and restoring the $1/n^3$ scaling, we have as the contribution from the hadronic vacuum polarization

$$\begin{aligned} \Delta E_{\text{VP-}h\text{-}A}(nS) &= \Delta E_\pi + \Delta E_\omega + \Delta E_\phi + \Delta E_{>} \\ &= \frac{\alpha}{\pi} [-0.080(9)] \frac{E_F}{n^3}. \end{aligned} \quad (40)$$

We estimate model-dependent uncertainties in the hadronic vacuum polarization to be of the order of 11%. The hadronic term is included in Table II (the result for VP- h -A).

The sum of all corrections (see Table II) to the hyperfine splitting in dimuonium amounts to

$$\Delta E_{\text{hfs}}(1S) = \frac{\alpha}{\pi} 0.689(9) E_F, \quad \Delta E_{\text{hfs}}(2S) = \frac{\alpha}{\pi} 0.556(9) \frac{E_F}{8}. \quad (41)$$

In the final results for hyperfine splitting, we estimate higher-order corrections to enter at the 5% level of the next-to-leading-order contributions. We obtain

$$E_{\text{hfs}}(1S) = 4.232 \ 83(35) \times 10^7 \text{ MHz} \quad (42)$$

and

$$E_{\text{hfs}}(2S) = 5.289 \ 41(34) \times 10^6 \text{ MHz}. \quad (43)$$

Results for the Lamb shift and the hyperfine structure of low-lying levels are presented in Tables I and II. The largest deviation from the scaling appeared in VPC- T and VPC- A contributions (16%) because the wave function [see Eqs. (21) and (22)] is more sensitive to the behavior of the potential about the origin.

An important point for a possible investigation of the spectrum is the lifetime of levels. As one can see from Eqs. (13) and (27), the annihilation lifetimes of S levels are much shorter than the lifetime of the free muon, $\tau_\mu = 2.20 \times 10^{-6}$ s. By contrast, the annihilation decay rate for P states includes an extra α^2 and the lifetime is of the same order of magnitude as that of the free muon.

The decay rates of excited states have to be compared with the atomic transition rates, which are also of the order of $\alpha^5 m_\mu$ (see, e.g., [16]). We obtain

$$\tau(2P \rightarrow 1S) = 1.54 \times 10^{-11} \text{ s}. \quad (44)$$

Annihilation of P states is suppressed by two orders of α compared to S states due to the behavior of the P wave function near the origin. Therefore, the annihilation lifetime of the $2P$ state can be estimated to be of the order of 10^{-7} s and it is seen that the atomic transition, not the annihilation, determines the lifetime of the excited $2P$ state in dimuonium.

This situation is different for S states, where the annihilation dominates over atomic transitions. The annihilation

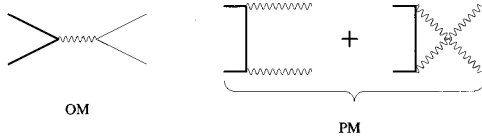


FIG. 6. Decay channels of orthodimuonium (one-photon decay into an electron positron pair, OM) and paradimuonium (two-photon decay, PM). The decay channels depicted yield the main contribution to the decay of the system (Γ of order $\alpha^5 m_\mu$). Bold fermionic lines denote muons and thin lines denote electrons and positrons.

lifetimes of $1S$ and $2S$ states lie between 0.6×10^{-12} s and 14×10^{-12} s and we expect that the decay rates could be measured via detection of the decay products. Radiative corrections to the decay rates are considered in the following section.

IV. DECAY CHANNELS OF ORTHODIMUONIUM

The leading-order contributions to the orthodimuonium and paradimuonium decay rate can be extracted as the imaginary parts of the energy corrections to the hyperfine structure

$$\Gamma^{(0)}(n^3S_1) = \frac{\alpha^5 m_\mu}{6n^3}, \quad \Gamma^{(0)}(n^1S_0) = \frac{\alpha^5 m_\mu}{2n^3}. \quad (45)$$

The calculations at leading order were presented above [Eqs. (13) and (27)] (see also Fig. 3). The above results can also be found in [3,4].

We begin the consideration of radiative corrections with orthodimuonium and provide results for the $1S$ state here. Results for the $2S$ state (and $1S$) are summarized in Table III. The diagrams contributing in next-to-leading order for orthostates are depicted in Fig. 7.

The VPC-A correction can be interpreted as a modification of the wave function at the origin. The energy shift and decay rate are both proportional to $|\psi(0)|^2$. The annihilation diagram contributes $(3/7)E_F$ to the first-order result for the hyperfine splitting. Hence the VPC-A diagram yields a correction of

TABLE III. Corrections of relative order α/π to the orthodimuonium decay. Results are given in relative units $\Delta\Gamma(n^3S_1) = \alpha/\pi C\Gamma^{(0)}(1^3S_1)$.

Diagram	$C(1S)$	$8C(2S)$
Vert-A	-4.00	-4.00
VPC-A	1.06	0.92
VP- μ -A	-1.78	-1.78
VP-e-A	6.92	6.92
VP-h-A	-0.37(4)	-0.37(4)
BeA + ReA	0.75	0.75
3A	1.16	1.16
Sum	3.74(4)	3.60(4)

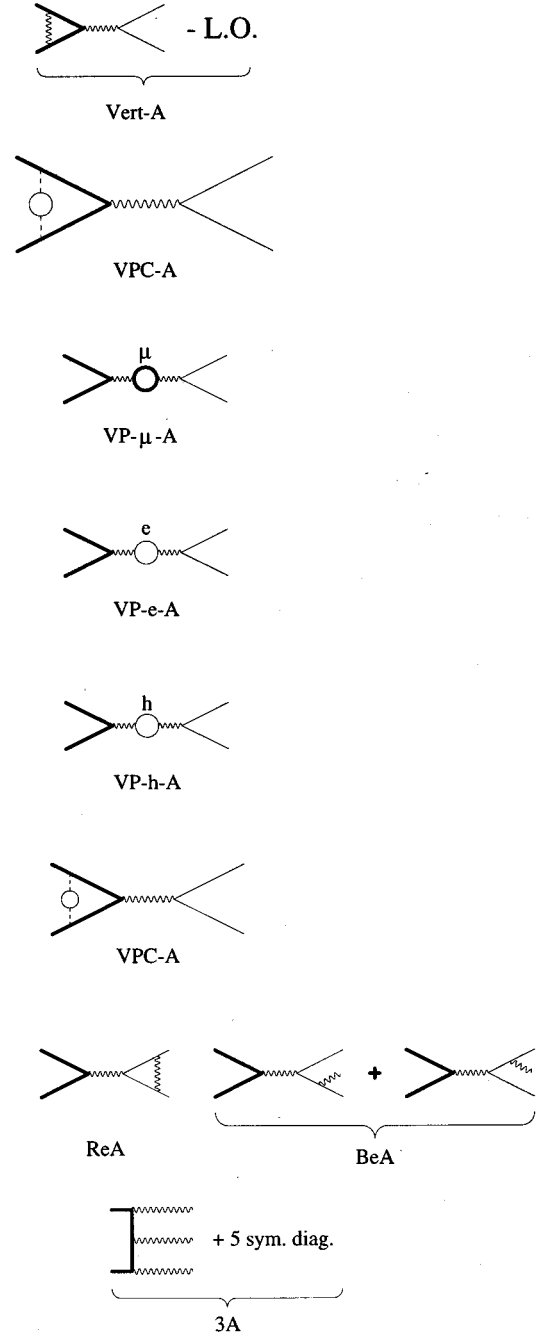


FIG. 7. Order α/π corrections to the decay channels of orthodimuonium. The five symmetrical diagrams originate from the symmetrization of photon wave functions.

$$\begin{aligned} \Delta\Gamma_{\text{VPC-A}}(1^3S_1) &= \frac{\alpha}{\pi} \left[\frac{7}{3} \times 0.454 \right] \Gamma^{(0)}(1^3S_1) \\ &= \frac{\alpha}{\pi} 1.06 \Gamma^{(0)}(1^3S_1) \end{aligned} \quad (46)$$

to the decay rate. Analogous considerations are true for the Vert-A correction because the diagrams consist of separated blocks and hence the correction to the hfs and to the decay rate can be traced back to the same matrix element of the wave function. We obtain

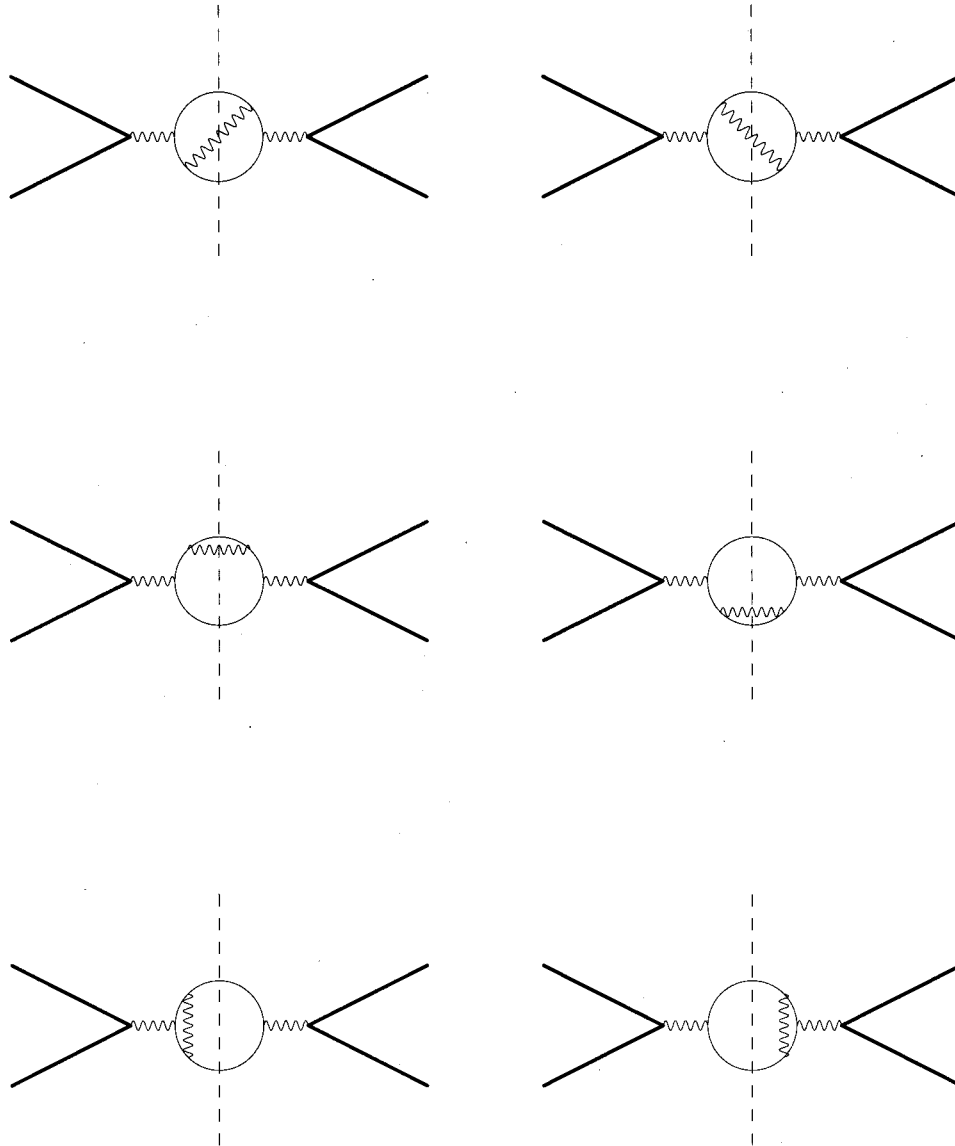


FIG. 8. Evaluation of bremsstrahlung and electron vertex corrections to orthodimuonium decay as the imaginary part of the two-loop vacuum polarization insertion in the photon line.

$$\Delta\Gamma_{\text{Vert-A}}(n^3S_1) = -4\frac{\alpha}{\pi}\Gamma^{(0)}(n^3S_1). \quad (47)$$

The diagrams VP- μ -A, VP- e -A, and VP- h -A have to be interpreted as modifications of the photon propagator. Because the energy shift is proportional to the amplitude of the propagator, but the decay rate is proportional to its square, we have to multiply the relative correction to the hfs by a factor of 2 in order to obtain the relative correction to the decay rate. Hence the VP- μ -A, VP- e -A, and VP- h -A diagrams yield a total correction of

$$\frac{\alpha}{\pi} \left[-\frac{16}{9} + \frac{4}{3} \ln \left(2 \frac{m_\mu}{m_e} \right) - \frac{10}{9} - 0.37(4) \right] \Gamma^{(0)}(n^3S_1)$$

to the decay rate.

We now consider the one-loop radiative corrections to the electron line (ReA) and emission of a photon by an electron, i.e., bremsstrahlung (BeA). The sum of $BeA + ReA$ can be easily obtained from the diagrams in Fig. 8 for the two-loop electron polarization correction to the hyperfine splitting, which is completely determined by the asymptotic behavior of the two-loop vacuum polarization (see, e.g., [13]). For the correction to the hyperfine splitting, we obtain

$$\Delta E_{\text{VP-2}}(nS) = \frac{\alpha^2}{\pi^2} \frac{3}{7} \left[\frac{1}{4} \ln \frac{q^2}{m_e^2} + \left(\zeta(3) - \frac{5}{24} \right) - \frac{\pi}{4} i \right] \frac{1}{n^3} E_F, \quad (48)$$

where $q^2 = 4m_\mu^2$ and $\zeta(3) = 1.202\dots$ is the Riemann zeta function of argument 3. This correction is of relative order α^2/π^2 with respect to E_F and therefore was not considered in Sec. III. The imaginary part of this contribution is just the

result for the sum of the diagrams $ReA + BeA$. The correction to the decay rate can now be evaluated easily. The sum of $ReA + BeA$ yields a correction of

$$\frac{3}{4} \frac{\alpha}{\pi} \Gamma^{(0)}(n^3 S_1)$$

relative to the leading-order result [cf. Eqs. (26), and (45)].

The last term in relative order α/π is the three-photon annihilation. The result is known from orthopositronium calculations [28] (see also [16,12,29]):

$$\begin{aligned} \Delta\Gamma_{3A}(n^3 S_1) &= \frac{2\alpha^6}{\pi} \frac{1}{n^3} \frac{\pi^2 - 9}{9} m_\mu \\ &= \frac{\alpha}{\pi} \left[\frac{4}{3} (\pi^2 - 9) \right] \Gamma^{(0)}(n^3 S_1). \end{aligned} \quad (49)$$

The final result for the $1S$ decay (see also Table III) is

$$\begin{aligned} \Delta\Gamma(1^3 S_1) &= \frac{\alpha}{\pi} \left[\left(\frac{4}{3} \ln \frac{2m_\mu}{m_e} - \frac{221}{36} \right) + 0.68(4) + \frac{4(\pi^2 - 9)}{3} \right] \\ &\quad \times \Gamma^{(0)}(1^3 S_1) \\ &\approx \frac{\alpha}{\pi} [1.90 + 0.68(4) + 1.16] \Gamma^{(0)}(1^3 S_1) \\ &= \frac{\alpha}{\pi} 3.74(4) \Gamma^{(0)}(1^3 S_1). \end{aligned} \quad (50)$$

The first term corresponds to the sum of the analytically evaluated contributions Vert-A, VP- μ -A, VP- e -A, and $BeA + ReA$ and has the numerical value 1.90. The second term originates from the numerically evaluated contributions VPC-A and VP- h -A. The last term is associated with the three-photon decay. The result in Eq. (50) has to be compared with the earlier analysis of the decay rate of heavy leptonium [4]. The final result for the analytically evaluated contributions, which is identical to the first term in Eq. (50), whose analytical expression is $(4/3)\ln(2m_\mu/m_e) - 221/36$, is found to be in agreement with the pioneering investigations by Malenfant [4]. For the VPC-A correction, Malenfant's results are in slight numerical disagreement with ours [see Eqs. (77) and (78) in [4]]. We presume this disagreement can be traced to the fact that Malenfant has calculated the VPC-A correction with free Green's functions, whereas the evaluation in this work is done using bound Green's functions. In this context it is important to note that in the limit of $\alpha m_\mu/2m_e \rightarrow 0$, our result is in agreement with that of Malenfant. This can be seen as follows. The VPC-A correction may be rewritten as a correction $\Delta\psi(0)$ to the wave function at the origin. After this reinterpretation, we find

$$\Delta\psi(0) = \frac{\alpha}{\pi} \left(\frac{3\pi}{16} \kappa + O(\kappa^2) \right) \psi(0) \quad \text{for} \quad \kappa = \frac{\alpha m_\mu}{2m_e} \rightarrow 0, \quad (51)$$

which is in agreement with Eq. (80) of [4]. We can therefore conclude that in the limit of weak binding ($\kappa \rightarrow 0$), Malenfant's result is in agreement with ours. However, we hold the view that bound Green's functions should rather be used for

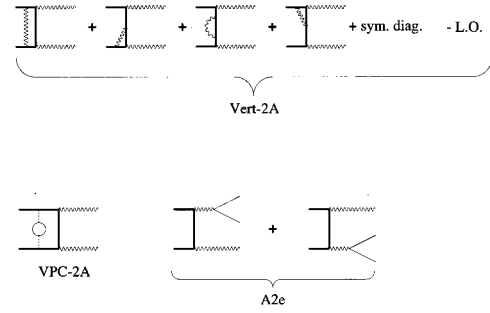


FIG. 9. Order α/π corrections to the decay channels of paradimuonium. The symmetrical diagrams originate from the symmetrization of the photon wave functions.

the VPC-A correction. The atomic momentum in dimuonium is of the order of $\alpha m_\mu/2 = \kappa m_e$ ($\kappa \approx 0.75$). This momentum is close to the mass of the loop particles (electrons and positrons) of electronic vacuum polarization. These particles determine the radius of the Uehling potential. Therefore, some of the momentum integration for the VPC-A correction is performed in an area about m_e , where the bound Coulomb Green's function cannot be approximated by the free Green's function (because the effect of the binding Coulomb potential, in momentum space, is inversely proportional to the square of the momentum transfer). This consideration should explain the slight numerical disagreement for the VPC-A correction between this work and the result in [4].

The sum of next-to-leading order corrections to the decay rate for the $2S$ state is (see Table III)

$$\Delta\Gamma(2^3 S_1) = \frac{\alpha}{\pi} 3.60(4) \Gamma^{(0)}(2^3 S_1). \quad (52)$$

We estimate the higher-order corrections to be suppressed by an additional factor of α compared to the next-to-leading-order result. We obtain the following results for the decay rate of orthodimuonium, taking into account also the uncertainty from our model of the hadronic vacuum polarization:

$$\begin{aligned} \tau(1^3 S_1) &= 1.790\,73(77) \times 10^{-12} \text{ s}, \\ \tau(2^3 S_1) &= 14.3305(59) \times 10^{-12} \text{ s}. \end{aligned} \quad (53)$$

V. DECAY CHANNELS OF PARADIMUONIUM

The diagrams contributing to paradimuonium decay in next-to-leading order $\alpha^6 m_\mu$ are presented in Fig. 9 and the results are summarized in Table IV. The vertex correction

TABLE IV. Corrections of relative order α/π to the paradimuonium decay. Results are given in relative units $\Delta\Gamma(n^1 S_0) = \alpha/\pi C \Gamma^{(0)}(1^1 S_0)$.

Diagram	$C(1S)$	$8C(2S)$
Vert-2A	-2.53	-2.53
VPC-2A	1.06	0.92
A2e	6.26	6.26
sum	4.79	4.65

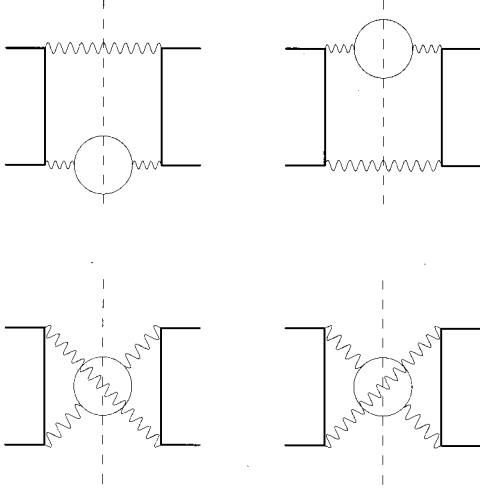


FIG. 10. Evaluation of corrections to the paradimuonium decay caused by the production of an electron positron pair ($A2e$ correction).

term (Vert-2A) is equivalent to the corresponding correction for parapositronium [30]

$$\Delta\Gamma_{\text{Vert-2A}}(n^1S_0) = -\frac{\alpha}{\pi} \frac{20 - \pi^2}{4} \Gamma^{(0)}(n^1S_0). \quad (54)$$

The correction to the wave function caused by the VPC-A diagram modifies the decay rate of the parasytem in the same way as the orthosystem (see Table IV) and introduces a deviation from the $1/n^3$ scaling.

For the parastate, there exists another correction to the decay rate corresponding to the decay into a photon and an electron-positron pair. The result for the $A2e$ correction can be obtained in the following way. We consider the one-loop vacuum polarization insertion into a two-photon annihilation diagram (Fig. 10). Because the vacuum polarization insertion evaluated for a real photon, must be equal to zero (gauge invariance of vacuum polarization), the imaginary part of the diagrams in Fig. 10 leads to the result we need. We can use Eq. (4) for the parametric form of the vacuum polarization insertion, and we consider the s integration as the final one. The integrand is now equivalent to the imaginary part of the 2A diagram for the hfs, but with one of the photons having a finite mass s (s corresponds to the sum of the four momenta of the emerging electron-positron pair). The vacuum polarization insertion normally fixes a gauge (the Landau gauge) for the virtual photon because the polarization insertion is proportional to the transverse projector. However, as demonstrated in [31], it is possible to substitute in Eq. (4) any covariant gauge. For convenience, we choose the Feynman gauge. As a result we have the expression

$$\Delta\Gamma_{A2e}(n^1S_0) = \frac{\alpha}{\pi} \int ds \rho(s) \Gamma^{(0)}(s,0), \quad (55)$$

where $\Gamma^{(0)}(s,0)$ is the decay rate to one real photon and a virtual photon with mass s . In order to obtain the correction

in relative units, we divide by the first-order result, which is given by $\Gamma^{(0)}(0,0)$. The correction relative to the first-order result $\Gamma^{(0)}(n^1S_0)$ is given by

$$\frac{\Delta\Gamma_{A2e}(n^1S_0)}{\Gamma^{(0)}(n^1S_0)} = 2 \frac{\alpha}{\pi} \int ds \rho(s) \frac{\Gamma^{(0)}(s,0)}{\Gamma^{(0)}(0,0)}. \quad (56)$$

An additional factor 2 appears because the insertion of a vacuum polarization operator doubles the number of non-equivalent diagrams contributing to the imaginary part. For the logarithmic coefficient we may neglect s in $\Gamma^{(0)}(s,0)$, approximate $\Gamma^{(0)}(s,0) \rightarrow \Gamma^{(0)}(0,0)$, and use the asymptotic form of the spectral function [cf. Eq. (2)]

$$\rho(s) \rightarrow \frac{1}{3s} \quad \text{for } s \rightarrow \infty.$$

We can thus easily obtain the logarithmic coefficient

$$\frac{\Delta\Gamma_{A2e}(n^1S_0)}{\Gamma^{(0)}(n^1S_0)} \approx 2 \int_{(2m_e)^2}^{\Lambda^2 = (2m_\mu)^2} ds \frac{1}{3s} = 2 \frac{1}{3} \ln \frac{m_\mu^2}{m_e^2} = \frac{4}{3} \ln \frac{m_\mu}{m_e}.$$

The full result requires a more detailed analysis of the dependence of $\Gamma^{(0)}(s,0)$ on s . It differs from the approximate analysis presented above only by an additive constant. The final result of the calculation is

$$\Delta\Gamma_{A2e}(n^1S_0) = \frac{\alpha}{\pi} \left(\frac{4}{3} \ln \frac{2m_\mu}{m_e} - \frac{16}{9} \right) \frac{1}{n^3} \Gamma^{(0)}(n^1S_0). \quad (57)$$

An independent evaluation of the $A2e$ correction using the standard S -matrix formalism is used to verify the result in Eq. (57). The treatment of the Dirac currents involved is simplified by the application of a symbolic program [32] developed for high-energy physics calculations by Hsieh and Yehudai. Some care must be taken during evaluation because one cannot assume the electrons to be massless in the final states (the result otherwise includes a logarithmic divergence in the electron mass). Proper regularization of the relevant expression then leads to the result in Eq. (57).

The final result for the next-to-leading-order radiative corrections to paradimuonium decay (see Table IV) is

$$\Delta\Gamma(1^1S_0) = \frac{\alpha}{\pi} 4.79 \Gamma^{(0)}(1^1S_0). \quad (58)$$

For the $2S$ state, we have

$$\Delta\Gamma(2^1S_0) = \frac{\alpha}{\pi} 4.65 \times \Gamma^{(0)}(2^1S_0). \quad (59)$$

Estimating higher-order corrections to enter at the level of 5% of the next-to-leading-order contributions, we obtain the theoretical values for the paradimuonium decay

$$\tau(1^1S_0) = 0.595 \, 47(33) \times 10^{-12} \text{ s},$$

and

$$\tau(2^1S_0) = 4.765 \, 3(25) \times 10^{-12} \text{ s}. \quad (60)$$

VI. CONCLUSION

We evaluate next-to-leading-order corrections to the spectrum, to the hyperfine splitting, and to the decay rate of low-lying levels of the dimuonic system. The results for the spectrum are given in Sec. II. We observe that for $2P$ states, the atomic decay into the $1S$ state dominates over annihilation processes. This would facilitate experimental observation of the atomic transition if dimuonium atoms can be produced in quantities sufficient to carry out spectroscopic measurements. We evaluate the hyperfine splitting of $1S$ and $2S$ states in next-to-leading order. The results are

$$E_{\text{hfs}}(1S) = \left[1 + \frac{\alpha}{\pi} 0.689(9) \right] \frac{7}{12} \alpha^4 m_\mu$$

$$= 4.232\,83(35) \times 10^7 \text{ MHz} \quad (61)$$

and

$$E_{\text{hfs}}(2S) = \left[1 + \frac{\alpha}{\pi} 0.556(9) \right] \frac{7}{12} \alpha^4 m_\mu$$

$$= 5.289\,41(34) \times 10^6 \text{ MHz}. \quad (62)$$

We present a complete evaluation of all next-to-leading-order radiative corrections to the lifetime of both the orthostate and the parastate of the dimuonic atom. In leading order, we reproduce the known results [3,4]

$$\Gamma(n^3S_1) = \frac{\alpha^5 m_\mu}{6n^3}, \quad \Gamma(n^1S_0) = \frac{\alpha^5 m_\mu}{2n^3} \quad (63)$$

as imaginary contributions to the hyperfine splitting [Eqs. (13) and (27)].

The results in next-to-leading order are, for orthodimuonium,

$$\Gamma(1^3S_1) = \left[1 + \frac{\alpha}{\pi} 3.74(4) \right] \Gamma^{(0)}(1^3S_1), \quad (64)$$

$$\Gamma(2^3S_1) = \left[1 + \frac{\alpha}{\pi} 3.60(4) \right] \Gamma^{(0)}(2^3S_1),$$

where the primary theoretical uncertainty is due to hadronic vacuum polarization. The lifetimes of orthostates are given by

$$\tau(1^3S_1) = 1.790\,73(77) \times 10^{-12} \text{ s}, \quad (65)$$

$$\tau(2^3S_1) = 14.3305(59) \times 10^{-12} \text{ s}.$$

For paradimuonium, we obtain

$$\Gamma(1^1S_0) = \left[1 + \frac{\alpha}{\pi} 4.79 \right] \Gamma^{(0)}(1^1S_0), \quad (66)$$

$$\Gamma(2^1S_0) = \left[1 + \frac{\alpha}{\pi} 4.65 \right] \Gamma^{(0)}(2^1S_0).$$

The lifetimes are given by

$$\tau(1^1S_0) = 0.595\,47(33) \times 10^{-12} \text{ s}, \quad (67)$$

$$\tau(2^1S_0) = 4.7653(25) \times 10^{-12} \text{ s}.$$

We estimate higher-order QED corrections to enter at the 5% level of the next-to-leading-order corrections considered in this work.

Lifetimes in the 10^{-12} s range can be measured by established methods of particle physics via detection of the decay products (electron-positron pairs in the case of orthodimuonium and two photons in the case of paradimuonium). We stress that accurate decay rate measurements can be accomplished with fewer individual atoms than would be needed for spectroscopic investigations.

One of the ways to investigate the hyperfine structure of $1S$ or $2S$ states could be based on the observation of the interference between paradimuonium and orthodimuonium in an infrared frequency field at resonance (the radio-frequency field would mix the two states and thus yield a modified decay rate of the statistical sample). We conclude that the dimuonic system offers the possibility to observe quantum electrodynamic effects in a previously unexplored kinematic region.

Note added in proof. After this work was submitted, Professor W. W. Repko informed us about an evaluation of the hadronic vacuum polarization contribution to the hyperfine splitting in the dimuonic system [33]. Our result in Eq. (40) is in agreement with this previous estimate.

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