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ADAPTIVE CRITIC-BASED NEURAL NETWORK CONTROLLER FOR UNCERTAIN NONLINEAR SYSTEMS WITH UNKNOWN DEADZONES[1]

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Abstract — A novel multilayer neural network (NN) controller in discrete-time is designed to deliver a desired tracking performance for a class of nonlinear systems with input deadzones. This multilayer NN controller has an adaptive critic NN architecture with two NNs for compensating the deadzone nonlinearity and a third NN for approximating the dynamics of the nonlinear system. Reinforcement learning scheme in discrete-time is proposed for the adaptive critic NN deadzone compensator, where the learning is performed based on a certain performance measure, which is supplied from a critic. The adaptive generating NN rejects the errors induced by the deadzone whereas a second NN based critic generates a signal, which is used to tune the weights of the action generating NN so that the deadzone compensation scheme becomes adaptive whereas a third multilayer NN simultaneously approximate the nonlinear dynamics of the system. Using the Lyapunov approach, the uniform ultimately boundedness (UUB) of the closed-loop tracking error and weight estimates of action generating NN, critic NN and the third NN are shown by using a novel weight updates.

I. INTRODUCTION

Several adaptive control schemes, both in continuous and discrete-time, were presented in the literature for the past decade [2-3,5,7]. Discrete-time adaptive control design is far complex than continuous-time [2] due primarily to the fact that discrete-time Lyapunov derivatives are quadratic in the state, not linear as in the continuous-case. Nonlinear systems, for instance robot manipulators, high power machinery and more, often have actuator nonlinearities such as deadzone, backlash, saturation etc. The deadzone parameters are unknown, which presents a significant challenge to the control engineer. Standard techniques for compensating deadzones are dithering [4] and so on. Recently, adaptive schemes have been used to compensate for deadzone in nonlinear systems [6]. In keeping with standard adaptive control techniques applied to discrete-time systems, a certainty equivalence assumption was then made to prove the stability.

Learning-based control methodology using neural networks (NN) and fuzzy logic has emerged as an alternative to adaptive control [7] since these NN and FL-based systems are nonlinear in the tunable parameters. The NN control research is now being pursued by several groups [2-3,5,7,8] since. While some work in the design of discrete-time NN controllers is available [2], however, most of the results are either based on Backpropagation [7] or using tracking error information [2] without any performance measure. The NN function approximation properties and ability of NN to discriminate information based on regions defined by the input variables (classification property), makes them as ideal candidates for compensation of nonanalytic actuator nonlinearities. On the other hand, the fuzzy logic-based deadzone compensator is given in [5].

In this paper, a novel adaptive critic neural network-based controller is developed to control a class of unknown nonlinear systems with unknown deadzones in discrete-time. The general case of nonsymmetric deadzone is treated. The action generating NN in the adaptive critic compensates the unknown deadzone and its weights are tuned based on a signal from the critic NN, whose weights are tuned by tracking error. In addition, a third NN approximates the nonlinear dynamics of the unknown nonlinear system. A rigorous design procedure is given that results in a outer PD tracking loop with an adaptive NN loop for the approximation of nonlinear dynamics, another adaptive critic-NN based feedforward loop for deadzone compensation. The approach is akin to the fuzzy logic deadzone compensator [5] but additional complexities arise due to the fact that several nonlinearities are simultaneously compensated. Closed-loop performance is shown using a Lyapunov-based analysis.

II. BACKGROUND

The following mathematical notion is required for the development of the neural network controller.

2.1 Properties of the NN Systems

A general function $f(x) \in C^{(s)}$ can be approximated using the two-layer neural network as

$$f(x(k)) = W^T \varphi_2\left(V^T \varphi_1(x(k))\right) + \varepsilon(k), \qquad (1)$$

where W and V are constant weights and $\varphi_2\left(V^T(k)\varphi_1(x(k)\right), \varphi_1(x(k))$ denote the vectors of activation functions at the instant k, with $\varepsilon(k)$ an NN functional reconstruction error vector. The net output is defined as

$$\hat{f}(x(k)) = \hat{W}^T \varphi_2\left(\hat{V}^T \varphi_1(x(k))\right) + \varepsilon(k) \qquad (2)$$

From now on $\varphi_1(x(k))$ is denoted as $\varphi_1(k)$ and $\varphi_2(\hat{V}^T \varphi_1(x(k)))$ is denoted as $\hat{\varphi}_2(k)$. It is shown in [8] if the inputs to the hidden layer weights are selected randomly and kept constant, $\varphi(k)$ forms a basis, then any nonlinear function can be approximated.

2.2 Nonlinear System Description

Consider the following nonlinear system, to be controlled, given in the following form

$$x_1(k+1) = x_2(k)$$
$$\vdots$$
$$x_n(k+1) = f(x(k)) + u(k) + d(k) \qquad (3)$$

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with state $x(k)=[x_1(k),x_2(k),\ldots,x_n(k)]^T$ with each $x_i(k)\in\Re^m$; i=1,..,n is the state at time instant k, $f(x(k))\in\Re^m$ is the unknown nonlinear function, $u(k)\in\Re^m$ is the input and $d(k)\in\Re^m$ is the unknown but bounded disturbance vector, whose bound is given by $\|d(k)\|\le d_M$. Given a trajectory, $x_{nd}(k)$, and its delayed values, $x_d(k)=[x_{1d}(k),x_{2d}(k),\ldots,x_{nd}(k)]^T$ define the tracking error

$$e_n(k)=x_n(k)-x_{nd}(k), \tag{4}$$

and the filtered tracking error, $r(k)\in\Re^m$, as

$$r(k)=[\Lambda\ 1]e(k)\ , \tag{5}$$

with $e(k)=[e_1(k),e_2(k),\ldots,e_n(k)]^T$, $e_1(k+1)=e_2(k)$, where $e_1(k+1)$ is the future value for the error $e_1(k)$, $e_{n-1}(k),\ldots,e_1(k)$ are delayed values of the error $e_n(k)$ and $\Lambda=[\lambda_{n-1},\lambda_n,\ldots,\lambda_1]^T\in R^{mnxn}$ are constant diagonal positive definite matrices selected such that the eigen values are within a unit disc. Equation (3) can be expressed as

$$r(k+1)=f(x(k))-x_{nd}(k+1)+\lambda_1 e_n(k)+\cdots$$
$$+\lambda_{n-1}e_2(k)+u(k)+d(k) \tag{6}$$

2.3 Compensation of Deadzone Nonlinearity

In this section, an adaptive critic NN precompensator is designed for nonsymmteric deadzone nonlinearities in the actuation systems of systems in the class (3).

To offset the deleterious effects of deadzones, a precompensator as illustrated in Figure 1 is required. There, the desired function of the precompensator is to cause the composite throughput from p(k) to u to be unity. For a scalar system, the nonsymmetric deadzone given by

$$u=D_d(\tau)=\tau-sat_d(\tau), \tag{7}$$

where

$$\begin{aligned} sat_d(\tau) &= -d_-,\ \tau<-d_- \\ &=\tau,\ -d_-\le\tau<d_+ \\ &=d_+,\ d_+\le\tau \end{aligned} \tag{8}$$

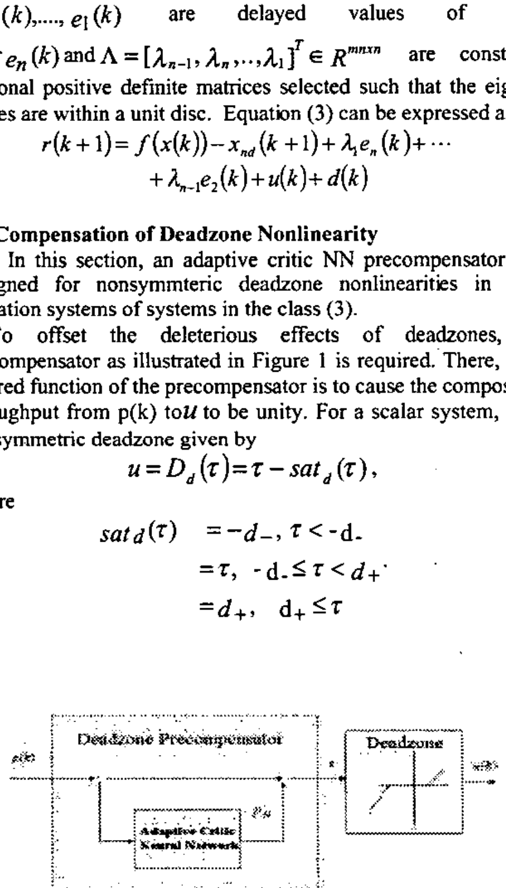

Fig 1: Deadzone compensation scheme.

The parameter vector $d=[d_+\ d_-]^T$ characterizes the width of the motion dead band. For a n-dimensional practical systems, $p(k),\tau(k),u(k)$ are generally n vectors, which can be written in vector form

$$u(k)=D_D(\tau)=\tau-sat_D(\tau), \tag{9}$$

where $D\equiv diag\{d_1,d_2,\ldots,d_n\}\in\Re^{2nxn}$. Then one can use a NN for each dimension or one NN for all the channels. Define the estimate vector $\hat{d}_i=[\hat{d}_{i+}\ \hat{d}_{i-}]^T$ and $\hat{D}\equiv diag\{\hat{d}_1,\hat{d}_2,\ldots,\hat{d}_n\}$. According to [5], the throughput of the compensator plus the deadzone for vectors $p,\tau,u\in\Re^n$ is

$$u=p+\tilde{D}^T\phi(k), \tag{10}$$

The vector of basis functions is given as $\phi(k)$.

III. DISCRETE-TIME ADAPTIVE-CRITIC MULTILAYER NN CONTROLLER

In this section, it is shown how to tune or learn the deadzone width estimates, $\hat{D}$ on-line using a critic signal so that the tracking error is guaranteed small and all internal states are bounded.

Assumption (Constant bounded deadzone): It is assumed that the unknown deadzone widths are bounded so that $\|D\|\le D_{\max}$ for some known scalar $D_{\max}$. Moreover, it is assumed that $D(k+1)=D(k)$.

3.1 Tracking Controller with NN Deadzone Compensation

A compensation scheme for unknown f(x(k)) is provided using a third NN by selecting the tracking controller as

$$p(k)=x_{nd}(k+1)-\hat{f}(x(k))+k_v r(k)-\lambda_1 e_n(k)$$
$$-\cdots-\lambda_{n-1}e_2(k)+v(k) \tag{11}$$

where $\hat{f}(x(k))$ is an estimate of the unknown function f(x(k)), v(k) is an auxiliary input, and k_v is a diagonal gain matrix. The deadzone compensation is provided using

$$\tau(k)=p(k)+\hat{w}_2^T(k)\phi(v_2^T k), \tag{12}$$

where $\hat{w}_2$ is the NN weight matrix and $\phi(v_2^T k)\equiv\phi(k)$ is defined as a basis for the action generating NN, which gives the overall feedforward throughput. Substituting (11), (7),(10) in (6) yields the closed-loop dynamics

$$r(k+1)=k_v r(k)+\tilde{f}(x(k))+\tilde{w}_2^T(k)\phi(k)+v(k)+d(k) \tag{13}$$

where the nonlinear functional estimation error is given by $\tilde{f}(x(k))=f(x(k))-\hat{f}(x(k))$. Equation (13) relates the filtered tracking error with the functional estimation error, deadzone width estimation errors and bounded disturbances. In the remainder of this paper, (13) is used to focus on selecting NN tuning algorithms that guarantee the stability of the filtered tracking error r(k). If the functional estimation error $\tilde{f}(x(k))$ is bounded above such that $\|\tilde{f}(x(k))\|\le f_M(x(k))$, for some known bounding function $f_M(x(k))$ and the deadzone widths are bounded above such that $\|\tilde{D}(k)\phi(k)\|\le D_M\phi_{\max}$, then next stability results hold, where $\tilde{D}(k)=\tilde{w}_2(k)$.

Theorem 2.1: Consider the system given by (3). Let the control action is provided by (10) with the auxiliary input $v(k)=-[f_M(x(k))+D_M\phi_{\max}+D_M\delta_N+d_M]$. The closed-loop tracking error system (18) is stable provided

$$k_v^T k_v<I \tag{14}$$

Proof: Let us consider the following Lyapunov function candidate $J = r(k)^T r(k)$. The first difference is given by

$$\Delta J = r(k+1)^T r(k+1) - r(k)^T r(k). \quad (15)$$

Substituting the tracking error dynamics (13) along with the auxiliary input given above results in

$$\Delta J \le -(1 - k_{v\max}^2)\|r(k)\|^2. \quad (16)$$

The closed-loop system is globally asymptotically stable.

3.2. NN Controller Design

The structure of the proposed adaptive critic NN controller is shown in Figure 2 An inner action generating NN loop eliminates the nonlinear dynamics of the system. The adaptive critic NN feedforward loop compensates for the deadzone. The outer loop designed via Lyapunov guarantees the stability and accuracy in following the desired trajectory. The tracking error, r(k), can be viewed as the real-valued instantaneous utility function of the plant performance. When r(k) is small, system performance is good. It is now desired to tune the action generating NN in such a fashion that the tracking error r(t) is guaranteed to be small and the control input u(k) is bounded. Stability analysis using a Lyapunov direct method is carried out for the closed-loop system (13).

To accomplish this using a critic NN architecture, introduce a critic NN that manufactures a critic signal as

$$R(k) = \hat{w}_1^T(k)\sigma(v_1^T(r(k))) = \hat{w}_1^T(k)\sigma(.), \quad (17)$$

where $\hat{w}_1(k)$, v_1 are the matrix of weight estimates of the critic NN. The input to the critic NN is the signal r(k) which contains performance of the system. The critic signal R must be used to tune the weights $\hat{w}_2(k)$ of the action generating NN, which is defined next. Defining the action generating NN functional estimate as

$$p_N = \hat{w}_2^T(k)\phi(v_2^T x(k)) = \hat{w}_2^T(k)\phi(.) \quad (18)$$

where $\hat{w}_2(k)$, v_2 are the matrix of weight estimates. Let the functional estimate provided by the third NN is given by

$$\hat{f}(x(k)) = \hat{w}_3^T(k)\varphi(v_3^T x(k)) = \hat{w}_3^T(k)\varphi(.) \quad (19)$$

where $\hat{w}_3(k)$, v_3 are the matrix of weight estimates for this NN. Let w_1, w_2 and w_3 are the unknown target NN weights for the action generating NN, critic NN and the third NN and assume that they are bounded so that

$$\|w_1\| \le w_{1\max}, \|w_2\| \le w_{2\max}, \|w_3\| \le w_{3\max} \quad (20)$$

where $w_{1\max}$, $w_{2\max}$, and $w_{3\max}$ are the maximum bound on the unknown weights. Then the error in weights during estimation is given by

$$\tilde{w}(k) = w - \hat{w}(k), \quad (21)$$

where $w = diag(w_1, w_2, w_3)$, and $\hat{w} = diag(\hat{w}_1, \hat{w}_2, \hat{w}_3)$

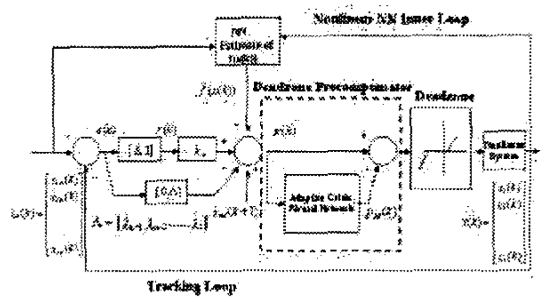

Fig 2: NN controller structure.

Assumption: The activation functions for all NN are bounded by known positive values so that

$$\|\phi(x(k))\| \le \phi_{\max}, \|\sigma(r)\| \le \sigma_{\max}, \|\varphi(x(k))\| \le \varphi_{\max} \quad (22)$$

Substituting the third NN estimate and the deadzone compensation scheme in (13) yields the closed-loop tracking dynamics

$$r(k+1) = k_v r(k) + \bar{e}_{li}(k) + \bar{e}_i(k) + v(k) + \varepsilon(k) + d(k) \quad (23)$$

where the functional estimation and deadzone width estimation *errors* are defined by

$$\bar{e}_{li}(k) = \tilde{w}_3^T(k)\varphi(k), \quad (24)$$

$$\bar{e}_i(k) = \tilde{w}_2^T(k)(\phi(k)) \quad (25)$$

3.3 Weight Updates for Guaranteed Performance

It is required to demonstrate that the performance criterion in terms of tracking error, r(k), is suitably small and that the NN weights, $\hat{w}_1(k)$, $\hat{w}_2(k)$, and $\hat{w}_3(k)$ remain bounded. In the following theorem, a discrete-time weight tuning algorithms based on the tracking error is given so that closed-loop stability is inferred.

Theorem 3.1: (General case): Let the desired trajectory, x_{nd}, and its delayed values are bounded. Let the approximation error is bounded above by ε_N, and the disturbance bound d_M are known constants. Let the critic NN weight tuning is given by

$$\hat{w}_1(k+1) = \hat{w}_1(k) - \alpha_1\sigma(.)\,(k_v r + R)^T - \Gamma_1\|I - \alpha_1\sigma(.)\sigma(.)^T\|\hat{w}_1(k) \quad (26)$$

with the action generating NN weight tuning provided by

$$\hat{w}_2(k+1) = \hat{w}_2(k) - \alpha_2\phi(.)\,(\hat{w}_2\phi + r(k+1) + AR)^T - \Gamma_2\|I - \alpha_2(.)\phi(.)\phi^T(.)\|\hat{w}_2(k) \quad (27)$$

and the weight tuning for the third NN is taken as

$$\hat{w}_3(k+1) = \hat{w}_3(k) - \alpha_3\varphi(k)\,(\hat{w}_3^T\varphi(k) + Bk_v r)^T - \Gamma_3\|I - \alpha_2\varphi(k)\varphi(k)^T\|\hat{w}_3(k) \quad (28)$$

with the auxiliary input given by

$$v(k) = \hat{w}_3^T(k)\varphi(k) + \hat{w}_2^T(k)\phi(k) - CR, \quad (29)$$

where the critic signal, R(k), is specified in (17), matrix B is a design matrix whose bound is given by $\|B\| \le \kappa_1, \|A\| \le \kappa_2, \|C\| \le \kappa$ and $\alpha_1, \alpha_2, \alpha_3$ are adaptation gains with $\Gamma_1, \Gamma_2, \Gamma_3$ are design parameters. Then the filtered tracking error, $r(k)$, the NN weight estimates, $\hat{w}_1(k), \hat{w}_2(k), \hat{w}_3(k)$ are *UUB*, with the bounds specifically given by (A.11) and (A.12) provided the design parameters are selected as:

$$(1)\ \alpha_1 \|\sigma(.)\|^2 < 1, \alpha_2 \|\phi(.)\|^2 < 1, \alpha_3 \|\varphi(.)\|^2 < 1, \quad (30)$$

$$(2)\ 0 < \Gamma_1 < 1, 0 < \Gamma_2 < 1, 0 < \Gamma_3 < 1, \quad (31)$$

$$(3)\ k_{v\max} < \frac{1}{\sqrt{\bar{\sigma}}}; \quad (32)$$

where $k_{v\max}$ is the maximum singular value of the gain matrix, k_v and the parameter $\bar{\sigma}$ is presented during the proof. Moreover, the performance measure r(k) can be made arbitrarily small by placing the poles using the gain matrix, k_v.

Note: Note it is very easy to verify conditions (1) through (3) and therefore the proof is omitted.

Proof: *See Appendix.* ■

Remark 1: It is important to note that in this theorem there is no certainty equivalence (CE) assumption, in contrast to standard work in discrete-time adaptive control.

Remark 2: The mutual dependence between the two NNs in the adaptive critic NN architecture result in coupled tuning law equations. As a result, additional complexities arise due to the addition of the third NN with the adaptive critic architecture since this addition causes further interaction among the NNs.

Remark 3: Cross product terms are cancelled by the auxiliary input v(k).

IV. SIMULATION RESULTS

Consider the following nonlinear plant:

$$x_1(k+1) = x_2(k), \quad x_2(k+1) = f(x(k)) + u(k) \quad (33)$$

where $x(k)) = [x_1(k), x_2(k)]^T$ and the unknown nonlinear dynamics is $f(x(k)) = -\frac{3}{16}\left[\frac{x_2(k)}{1+x_1^2(k)}\right] + x_2(k)$. The deadzone widths were selected as $d_+ = d_- = 0.8$. The reference signal used was selected to be

$$x_{2d} = \sin(\omega t_k + \zeta), \omega = 0.5, \zeta = \frac{\pi}{2}. \quad (34)$$

The initial estimates for the deadzone widths were selected as $d_+ = d_- = 0$ and the sampling interval of T=50msec. The learning rate is selected as $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$ and the design parameters are set at $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0.001$. The auxiliary input parameter matrices were selected as *A=0.5I, B=0.1I* and *C=0.008I,* where I is the appropriately dimensioned identity matrix. The conventional PD controller $(k_v = 0.99, \Lambda = 0.2)$ does a reasonable job on the tracking but large tracking error remains as shown in Figure 3. The system tracking performance is destroyed and the PD controller by itself is not capable of compensating for that. Figure 5 shows the performance of the PD controller plus the third NN that was used for approximating the unknown dynamics. Here the adaptive critic deadzone compensator was not included. Though the third NN compensates the unknown dynamics of the nonlinear system, it fails to compensate the deadzone resulting in considerable errors. Figure 7 shows the performance with the adaptive critic NN deadzone compensator, PD controller plus the third NN. The NN compensator takes care of the system deadzone and the tracking performance is as good as it was without deadzone. It is important to note that the discrete-time NN deadzone compensator does not take care of steady state tracking error remaining in the system after the PD controller action.

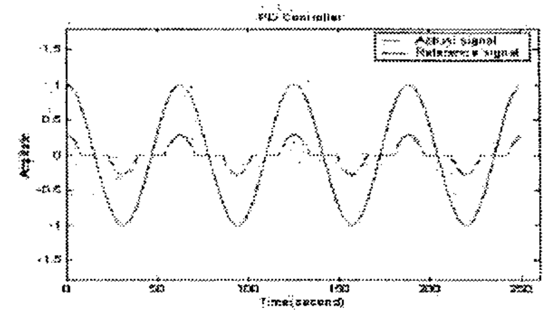

Fig 3: PD controller alone with deadzone.

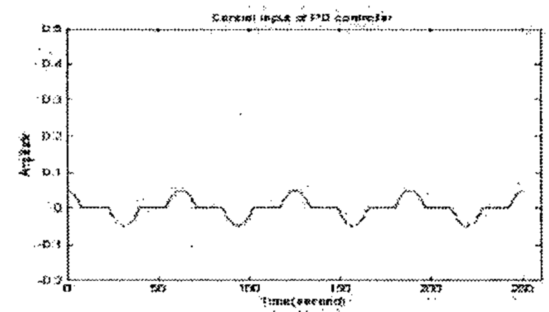

Fig 4: The control input for the PD controller alone.

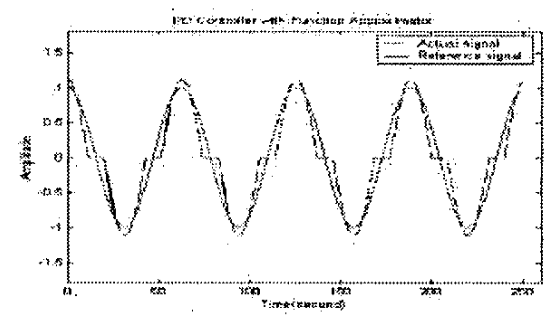

Fig 5: PD controller with a function approximator alone.

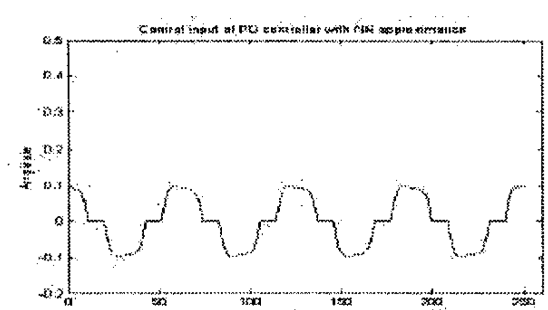

Fig 6: The control input.

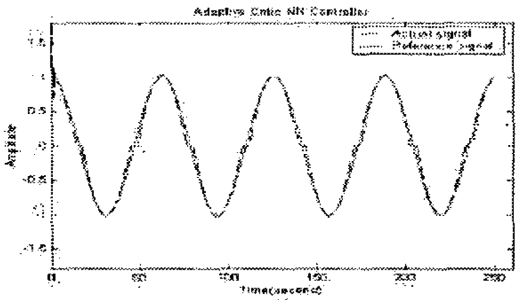

Fig 7: Performance of the adaptive critic NN compensator.

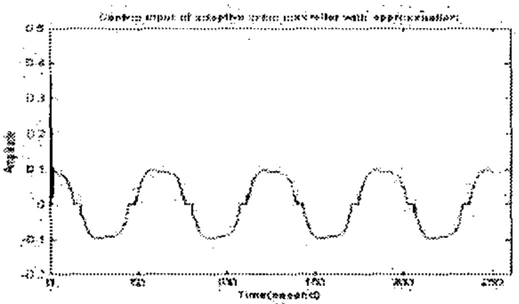

Fig 8: The control input.

V. CONCLUSIONS

This paper proposes a multilayer NN controller for a class of unknown nonlinear systems with unknown deadzones. This multilayer NN controller consists of an inner NN approximation loop, an adaptive critic NN based feedforward deadzone compensator and an outer PD tracking loop. This adaptive NN-based approach does not require the information about the system dynamics nor requires an initial learning phase. A novel weight tuning methods are derived for the NN algorithm using a rigorous mathematical analysis. The adaptive critic NN compensator includes an action generating NN for compensating the deadzone, a critic signal for tuning the action generating NN. The tuning of the action generating NN, critic NN and the inner NN for approximating the unknown nonlinear dynamics is performed online and guarantees performance as shown through the Lyapunov analysis. Since both the tracking error and the parameter estimation error are weighted in the same Lyapunov function, no certainty equivalence assumption is needed.

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APPENDIX

Proof of Theorem 3.1: Define the Lyapunov function candidate

$$J = r^T(k)r(k) + \sum_{i=1}^{3} \frac{1}{\alpha_i} tr\left(\tilde{w}_i^T(k)\tilde{w}_i(k)\right),$$

whose first difference is

$$\Delta J = \Delta J_1 + \Delta J_2 + \Delta J_3 + \Delta J_4 . \qquad (A.1)$$

The first difference ΔJ_1 is obtained using (23) and (29)as

$$\Delta J_1 = \left(k_v r(k) + d_1(k)\right)^T \left(k_v r(k) + d_1(k)\right) + \|CR\|^2 - 2(CR)^T\left(k_v r(k) + d_1(k)\right) - r^T(k)r(k) \qquad (A.2)$$

where $d_1(k) = \bar{e}_{1i}(k) + \bar{e}_i(k) + \varepsilon(k) + d(k) + \hat{w}_3^T\varphi + \hat{w}_2^T\varphi$

Now taking the second term in the first difference (A.1) and substituting the NN weight updates from (26) into it and combining

$$\begin{aligned}\Delta J_2 =& -\left(2 - \alpha_1\sigma(.)^T\sigma(.)\right)\left(\tilde{w}_1^T\sigma(.)\right)^T\left(\tilde{w}_1^T\sigma(.)\right) \\ &+ \frac{1}{\alpha_1}\left\|I - \alpha_1\sigma(.)\,\sigma^T(.)\right\|^2 tr[\Gamma_1^2\hat{w}_1^T\hat{w}_1 + 2\Gamma_1\tilde{w}_1^T\hat{w}_1] \\ &+ 2\left(1 - \alpha_1\sigma(.)^T\sigma(.) - \Gamma_1\left\|I - \alpha_1\sigma(.)\sigma^T(.)\right\|\right)\left(\tilde{w}_1^T\sigma(.)\right)^T\left(k_v r(k) + w_1^T\sigma(.)\right) \\ &+ \alpha_1\sigma(.)^T\sigma(.)\left\|k_v r(k) + w_1^T\sigma(.)\right\|^2 \\ &+ 2\Gamma_1\left\|I - \alpha_1\sigma(.)\,\sigma^T(.)\right\|\left(w_1^T\sigma(.)\right)^T\left(k_v r(k) + w_1^T\sigma(.)\right) \qquad (A.3)\end{aligned}$$

Taking the third term in (A.1) and substituting the weight updates for the NN (27) and simplifying to get

$$\begin{aligned}\Delta J_3 =& -(2 - \alpha_2\,\phi^T(.)\phi(.))\times \\ &\left\|\tilde{e}_i + \frac{[(1-\alpha_2\,\phi^T(.)\phi(.)) - \Gamma_2\left\|I - \alpha_2\,\phi(.)\phi(.)^T\right\|]^2}{(2-\alpha_2\,\phi^T(.)\phi(.))}(k_v r(k) + d_2 + ER)\right\|^2 \\ &+ \frac{1}{\alpha_2}\left\|I - \alpha_2\phi(.)\,\phi^T(.)\right\|^2 tr[\Gamma_2^2\hat{w}_2^T(k)\hat{w}_2(k) + 2\Gamma_2\tilde{w}_2^T(k)\hat{w}_2(k)] \\ &+ 2\Gamma_2\left\|I - \alpha_2\,\phi(.)\phi(.)^T\right\|\left(w_2^T\phi\right)^T(k_v r(k) + d_2 + ER) \\ &+ \gamma_1\left\|k_v r(k) + d_2 + ER\right\|^2 \qquad (A.4)\end{aligned}$$

where

$$\gamma_1 = \alpha_2\phi^T\phi + \frac{[(1-\alpha_2\,\phi^T(.)\phi(.)) - \Gamma_2\left\|I - \alpha_2\,\phi(.)\phi(.)^T\right\|]^2}{(2-\alpha_2\,\phi^T(.)\phi(.))}$$

and $d_2 = d_1 + w_2^T\phi$, $E = A - C$

Now taking the third term in (A.1) and substituting the weight updates for the NN (28) and simplifying to get

$$\Delta J_4 = -(2-\alpha_3\varphi^T(k)\varphi(k))\times \left\|\tilde{w}_3^T\varphi(k) - \frac{[(1-\alpha_3\varphi^T(k)\varphi(k)) - \Gamma_3\|I-\alpha_3\varphi(.)\varphi^T(.)\|(w_3^T\varphi(k)+Bk_v r)}{(2-\alpha_3\varphi^T(k)\varphi(k))}\right\|^2$$

$$+\frac{1}{\alpha_3}\|I-\alpha_3\varphi(k)\varphi^T(k)\|^2 tr[\Gamma_3^2\hat{w}_3^T\hat{w}_3 + 2\Gamma_3\tilde{w}_3^T\hat{w}_3]$$

$$+2k_{v\max}\|r(k)\|(\beta_3+\Gamma_3\|I-\alpha_3\varphi(k)\varphi^T(k)\|)w_{3\max}\varphi_{\max}\kappa_1$$

$$+k_{v\max}^2\|r(k)\|^2\kappa_1^2\beta_3 + (\beta_3+2\Gamma_3\|I-\alpha_3\varphi(k)\varphi^T(k)\|)\varphi_{\max}^2 w_{3\max}^2 \quad (A.5)$$

where

$$\beta_3 = \alpha_3\varphi_{\max}^2 + \frac{\left((1-\alpha_3\varphi^T(k)\varphi(k)) - \Gamma_3\|I-\alpha_3\varphi(.)\varphi^T(.)\|\right)^2}{(2-\alpha_3\varphi^T\varphi)}$$

Combining (A.2), (A.3), (A.4), and (A.5) and auxiliary input (34) to get the first difference and simplifying to get

$$\Delta J \le -(1-\bar{\sigma}k_{v\max}^2)\|r(k)\|^2 + 2k_{v\max}\gamma\|r(k)\| + \rho$$

$$-(1-\alpha_1\sigma^T(.)\sigma(.)-\gamma_1\|D\|)\left\|\tilde{w}_1^T\sigma(.) - \frac{d_3+d_4(k_v r(k))}{(1-\alpha_1\sigma^T(.)\sigma(.)-\gamma_1\|D\|)}\right\|^2$$

$$-(2-\alpha_2\phi^T(.)\phi(.))$$

$$\left\|\tilde{e}_i + \frac{[(1-\alpha_2\phi^T(.)\phi(.)) - \Gamma_2\|I-\alpha_2\phi(.)\phi(.)^T\|]^2}{(2-\alpha_2\phi^T(.)\phi(.))}(k_v r(k)+d_2+ER)\right\|^2$$

$$-(2-\alpha_3\varphi^T(k)\varphi(k))\times$$

$$\left\|\tilde{w}_3^T\varphi(k) - \frac{[(1-\alpha_3\varphi^T(k)\varphi(k)) - \Gamma_3\|I-\alpha_3\varphi(.)\varphi^T(.)\|(w_3^T\varphi(k)+Bk_v r)}{(2-\alpha_3\varphi^T(k)\varphi(k))}\right\|^2$$

$$-\alpha_{\max}c_{\min}[\|\tilde{w}(k)\| - \frac{c_{\max}}{c_{\min}}w_{\max}]^2 + (\frac{c_{\max}^2 - c_{0\max}^2}{c_{\min}^2})w_{\max}^2] \quad (A.6)$$

where

$$d_3 = -(\gamma_1\|D\|^2 + \alpha_1\sigma(.)^T\sigma(.) + \Gamma_1\|I-\alpha_1\sigma(.)\sigma^T(.)\|)(w_1^T\sigma(.)) + d_1$$
$$-\gamma_1 D^T d_2 - \Gamma_2\|I-\alpha_2\phi(.)\phi(.)^T\|(Dw_2^T\phi(.)),$$

$$d_4 = (2-\alpha_1\sigma(.)^T\sigma(.) - \Gamma_1\|I-\alpha_1\sigma(.)\sigma^T(.)\|)I - \gamma_1 D^T \quad (A.7)$$

$$\bar{\sigma} = \left(1+\gamma_1+\alpha_1\sigma_{\max}^2+\kappa_1^2\beta_3+\frac{\|d_{4\max}\|}{(1-\alpha_1\sigma_{\max}^2-\gamma_{1\max}\|D\|_{\max})}\right)$$

$$\gamma = d_{1\max} + \gamma_{1\max}d_{2\max} + w_{1\max}\sigma_{\max}\|I-\gamma_1 D^T\|_{\max}$$
$$+(\beta_3+\Gamma_3\|I-\alpha_3\varphi(.)\varphi^T(.)\|)w_{3\max}\varphi_{\max}\kappa_1 + \frac{d_{3\max}d_{4\max}}{(1-\alpha_1\sigma_{\max}^2-\gamma_{1\max}\|D\|_{\max})}$$
$$+2\Gamma_2\|I-\alpha_2\phi(.)\phi(.)^T\|w_{2\max}\phi_{\max} \quad (A.8)$$

$$\rho = d_{1\max}^2 + \gamma_{1\max}d_{2\max}^2 + (1+\gamma_1\|\beta\|^2)(w_{1\max}\sigma_{\max})^2$$
$$+2\Gamma_2\|I-\alpha_2\phi(.)\phi(.)^T\|w_{2\max}\phi_{\max}d_{2\max}$$
$$\left((\beta_3+2\Gamma_3\|I-\alpha_3\varphi(.)\varphi^T(.)\|)w_{3\max}^2\varphi_{\max} + \frac{d_{3\max}^2}{(1-\alpha_1\sigma_{\max}^2-\gamma_{1\max}\|D\|_{\max})}\right)$$
$$+2\Gamma_1\|I-\alpha_1\sigma(.)\sigma^T(.)\|(w_{1\max}\sigma_{\max})^2 \quad (A.9)$$

$$\alpha_{\max} = \max eig(\alpha), c_{0\max} = \max eig(\beta),$$
$$c_{\max} = \max eig(\eta+\beta), \quad c_{\min} = \min eig(2\eta-\beta),$$

$$\alpha = \begin{bmatrix} \frac{1}{\alpha_1}\|I-\alpha_1\sigma(.)\sigma^T(.)\|^2 & 0 & 0 \\ 0 & \frac{1}{\alpha_2}\|I-\alpha_2\phi(.)\phi^T(.)\|^2 & 0 \\ 0 & 0 & \frac{1}{\alpha_3}\|I-\alpha_3\varphi(.)\varphi^T(.)\|^2 \end{bmatrix},$$

$$\beta = \begin{bmatrix} \Gamma_1^2 & 0 & 0 \\ 0 & \Gamma_2^2 & 0 \\ 0 & 0 & \Gamma_3^2 \end{bmatrix},$$

$$\eta = diag(\Gamma_1, \Gamma_2, \Gamma_3), \quad (A.10)$$

Note: $2\eta - \beta > 0$ since $0 < \Gamma_1, \Gamma_2 < 1$. This implies that $\Delta J \le 0$ as long as (36) through (38) holds and

$$\|r(k)\| > \frac{1}{(1-\bar{\sigma}k_{v\max}^2)}[\gamma k_{\max} + \sqrt{\gamma^2 k_{v\max}^2 + (\rho + \frac{\alpha_{\max}(c_{\max}^2 - c_{0\max}^2)w_{\max}^2}{c_{\min}})(1-\bar{\sigma}k_{v\max}^2)} \quad (A.11)$$

or

$$\|\tilde{w}(k)\| > \frac{2\frac{c_{\max}}{c_{\min}}w_{\max} + \sqrt{\frac{4c_{\max}^2}{c_{\min}}w_{\max}^2 + 4c}}{\frac{c_{\max}}{c_{\min}}w_{\max} + \sqrt{\frac{c_{\max}^2}{c_{\min}}w_{\max}^2 + c}}, \quad (A.12)$$

where $c = \frac{c_{0\max}}{c_{\min}}w_{\max}^2 + \frac{\gamma^2 k_{v\max}^2}{(1-\bar{\sigma}k_{v\max}^2)}$. In general $\Delta J \le 0$ in a compact set as long as (30) through (32) are satisfied and (A.11) or (A.12) holds. According to a standard Lyapunov extension theorem [1], this demonstrates that the filtered tracking error and the error in weight estimates are UUB. The boundedness of $\tilde{w}_1(k)$, $\tilde{w}_2(k)$, and $\tilde{w}_3(k)$ implies that the weight estimates $\hat{w}_1(k)$, $\hat{w}_2(k)$, and $\hat{w}_3(k)$ are bounded. ∎