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SNEAK–OUT PRINCIPLE ON TIME SCALES

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Abstract. In this paper, we show that the so-called "sneak-out principle" for discrete inequalities is valid also on a general time scale. In particular, we prove some new dynamic inequalities on time scales which as special cases contain discrete inequalities obtained by Bennett and Grosse-Erdmann. The main results also are used to formulate the corresponding continuous integral inequalities, and these are essentially new. The techniques employed in this paper are elementary and rely mainly on the time scales integration by parts rule, the time scales chain rule, the time scales Hölder inequality, and the time scales Minkowski inequality.

1. Introduction

In [4], Bennett and Grosse-Erdmann studied the problem of deducing the convergence of one series from that of another during the course of their investigations on Hardy type inequalities. In particular, they have considered general types of series which cannot be treated by any of the usual convergence tests (ratio test, comparison test, Raabe's test, etc.). After they had several successes, they noticed something quite remarkable in a subject as old as this: The emergence of new techniques. They called these the "five series theorem", the "sneak-out principle" and the "heads/tails option" (see [3]). All three are valid in considerable generality and all have significant applications beyond Hardy's inequality.

Now the following question arises: Is it possible to extend these techniques to time scales? In other words, is it possible to extend these techniques to contain corresponding integral inequalities and summation inequalities as special cases? Our aim in this paper is to give an affirmative answer for the second one which is the sneak-out principle. The obtained results will support the advice of Hardy, Littlewood and Pólya [12, page 11] by giving a unification of the continuous and the discrete inequalities and showing that what goes for sums goes for integrals and vise versa.

The sneak-out principle that has been considered by Bennett and Grosse-Erdmann [3] was concerned with the equivalence of the two series

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} x_k \right)^p \tag{1.1}$$

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and

$$\sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} x_k \right)^p .$$

In other words, when is it possible to sneak the term A_k^α out of the inner sum in (1.1). Bennett and Grosse-Erdmann, using this principle, proved several inequalities of the form

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^\alpha x_k \right)^p \leqslant K(\alpha, p) \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} x_k \right)^p$$

and their inverses for different values of p and α. Our aim in this paper is concerned with the equivalence of the two time scales integrals

$$\int_{t_0}^{\infty} a(t) \left(\int_t^{\infty} A^\alpha(\sigma(s)) x(s) \Delta s \right)^p \Delta t \tag{1.2}$$

and

$$\int_{t_0}^{\infty} a(t) A^{\alpha p}(\sigma(s)) \left(\int_t^{\infty} x(s) \Delta s \right)^p \Delta t,$$

where the domain of the unknown function is a so-called time scale $\mathbb{T}$ (which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$). In other words, we want to determine precisely when it is possible to sneak $A^\alpha(\sigma(s))$ outside of the inner integral in (1.2). More precisely, we are concerned with new dynamic inequalities of the form

$$\int_{t_0}^{\infty} a(t) \left(\int_t^{\infty} A^\alpha(\sigma(s)) x(s) \Delta s \right)^p \Delta t \leqslant K \int_{t_0}^{\infty} a(t) A^{\alpha p}(\sigma(s)) \left(\int_t^{\infty} x(s) \Delta s \right)^p \Delta t \tag{1.3}$$

and their converses on time scales for different values of p and α, which as special cases with $\mathbb{T} = \mathbb{N}$ contain the discrete inequalities obtained by Bennett and Grosse-Erdmann [3, Section 6] and can be applied also with $\mathbb{T} = \mathbb{R}$ to formulate the corresponding integral inequalities. The results also can be applied to other time scales such as $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. We include the details of the proofs since they provide a strategy which can be used in other situations. For related dynamic inequalities on time scales, we refer the reader to [1, 2, 5, 8, 13–17].

This paper is organized as follows: In Section 2, we provide some auxiliary results such as the time scales integration by parts rule, the time scales chain rule, the time scales Hölder inequality, and the time scales Minkowski inequality. Section 3 features two new dynamic inequalities of Copson type that are needed in the proofs of our main results. In Section 4, we present our main results, which are three dynamic inequalities of the type (1.3) for different values of $p \geqslant 1$ and α. First the case $\alpha \geqslant 1$ is treated (see Theorem 4.1 below), then the case $0 \leqslant \alpha \leqslant 1$ (see Theorem 4.4 below), and finally the case $-1/p < \alpha \leqslant 0$ (see Theorem 4.6 below). The corresponding cases when $0 < p < 1$ are still open problems and will be considered in the future.

2. Auxiliary results

For completeness, we recall the following concepts related to the notion of time scales. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [6, 7]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ represent the classical theories of differential and difference calculus. In this paper, we assume that $\sup\mathbb{T}=\infty$ and define the forward jump operator by

$$\sigma(t) := \inf\{s\in\mathbb{T}:\ s>t\}.$$

For any function $f:\mathbb{T}\to\mathbb{R}$, we write f^σ for $f\circ\sigma$. For $t\in\mathbb{T}$, we define $f^\Delta(t)$ to be the number (if it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t with

$$\left|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)\right| \leqslant \varepsilon|\sigma(t)-s| \quad \text{for all} \quad s\in U.$$

In this case, we say $f^\Delta(t)$ is the (delta) derivative of f at t. If f is (delta) differentiable at any $t\in\mathbb{T}$, then $f^\Delta:\mathbb{T}\to\mathbb{R}$ is called the delta derivative of f. Next, if $F:\mathbb{T}\to\mathbb{R}$ is an antiderivative of f, i.e., $F^\Delta=f$, then the Cauchy delta integral of f is defined by

$$\int_a^t f(s)\Delta s := F(t)-F(a),$$

where $a\in\mathbb{T}$ is fixed. It is known [6, Theorem 1.74] that so-called rd-continuous functions always possess antiderivatives.

EXAMPLE 2.1. Note that if $\mathbb{T}=\mathbb{R}$, then

$$\sigma(t)=t, \quad f^\Delta=f', \quad \text{and} \quad \int_a^b f(t)\Delta t=\int_a^b f(t)\mathrm{d}t$$

for $a,b\in\mathbb{R}$ with $a<b$. If $\mathbb{T}=\mathbb{Z}$, then

$$\sigma(t)=t+1, \quad f^\Delta=\Delta f, \quad \text{and} \quad \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1} f(t)$$

for $a,b\in\mathbb{Z}$ with $a<b$. If $\mathbb{T}=h\mathbb{Z}$ with $h>0$, then

$$\sigma(t)=t+h \quad \text{and} \quad \int_a^b f(t)\Delta t=h\sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)$$

for $a,b\in h\mathbb{Z}$. If $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, then

$$\sigma(t)=qt \quad \text{and} \quad \int_a^b f(t)\Delta t=(q-1)\sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f(q^k)$$

for $a,b\in q^{\mathbb{N}_0}$.

Now we collect those known time scales results that will be used frequently throughout this paper. The product and quotient rules [6, Theorem 1.20] for the derivative of the product fg and the quotient f/g (with $g(t) \neq 0$ for all $t \in \mathbb{T}$) of two differentiable functions $f, g : \mathbb{T} \to \mathbb{R}$ state

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma} \quad \text{and} \quad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

The chain rule [6, Theorem 1.90] for the γ-th power ($\gamma \in \mathbb{R}$) of a differentiable function $f : \mathbb{T} \to \mathbb{R}$ says (see [6, Theorem 1.90])

$$(f^{\gamma})^{\Delta} = \gamma f^{\Delta} \int_0^1 (hf^{\sigma} + (1-h)f)^{\gamma-1} \, \mathrm{d}h. \tag{2.2}$$

The integration by parts formula [6, Theorem 1.77] for two differentiable functions $f, g : \mathbb{T} \to \mathbb{R}$ is given ($a, b \in \mathbb{T}$) by

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = f(t)g(t)\Big|_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t. \tag{2.3}$$

Hölder's inequality [6, Theorem 6.13] states that two rd-continuous functions $f, g : \mathbb{T} \to \mathbb{R}$ satisfy

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q \Delta t\right\}^{\frac{1}{q}}, \tag{2.4}$$

where $p > 1$, $q = p/(p-1)$, and $a, b \in \mathbb{T}$. Minkowski's inequality [6, Theorem 6.16] asserts that three rd-continuous functions $f, g, h : \mathbb{T} \to \mathbb{R}$ satisfy

$$\left\{\int_a^b |h(t)||f(t)+g(t)|^p \Delta t\right\}^{\frac{1}{p}} \leqslant \left\{\int_a^b |h(t)||f(t)|^p \Delta t\right\}^{\frac{1}{p}} + \left\{\int_a^b |h(t)||g(t)|^p \Delta t\right\}^{\frac{1}{p}}, \tag{2.5}$$

where $p > 1$ and $a, b \in \mathbb{T}$.

Inequalities (2.6) and (2.7) below are simple consequences of the chain rule (2.2), but for convenience of further reference, we now state these four important inequalities (which are "substitutes" for the power rule from differential calculus) in the following lemma, supplemented by two new inequalities which are, however, merely simple consequences of the product rule (2.1).

LEMMA 2.2. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is differentiable and positive. Let $\gamma \in \mathbb{R}$. If f^{Δ} is either always positive or always negative, then*

$$\gamma f^{\Delta} \left(f^{\gamma-1}\right)^{\sigma} \leqslant (f^{\gamma})^{\Delta} \leqslant \gamma f^{\Delta} f^{\gamma-1} \quad \textit{if} \quad 0 \leqslant \gamma \leqslant 1 \tag{2.6}$$

and

$$\gamma f^{\Delta} f^{\gamma-1} \leqslant (f^{\gamma})^{\Delta} \leqslant \gamma f^{\Delta} \left(f^{\gamma-1}\right)^{\sigma} \quad \textit{if} \quad \gamma \geqslant 1. \tag{2.7}$$

If f^Δ *is always positive, then*

$$(f^\gamma)^\Delta \leqslant f^\Delta \left(f^{\gamma-1}\right)^\sigma \quad \text{if} \quad 0 \leqslant \gamma \leqslant 1 \tag{2.8}$$

and

$$(f^\gamma)^\Delta \geqslant f^\Delta \left(f^{\gamma-1}\right)^\sigma \quad \text{if} \quad \gamma \geqslant 1. \tag{2.9}$$

Proof. Inequalities (2.6) and (2.7) follow directly from (2.2). Next, if f is increasing and if $0 \leqslant \gamma \leqslant 1$, then $f^{\gamma-1}$ is decreasing and thus $\left(f^{\gamma-1}\right)^\Delta < 0$ so that

$$(f^\gamma)^\Delta = \left(f f^{\gamma-1}\right)^\Delta \overset{(2.1)}{=} f^\Delta \left(f^{\gamma-1}\right)^\sigma + f \left(f^{\gamma-1}\right)^\Delta.$$

This shows (2.8), and (2.9) follows similarly. □

3. Dynamic inequalities of Copson type

In this section, we prove two new dynamic inequalities of Copson type (see [9, 10] for the original Copson inequalities). These will be used in the proofs of our main results in the next section. Throughout, we are using the following assumptions:

$$\begin{cases} \sup \mathbb{T} = \infty, \quad t_0 \in \mathbb{T}, \\ a : \mathbb{T} \to (0,\infty) \quad \text{is rd-continuous}, \\ A(t) := \int_{t_0}^t a(s)\Delta s, \quad t \in \mathbb{T}. \end{cases} \tag{3.1}$$

THEOREM 3.1. *Assume* (3.1). *Suppose* $\varphi : \mathbb{T} \to \mathbb{R}$ *is such that*

$$\Phi(t) := \int_t^\infty \frac{a(s)}{A(\sigma(s))} \varphi(s)\Delta s, \quad t \in \mathbb{T}$$

is well defined. Let $k \geqslant 1$. *Then*

$$\int_{t_0}^\infty a(t)\Phi^k(t)\Delta t \leqslant k^k \int_{t_0}^\infty a(t)\varphi^k(t)\Delta t. \tag{3.2}$$

Proof. We use integration by parts (2.3), the left part of the inequality (2.7) with $f = \Phi$ and $\gamma = k$, and Hölder's inequality (2.4) with $p = k$ and $q = k/(k-1)$ (unless $k = 1$ in which case (2.4) is not needed) to obtain

$$\begin{aligned} \int_{t_0}^\infty a(t)\Phi^k(t)\Delta t &= \int_{t_0}^\infty A^\Delta(t)\Phi^k(t)\Delta t \\ &\overset{(2.3)}{=} A(t)\Phi^k(t)\Big|_{t_0}^\infty - \int_{t_0}^\infty A(\sigma(t))\left(\Phi^k\right)^\Delta(t)\Delta t \\ &= -\int_{t_0}^\infty A(\sigma(t))\left(\Phi^k\right)^\Delta(t)\Delta t \\ &\overset{(2.7)}{\leqslant} -\int_{t_0}^\infty A(\sigma(t))k\Phi^\Delta(t)\Phi^{k-1}(t)\Delta t \end{aligned}$$

$$
\begin{aligned}
&= k\int_{t_0}^{\infty} a(t)\varphi(t)\Phi^{k-1}(t)\Delta t \\
&= k\int_{t_0}^{\infty} \left[a^{\frac{1}{k}}(t)\varphi(t)\right]\left[a^{\frac{k-1}{k}}(t)\Phi^{k-1}(t)\right]\Delta t \\
&\overset{(2.4)}{\leqslant} k\left\{\int_{t_0}^{\infty}\left[a^{\frac{1}{k}}(t)\varphi(t)\right]^{k}\Delta t\right\}^{\frac{1}{k}}\left\{\int_{t_0}^{\infty}\left[a^{\frac{k-1}{k}}(t)\Phi^{k-1}(t)\right]^{\frac{k}{k-1}}\Delta t\right\}^{\frac{k-1}{k}} \\
&= k\left\{\int_{t_0}^{\infty} a(t)\varphi^{k}(t)\Delta t\right\}^{\frac{1}{k}}\left\{\int_{t_0}^{\infty} a(t)\Phi^{k}(t)\Delta t\right\}^{1-\frac{1}{k}}.
\end{aligned}
$$

Dividing the entire inequality by the right-hand factor of the last expression and then raising the resulting inequality to the k-th power confirms the validity of (3.2). □

REMARK 3.2. It is worth to mention here that our technique of the proof of Theorem 3.1 is different from the technique due to Copson to prove the discrete form of (3.2). In particular, he followed the technique due to Elliott [11] that has been used to prove the Hardy inequality.

THEOREM 3.3. *Assume* (3.1). *Suppose* $\varphi : \mathbb{T} \to \mathbb{R}$ *is such that*

$$\Phi(t) := \int_{t}^{\infty} a(s)\varphi(s)\Delta s, \quad t \in \mathbb{T}$$

is well defined. Let $k \geqslant 1$ *and* $0 \leqslant c < 1$. *Then*

$$\int_{t_0}^{\infty} \frac{a(t)}{A^{c}(\sigma(t))}\Phi^{k}(t)\Delta t \leqslant \left(\frac{k}{1-c}\right)^{k}\int_{t_0}^{\infty} a(t)A^{k-c}(\sigma(t))\varphi^{k}(t)\Delta t. \tag{3.3}$$

Proof. First we define an auxiliary function $\tilde{A} : \mathbb{T} \to \mathbb{R}$ by

$$\tilde{A}(t) := \int_{t_0}^{t} \frac{a(s)}{A^{c}(\sigma(s))}\Delta s, \quad t \in \mathbb{T}.$$

Note that the left part of the inequality (2.6) with $f = A$ and $\gamma = 1-c$ implies

$$\tilde{A}(\sigma(t)) = \int_{t_0}^{\sigma(t)} \frac{a(s)}{A^{c}(\sigma(s))}\Delta s \leqslant \int_{t_0}^{\sigma(t)} \frac{\left(A^{1-c}\right)^{\Delta}(s)}{1-c}\Delta s = \frac{A^{1-c}(\sigma(t))}{1-c}. \tag{3.4}$$

Now we use integration by parts (2.3), the left part of the inequality (2.7) with $f = \Phi$ and $\gamma = k$, (3.4), and Hölder's inequality (2.4) with $p = k$ and $q = k/(k-1)$ (unless

$k=1$ in which case (2.4) is not needed) to obtain

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{a(t)}{A^c(\sigma(t))}\Phi^k(t)\Delta t &= \int_{t_0}^{\infty} \tilde{A}^{\Delta}(t)\Phi^k(t)\Delta t \\
&\overset{(2.3)}{=} \tilde{A}(t)\Phi^k(t)\Big|_{t_0}^{\infty} - \int_{t_0}^{\infty} \tilde{A}(\sigma(t))\left(\Phi^k\right)^{\Delta}(t)\Delta t \\
&= -\int_{t_0}^{\infty} \tilde{A}(\sigma(t))\left(\Phi^k\right)^{\Delta}(t)\Delta t \\
&\overset{(3.4)}{\underset{(2.7)}{\leqslant}} -\int_{t_0}^{\infty} \frac{A^{1-c}(\sigma(t))}{1-c}k\Phi^{\Delta}(t)\Phi^{k-1}(t)\Delta t \\
&= \frac{k}{1-c}\int_{t_0}^{\infty} a(t)A^{1-c}(\sigma(t))\varphi(t)\Phi^{k-1}(t)\Delta t \\
&= \frac{k}{1-c}\int_{t_0}^{\infty} \left[a^{\frac{1}{k}}(t)A^{1-\frac{c}{k}}(\sigma(t))\varphi(t)\right]\left[\frac{a^{\frac{k-1}{k}}(t)}{A^{\frac{c(k-1)}{k}}(\sigma(t))}\Phi^{k-1}(t)\right]\Delta t \\
&\overset{(2.4)}{\leqslant} \frac{k}{1-c}\left\{\int_{t_0}^{\infty}\left[a^{\frac{1}{k}}(t)A^{1-\frac{c}{k}}(\sigma(t))\varphi(t)\right]^k\Delta t\right\}^{\frac{1}{k}} \\
&\qquad \times\left\{\int_{t_0}^{\infty}\left[\frac{a^{\frac{k-1}{k}}(t)}{A^{\frac{c(k-1)}{k}}(\sigma(t))}\Phi^{k-1}(t)\right]^{\frac{k}{k-1}}\Delta t\right\}^{\frac{k-1}{k}} \\
&= \frac{k}{1-c}\left\{\int_{t_0}^{\infty} a(t)A^{k-c}(\sigma(t))\varphi^k(t)\Delta t\right\}^{\frac{1}{k}}\left\{\int_{t_0}^{\infty}\frac{a(t)}{A^c(\sigma(t))}\Phi^k(t)\Delta t\right\}^{1-\frac{1}{k}}.
\end{aligned}$$

Dividing the entire inequality by the right-hand factor of the last expression and then raising the resulting inequality to the k-th power confirms the validity of (3.3). □

4. Dynamic sneak-out inequalities

In this section, we prove the main results of this paper. We also apply these results to the special time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. The results for $\mathbb{T}=\mathbb{Z}$ are known (see [3]) while the results for $\mathbb{T}=\mathbb{R}$ are new. Throughout this section, we let $p \geqslant 1$. For the cases $\alpha \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, and $-1/p < \alpha \leqslant 0$, we present three distinct inequalities. In addition to (3.1), for a given value of α, we require the assumptions

$$\begin{cases} x:\mathbb{T}\to(0,\infty) \quad \text{is rd-continuous,} \\ y(t):=\int_t^{\infty} x(s)\Delta s \quad \text{and} \quad \Psi(t):=\int_t^{\infty} A^{\alpha}(\sigma(s))x(s)\Delta s, \quad t\in\mathbb{T} \\ \text{are well defined.}\end{cases} \tag{4.1}$$

THEOREM 4.1. *Let $p \geqslant 1$ and $\alpha \geqslant 1$. Assume (3.1) and (4.1). Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leqslant (1+\alpha p)^p \int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t. \tag{4.2}$$

Proof. We use integration by parts (2.3) and the right part of the inequality (2.7) with $f = A$ and $\gamma = \alpha$ to obtain

$$\begin{aligned}
\Psi(t) &= -\int_t^\infty A^\alpha(\sigma(s))y^\Delta(s)\Delta s \\
&\overset{(2.3)}{=} -\left\{ A^\alpha(s)y(s)\Big|_t^\infty - \int_t^\infty (A^\alpha)^\Delta(s)y(s)\Delta s \right\} \\
&= A^\alpha(t)y(t) + \int_t^\infty (A^\alpha)^\Delta(s)y(s)\Delta s \\
&\overset{(2.7)}{\leqslant} A^\alpha(t)y(t) + \int_t^\infty \alpha A^\Delta(s)A^{\alpha-1}(\sigma(s))y(s)\Delta s \\
&\leqslant A^\alpha(\sigma(t))y(t) + \alpha \int_t^\infty a(s)A^{\alpha-1}(\sigma(s))y(s)\Delta s,
\end{aligned}$$

where we have utilized $A \leqslant A^\sigma$ in the last inequality. Now we use Minkowski's inequality (2.5) and Theorem 3.1 with $k = p$ and $\varphi = (A^\alpha)^\sigma y$ to find the estimate

$$\begin{aligned}
&\left\{ \int_{t_0}^\infty a(t)\Psi^p(t)\Delta t \right\}^{\frac{1}{p}} \\
&\leqslant \left\{ \int_{t_0}^\infty a(t) \left[A^\alpha(\sigma(t))y(t) + \alpha \int_t^\infty a(s)A^{\alpha-1}(\sigma(s))y(s)\Delta s \right]^p \Delta t \right\}^{\frac{1}{p}} \\
&\overset{(2.5)}{\leqslant} \left\{ \int_{t_0}^\infty a(t)\left[A^\alpha(\sigma(t))y(t)\right]^p \Delta t \right\}^{\frac{1}{p}} + \left\{ \int_{t_0}^\infty a(t) \left[\alpha \int_t^\infty a(s)A^{\alpha-1}(\sigma(s))y(s)\Delta s \right]^p \Delta t \right\}^{\frac{1}{p}} \\
&= \left\{ \int_{t_0}^\infty a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t \right\}^{\frac{1}{p}} + \alpha \left\{ \int_{t_0}^\infty a(t) \left[\int_t^\infty \frac{a(s)}{A(\sigma(s))} A^\alpha(\sigma(s))y(s)\Delta s \right]^p \Delta t \right\}^{\frac{1}{p}} \\
&\overset{(3.2)}{\leqslant} \left\{ \int_{t_0}^\infty a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t \right\}^{\frac{1}{p}} + \alpha \left\{ p^p \int_{t_0}^\infty a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t \right\}^{\frac{1}{p}} \\
&= (1+\alpha p) \left\{ \int_{t_0}^\infty a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t \right\}^{\frac{1}{p}}.
\end{aligned}$$

Raising this inequality to the p-th power confirms the validity of (4.2). □

EXAMPLE 4.2. Let $\mathbb{T} = \mathbb{Z}$ and $t_0 = 1$. As a special case of Theorem 4.1, we have from (4.2) the inequality

$$\sum_{n=1}^\infty a(n) \left(\sum_{k=n}^\infty A^\alpha(k+1)x(k) \right)^p \leqslant (1+\alpha p)^p \sum_{n=1}^\infty a(n)A^{\alpha p}(n+1) \left(\sum_{k=n}^\infty x(n) \right)^p, \quad (4.3)$$

where

$$A(n) = \sum_{k=1}^{n-1} a(k), \quad n \in \mathbb{N}.$$

Note that (4.3) is the discrete inequality [3, Theorem 8] due to Bennett and Grosse-Erdmann.

EXAMPLE 4.3. Let $\mathbb{T}=\mathbb{R}$. As a special case of Theorem 4.1, we have from (4.2) the inequality

$$\int_{t_0}^{\infty} a(t)\left(\int_t^{\infty} A^{\alpha}(s)x(s)\mathrm{d}s\right)^p \mathrm{d}t \leqslant (1+\alpha p)^p \int_{t_0}^{\infty} a(t)A^{\alpha p}(t)\left(\int_t^{\infty} x(s)\mathrm{d}s\right)^p \mathrm{d}t, \quad (4.4)$$

where

$$A(t)=\int_{t_0}^{t} a(s)\mathrm{d}s, \quad t\in\mathbb{R}.$$

Note that (4.4) is a new inequality.

THEOREM 4.4. *Let* $p\geqslant 1$ *and* $0\leqslant\alpha\leqslant 1$. *Assume* (3.1) *and* (4.1). *Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leqslant (1+p)^p \int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t. \quad (4.5)$$

Proof. In this case, we cannot use the right part of the inequality (2.7) with $f=A$ and $\gamma=\alpha$ as we did in the proof of Theorem 4.1. However, we can use inequality (2.8) with $f=A$ and $\gamma=\alpha$ and follow the exact same steps as in the proof of Theorem 4.1 to obtain (4.5). □

REMARK 4.5. For $\mathbb{T}=\mathbb{Z}$, Theorem 4.4 reduces to the corresponding discrete inequality by Bennett and Grosse-Erdmann [3, Theorem 9]. For $\mathbb{T}=\mathbb{R}$, (4.5) is new.

THEOREM 4.6. *Let* $p\geqslant 1$ *and* $-1/p<\alpha\leqslant 0$. *Assume* (3.1) *and* (4.1). *Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \geqslant \left(\frac{1+\alpha p}{1+\alpha p+p}\right)^p \int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t. \quad (4.6)$$

Proof. We use integration by parts (2.3) and the inequality (2.8) with $f=A$ and $\gamma=-\alpha$ to obtain

$$\begin{aligned}
y(t) &= -\int_t^{\infty} A^{-\alpha}(\sigma(s))\Psi^{\Delta}(s)\Delta s \\
&\overset{(2.3)}{=} -\left\{A^{-\alpha}(s)\Psi(s)\Big|_t^{\infty} - \int_t^{\infty}\left(A^{-\alpha}\right)^{\Delta}(s)\Psi(s)\Delta s\right\} \\
&= A^{-\alpha}(t)\Psi(t) + \int_t^{\infty}\left(A^{-\alpha}\right)^{\Delta}(s)\Psi(s)\Delta s \\
&\overset{(2.8)}{\leqslant} A^{-\alpha}(t)\Psi(t) + \int_t^{\infty} A^{\Delta}(s)A^{-\alpha-1}(\sigma(s))\Psi(s)\Delta s \\
&\leqslant A^{-\alpha}(\sigma(t))\Psi(t) + \int_t^{\infty} a(s)A^{-\alpha-1}(\sigma(s))\Psi(s)\Delta s,
\end{aligned}$$

where we have utilized $A\leqslant A^{\sigma}$ in the last inequality. Now we use Minkowski's inequality (2.5) and Theorem 3.3 with $k=p$, $c=-\alpha p$, and $\varphi=\left(A^{-\alpha-1}\right)^{\sigma}\Psi$ to find the estimate

$$\begin{aligned}
&\left\{\int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))y^p(t)\Delta t\right\}^{\frac{1}{p}} \\
&\leqslant \left\{\int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))\left[A^{-\alpha}(\sigma(t))\Psi(t)+\int_t^{\infty} a(s)A^{-\alpha-1}(\sigma(s))\Psi(s)\Delta s\right]^p \Delta t\right\}^{\frac{1}{p}} \\
&\overset{(2.5)}{\leqslant} \left\{\int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))\left[A^{-\alpha}(\sigma(t))\Psi(t)\right]^p \Delta t\right\}^{\frac{1}{p}} \\
&\quad + \left\{\int_{t_0}^{\infty} a(t)A^{\alpha p}(\sigma(t))\left[\int_t^{\infty} a(s)A^{-\alpha-1}(\sigma(s))\Psi(s)\Delta s\right]^p \Delta t\right\}^{\frac{1}{p}} \\
&= \left\{\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t\right\}^{\frac{1}{p}} + \left\{\int_{t_0}^{\infty} \frac{a(t)}{A^{-\alpha p}(\sigma(t))}\left[\int_t^{\infty} a(s)A^{-\alpha-1}(\sigma(s))\Psi(s)\Delta s\right]^p \Delta t\right\}^{\frac{1}{p}} \\
&\overset{(3.3)}{\leqslant} \left\{\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t\right\}^{\frac{1}{p}} + \left\{\left(\frac{p}{1+\alpha p}\right)^p \int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t\right\}^{\frac{1}{p}} \\
&= \left(\frac{1+\alpha p+p}{1+\alpha p}\right)\left\{\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t\right\}^{\frac{1}{p}}.
\end{aligned}$$

Raising this inequality to the p-th power confirms the validity of (4.6). □

EXAMPLE 4.7. Let $\mathbb{T}=\mathbb{Z}$ and $t_0=1$. As a special case of Theorem 4.6, we have from (4.6) the inequality

$$\sum_{n=1}^{\infty} a(n)\left(\sum_{k=n}^{\infty} A^{\alpha}(k+1)x(k)\right)^p \geqslant \left(\frac{1+\alpha p}{1+\alpha p+p}\right)^p \sum_{n=1}^{\infty} a(n)A^{\alpha p}(n+1)\left(\sum_{k=n}^{\infty} x(n)\right)^p, \tag{4.7}$$

where

$$A(n)=\sum_{k=1}^{n-1} a(k), \quad n\in\mathbb{N}.$$

Note that (4.7) is the discrete inequality [3, Theorem 10] due to Bennett and Grosse-Erdmann.

EXAMPLE 4.8. Let $\mathbb{T}=\mathbb{R}$. As a special case of Theorem 4.6, we have from (4.6) the inequality

$$\int_{t_0}^{\infty} a(t)\left(\int_t^{\infty} A^{\alpha}(s)x(s)ds\right)^p \mathrm{d}t \geqslant \left(\frac{1+\alpha p}{1+\alpha p+p}\right)^p \int_{t_0}^{\infty} a(t)A^{\alpha p}(t)\left(\int_t^{\infty} x(s)\mathrm{d}s\right)^p \mathrm{d}t, \tag{4.8}$$

where

$$A(t)=\int_{t_0}^{t} a(s)\mathrm{d}s, \quad t\in\mathbb{R}.$$

Note that (4.8) is a new inequality.

REFERENCES

[1] R. P. AGARWAL, M. BOHNER, AND S. H. SAKER, *Dynamic Littlewood-type inequalities*, Proc. Amer. Math. Soc., **143** (2): 667–677, 2015.
[2] M. R. S. AMMI AND D. F. M. TORRES, *Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales*, An. Univ. Craiova Ser. Mat. Inform., **37** (1): 1–11, 2010.
[3] G. BENNETT AND K.-G. GROSSE-ERDMANN, *On series of positive terms*, Houston J. Math., **31** (2): 541–586, 2005.
[4] G. BENNETT AND K.-G. GROSSE-ERDMANN, *Weighted Hardy inequalities for decreasing sequences and functions*, Math. Ann., **334** (3): 489–531, 2006.
[5] M. BOHNER, R. A. FERREIRA, AND D. F. M. TORRES, *Integral inequalities and their applications to the calculus of variations on time scales*, Math. Inequal. Appl., **13** (3): 511–522, 2010.
[6] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[7] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[8] M. BOHNER AND A. ZAFER, *Lyapunov type inequalities for planar linear Hamiltonian systems on time scales*, Appl. Anal. Discrete Math., **7** (1): 129–142, 2013.
[9] E. T. COPSON, *Note on series of positive terms*, J. London Math. Soc., S1–2 (1): 9–12, 1927.
[10] E. T. COPSON, *Note on series of positive terms*, J. London Math. Soc., S1–3 (1): 49–51, 1928.
[11] E. B. ELLIOTT, *A simple expansion of some recently proved facts as to convergency*, J. London Math. Soc., S1–1 (1): 93–96, 1926.
[12] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1952.
[13] U. M. ÖZKAN AND H. YILDIRIM, *Hardy-Knopp-type inequalities on time scales*, Dynam. Systems Appl., **17** (3–4): 477–486, 2008.
[14] S. H. SAKER, *Hardy–Leindler type inequalities on time scales*, Appl. Math. Inf. Sci., **8** (6): 2975–2981, 2014.
[15] S. H. SAKER AND J. GRAEF, *A new class of dynamic inequalities of Hardy's type on time scales*, Dynam. Systems Appl., **23** (1): 83–99, 2014.
[16] S. H. SAKER, D. O'REGAN, AND R. AGARWAL, *Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales*, Math. Nachr., **287** (5–6): 686–698, 2014.
[17] A. TUNA AND S. KÜTÜKÇÜ, *Some integral inequalities on time scales*, Appl. Math. Mech. (English Ed.), **29** (1): 23–29, 2008.

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