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## A further extension of the extended Riemann–Liouville fractional derivative operator

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**Abstract:** The main objective of this paper is to establish the extension of an extended fractional derivative operator by using an extended beta function recently defined by Parmar et al. by considering the Bessel functions in its kernel. We also give some results related to the newly defined fractional operator, such as Mellin transform and relations to extended hypergeometric and Appell's function via generating functions.

**Key words:** Hypergeometric function, extended hypergeometric function, Mellin transform, fractional derivative, Appell's function

### 1. Introduction

Recently, the applications and importance of fractional calculus have received more attention. In the field of mathematical analysis, fractional calculus is a powerful tool. Various extensions and generalizations of fractional derivative operators were recently investigated in [7, 8, 10, 16].

Euler's beta function is defined by

$$B(\sigma_1, \sigma_2) = \int_0^1 t^{\sigma_1-1} (1-t)^{\sigma_2-1} dt, \quad \operatorname{Re}(\sigma_1) > 0, \quad \operatorname{Re}(\sigma_2) > 0, \quad (1.1)$$

and its relation with the gamma function is given by

$$B(\sigma_1, \sigma_2) = \frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1 + \sigma_2)}.$$

The Gauss hypergeometric and the confluent hypergeometric functions are defined (see [15]) by

$${}_2F_1(\sigma_1, \sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{(\sigma_1)_n (\sigma_2)_n}{(\sigma_3)_n} \frac{z^n}{n!}, \quad |z| < 1, \quad \sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}, \quad -\sigma_3 \notin \mathbb{N}_0 \quad (1.2)$$

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and

$${}_1\Phi_1(\sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{(\sigma_2)_n z^n}{(\sigma_3)_n n!}, \quad |z| < 1, \quad \sigma_2, \sigma_3 \in \mathbb{C}, -\sigma_3 \notin \mathbb{N}_0. \quad (1.3)$$

The Appell series or bivariate hypergeometric series is defined by

$$F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y) = \sum_{m,n=0}^{\infty} \frac{(\sigma_1)_{m+n} (\sigma_2)_m (\sigma_3)_n x^m y^n}{(\sigma_4)_{m+n} m! n!}, \quad (1.4)$$

for all  $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{C}$ ,  $-\sigma_4 \notin \mathbb{N}_0$ ,  $|x| < 1$ ,  $|y| < 1$ . The integral representations of (1.2), (1.3), and (1.4) are respectively given by

$${}_2F_1(\sigma_1, \sigma_2; \sigma_3; z) = \frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-1} (1-t)^{\sigma_3-\sigma_2-1} (1-zt)^{-\sigma_1} dt, \quad (1.5)$$

$\operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0$ ,  $|\arg(1-z)| < \pi$ , and

$${}_1\Phi_1(\sigma_2; \sigma_3; z) = \frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-1} (1-t)^{\sigma_3-\sigma_2-1} e^{zt} dt, \quad (1.6)$$

$\operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0$ ,

$$F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y) = \frac{\Gamma(\sigma_4)}{\Gamma(\sigma_1)\Gamma(\sigma_4 - \sigma_1)} \int_0^1 t^{\sigma_1-1} (1-t)^{\sigma_4-\sigma_1-1} (1-xt)^{-\sigma_2} (1-yt)^{-\sigma_3} dt. \quad (1.7)$$

Chaudhry et al. [2] introduced the extended beta function as

$$B(\sigma_1, \sigma_2; p) = B_p(\sigma_1, \sigma_2) = \int_0^1 t^{\sigma_1-1} (1-t)^{\sigma_2-1} e^{-\frac{p}{t(1-t)}} dt, \quad (1.8)$$

where  $\operatorname{Re}(p) > 0$ ,  $\operatorname{Re}(\sigma_1) > 0$ ,  $\operatorname{Re}(\sigma_2) > 0$ . If  $p = 0$ , then  $B(\sigma_1, \sigma_2; 0) = B(\sigma_1, \sigma_2)$ . The extended hypergeometric and confluent hypergeometric functions are defined in [3] as

$$F_p(\sigma_1, \sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_2 + n, \sigma_3 - \sigma_2)}{B(\sigma_2, \sigma_3 - \sigma_2)} (\sigma_1)_n \frac{z^n}{n!} \quad (1.9)$$

and

$$\Phi_p(\sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_2 + n, \sigma_3 - \sigma_2)}{B(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!}, \quad (1.10)$$

where  $p \geq 0$ . Also, in [3], the authors gave the integral representations of the extended hypergeometric and confluent hypergeometric functions as

$$F_p(\sigma_1, \sigma_2; \sigma_3; z) = \frac{1}{B(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-1} (1-t)^{\sigma_3-\sigma_2-1} (1-zt)^{-\sigma_1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \quad (1.11)$$

$$p \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0, \quad |\arg(1-z)| < \pi$$

and

$$\Phi_p(\sigma_2, \sigma_3; \gamma; z) = \frac{1}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-1} (1-t)^{\sigma_3-\sigma_2-1} \exp\left(zt - \frac{p}{t(1-t)}\right) dt, \quad (1.12)$$

$$p \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0.$$

The extended Appell function is defined (see [11]) by

$$F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y; p) = \sum_{n=0}^{\infty} \frac{B_p(\sigma_1 + m + n, \sigma_4 - \sigma_1)}{B(\sigma_1, \sigma_4 - \sigma_1)} (\sigma_2)_m (\sigma_3)_n \frac{x^m y^n}{m! n!}, \quad (1.13)$$

where  $p \geq 0$ , and its integral representation is given by

$$F_1(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y; p) = \frac{1}{B(\sigma_1, \sigma_4 - \sigma_1)} \int_0^1 t^{\sigma_1-1} (1-t)^{\sigma_4-\sigma_1-1} (1-xt)^{-\sigma_2} (1-yt)^{-\sigma_3} \exp\left(\frac{-p}{t(1-t)}\right) dt, \quad (1.14)$$

$$p \geq 0, \quad \operatorname{Re}(\sigma_4) > \operatorname{Re}(\sigma_1) > 0, \quad |\arg(1-x)| < \pi, \quad |\arg(1-y)| < \pi.$$

It is clear that if  $p = 0$ , then (1.9)–(1.14) reduce to the well-known hypergeometric, confluent hypergeometric, and Appell series and their integral representations, respectively.

For various extensions and generalizations, the readers may follow the recent work [1, 4, 9]. Parmar et al. [13] introduced the extended beta function as

$$\beta_v(\sigma_1, \sigma_2; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{\sigma_1-\frac{3}{2}} (1-t)^{\sigma_2-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \quad (1.15)$$

where  $K_{v+\frac{1}{2}}(\cdot)$  is the modified Bessel function of order  $v + \frac{1}{2}$ . Clearly, when  $v = 0$ , (1.15) reduces to (1.8) by using the fact that  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ . Also, the extended hypergeometric and confluent hypergeometric functions, and their integral representations, are as given in [13]:

$$F_{p,v}(\sigma_1, \sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} (\sigma_1)_n \frac{\beta_v(\sigma_2 + n, \sigma_3 - \sigma_2; p)}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!}, \quad (1.16)$$

$$p, v \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0, \quad |z| < 1,$$

$$\Phi_{p,v}(\sigma_2; \sigma_3; z) = \sum_{n=0}^{\infty} \frac{\beta_v(\sigma_2 + n, \sigma_3 - \sigma_2; p)}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!}, \quad (1.17)$$

$$p, v \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0,$$

$$F_{p,v}(\sigma_1, \sigma_2; \sigma_3; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2-\frac{3}{2}} (1-t)^{\sigma_3-\sigma_2-\frac{3}{2}} (1-zt)^{-\sigma_1} K_{v+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \quad (1.18)$$

$$p, v \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0, \quad |\arg(1-z)| < \pi,$$

and

$$\Phi_{p,v}(\sigma_2; \sigma_3; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{\beta(\sigma_2, \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2 - \frac{3}{2}} (1-t)^{\sigma_3 - \sigma_2 - \frac{3}{2}} \exp(zt) K_{v+\frac{1}{2}} \left( \frac{p}{t(1-t)} \right) dt, \quad (1.19)$$

$$p, v \geq 0, \quad \operatorname{Re}(\sigma_3) > \operatorname{Re}(\sigma_2) > 0.$$

They also obtained the transformation formula for extended confluent hypergeometric functions as

$$\Phi_{p,v}(\sigma_2, \sigma_3; \gamma; z) = e^z \Phi_{p,v}(\sigma_3 - \sigma_2; \sigma_3; -z). \quad (1.20)$$

It is clear that, when  $v = 0$ , (1.16)–(1.19) reduce to the extended hypergeometric and confluent hypergeometric functions, and their integral representations are given in (1.9)–(1.12) by using the fact that  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ .

Recently, Dar and Paris [5] introduced Appell's hypergeometric function by

$$\begin{aligned} F_{1,p,v}(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y) &= F_{1,v}(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y; p) \\ &= \sum_{m,n=0}^{\infty} (\sigma_2)_m (\sigma_3)_n \frac{\beta(\sigma_1 + m + n, \sigma_4 - \sigma_1)}{\beta(\sigma_1, \sigma_4 - \sigma_1)} \frac{x^m y^n}{n! m!}, \end{aligned} \quad (1.21)$$

where  $|x| < 1$ ,  $|y| < 1$ ,  $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{C}$ ,  $-\sigma_4 \notin \mathbb{N}_0$ .

In the same paper [5], they gave its integral representation as

$$\begin{aligned} &F_{1,p,v}(\sigma_1, \sigma_2, \sigma_3; \sigma_4; x, y) \\ &= \sqrt{\frac{2p}{\pi}} \frac{1}{\beta(\sigma_1, \sigma_4 - \sigma_1)} \int_0^1 t^{\sigma_1 - \frac{3}{2}} (1-t)^{\sigma_4 - \sigma_1 - \frac{3}{2}} (1-tx)^{-\sigma_2} (1-ty)^{-\sigma_3} K_{v+\frac{1}{2}} \left( \frac{p}{t(1-t)} \right) dt, \end{aligned} \quad (1.22)$$

where  $\operatorname{Re}(p) \geq 0$ ,  $v \geq 0$ ,  $\operatorname{Re}(\sigma_4) > \operatorname{Re}(\sigma_1) > 0$ ,  $|\arg(1-x)| < \pi$ , and  $|\arg(1-y)| < \pi$ . Obviously, when  $v = 0$  in (1.21) and (1.22), we get the extended Appell function and its integral representation (see (1.13) and (1.14)) by using the fact that  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ . Similarly, when  $v = 0$  and  $p = 0$ , (1.21) and (1.22) reduce to the well-known classical Appell function and its integral representation.

## 2. Extension of the fractional derivative operator

In this section, we define a further extension of the extended Riemann–Liouville fractional derivative.

**Definition 2.1** *The Riemann–Liouville fractional derivative of order  $\mu$  is defined by*

$$\mathfrak{D}_x^\mu \{f(x)\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t) (x-t)^{-\mu-1} dt, \quad \operatorname{Re}(\mu) > 0. \quad (2.1)$$

For the case  $m-1 < \operatorname{Re}(\mu) < m$ , where  $m \in \mathbb{N}$ , we have

$$\mathfrak{D}_x^\mu \{f(z)\} = \frac{d^m}{dx^m} \mathfrak{D}_x^{\mu-m} \{f(x)\} = \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^x f(t) (x-t)^{-\mu+m-1} dt \right\}, \quad \operatorname{Re}(\mu) > 0. \quad (2.2)$$

**Definition 2.2** (See [11]) *The extended Riemann–Liouville fractional derivative of order  $\mu$  is defined by*

$$\mathfrak{D}_x^\mu \{f(x); p\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} \exp\left(-\frac{px^2}{t(x-t)}\right) dt, \quad \operatorname{Re}(\mu) > 0. \quad (2.3)$$

For the case  $m-1 < \operatorname{Re}(\mu) < m$ , where  $m \in \mathbb{N}$ , we have

$$\begin{aligned} \mathfrak{D}_x^\mu \{f(z); p\} &= \frac{d^m}{dx^m} \mathfrak{D}_x^{\mu-m} \{f(x); p\} \\ &= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^x f(t)(x-t)^{-\mu+m-1} \exp\left(-\frac{px^2}{t(x-t)}\right) dt \right\}, \quad \operatorname{Re}(\mu) > 0. \end{aligned} \quad (2.4)$$

**Definition 2.3** (See [1]) *We define*

$$\mathfrak{D}_x^\mu \{f(x); p, q\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} \exp\left(-\frac{px}{t} - \frac{qx}{(x-t)}\right) dt, \quad \operatorname{Re}(\mu) > 0. \quad (2.5)$$

For the case  $m-1 < \operatorname{Re}(\mu) < m$ , where  $m \in \mathbb{N}$ , we have

$$\begin{aligned} \mathfrak{D}_x^\mu \{f(z); p, q\} &= \frac{d^m}{dx^m} \mathfrak{D}_x^{\mu-m} \{f(x); p, q\} \\ &= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^x f(t)(x-t)^{-\mu+m-1} \exp\left(-\frac{px}{t} - \frac{qx}{(x-t)}\right) dt \right\}, \quad \operatorname{Re}(\mu) > 0. \end{aligned} \quad (2.6)$$

A new extension of the Riemann–Liouville fractional derivative of order  $\mu$  as follows.

**Definition 2.4** *The extension of the extended Riemann–Liouville fractional derivative of order  $\mu$  is defined as*

$$\mathfrak{D}_x^\mu \{f(x); p, q, \lambda, \rho\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} {}_1F_1\left[\lambda; \rho; -\frac{px}{t}\right] {}_1F_1\left[\lambda; \rho; -\frac{qx}{(x-t)}\right] dt, \quad \operatorname{Re}(\mu) > 0. \quad (2.7)$$

For the case  $m-1 < \operatorname{Re}(\mu) < m$ , where  $m \in \mathbb{N}$ , we have

$$\begin{aligned} \mathfrak{D}_x^\mu \{f(z); p, q, \lambda, \rho\} &= \frac{d^m}{dx^m} \mathfrak{D}_x^{\mu-m} \{f(x); p, q, \lambda, \rho\} \\ &= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^x f(t)(x-t)^{-\mu+m-1} {}_1F_1\left[\lambda; \rho; -\frac{px}{t}\right] {}_1F_1\left[\lambda; \rho; -\frac{qx}{(x-t)}\right] dt \right\}, \end{aligned} \quad (2.8)$$

where  $\operatorname{Re}(\mu) > 0$ ,  $\operatorname{Re}(p) > 0$ , and  $\operatorname{Re}(q) > 0$ .

Next, we give an extension of the extended Riemann–Liouville fractional derivative operator (2.2) of order  $\mu$  as follows.

**Definition 2.5** *The extension of the extended Riemann–Liouville fractional derivative of order  $\mu$  is defined as*

$$\mathfrak{D}_x^\mu \{f(x); p, v\} = \sqrt{\frac{2px^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^x f(t)t^{-\frac{1}{2}}(x-t)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{px^2}{t(x-t)}\right) dt, \quad \operatorname{Re}(\mu) > 0. \quad (2.9)$$

For the case  $m - 1 < \operatorname{Re}(\mu) < m$ , where  $m \in \mathbb{N}$ , it follows that

$$\begin{aligned} \mathfrak{D}_x^\mu \{f(x); p, v\} &= \frac{d^m}{dx^m} \mathfrak{D}_x^{\mu-m} \{f(x); p, v\} \\ &= \frac{d^m}{dx^m} \left\{ \sqrt{\frac{2px^2}{\pi}} \frac{1}{\Gamma(-\mu+m)} \int_0^x f(t) t^{-\frac{1}{2}} (x-t)^{-\mu+m-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{px^2}{t(x-t)} \right) dt \right\}, \end{aligned} \quad (2.10)$$

where  $\operatorname{Re}(\mu) > 0$ ,  $\operatorname{Re}(p) > 0$  and  $v \geq 0$ .

**Remark 2.1** *The following observations are clear.*

(i) If  $v = 0$ , then Definition 2.5 reduces to the extended fractional derivative defined in Definition 2.2 by using the fact that  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ .

(ii) If  $v = 0$  and  $p = 0$ , then Definition 2.5 reduces to the Riemann–Liouville fractional derivative defined in Definition 2.1.

Now we prove a result concerning the extension of the fractional derivative.

**Theorem 2.1** *We have*

$$\mathfrak{D}_z^\mu \{z^\eta; p, v\} = \frac{\beta_v(\eta+1, -\mu; p)}{\Gamma(-\mu)} z^{\eta-\mu}, \quad \operatorname{Re}(\mu) > 0. \quad (2.11)$$

**Proof** From (2.9), we have

$$\begin{aligned} \mathfrak{D}_z^\mu \{z^\eta; p, v\} &= \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^z t^{\eta-\frac{1}{2}} (z-t)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{t(z-t)} \right) dt \\ &\stackrel{(t=zu)}{=} \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^1 (uz)^{\eta-\frac{1}{2}} (z-uz)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{uz(z-uz)} \right) z du \\ &= \frac{z^{\eta-\mu} \sqrt{\frac{2p}{\pi}}}{\Gamma(-\mu)} \int_0^1 u^{\eta-\frac{1}{2}} (1-u)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{p}{u(1-u)} \right) du, \end{aligned}$$

which shows (2.11). □

**Theorem 2.2** *Let  $\operatorname{Re}(\mu) > 0$  and suppose that  $f$  is analytic at the origin with its Maclaurin expansion given by  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , where  $|z| < \delta$  for some  $\delta > 0$ . Then*

$$\mathfrak{D}_z^\mu \{f(z); p, v\} = \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^\mu \{z^n; p, v\}. \quad (2.12)$$

**Proof** Using the series expansion of  $f$  in (2.7) gives

$$\mathfrak{D}_z^\mu \{f(z); p, v\} = \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^z \sum_{n=0}^{\infty} a_n t^{n-\frac{1}{2}} (z-t)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{t(z-t)} \right) dt.$$

As the series is uniformly convergent on any closed disk centered at the origin with its radius smaller than  $\delta$ , the series is on the line segment from 0 to a fixed  $z$  for  $|z| < \delta$ . Thus, we may integrate term by term to find

$$\begin{aligned}\mathfrak{D}_z^\mu\{f(z); p, v\} &= \sum_{n=0}^{\infty} a_n \left\{ \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^z t^{n-\frac{1}{2}} (z-t)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{t(z-t)} \right) dt \right\} \\ &= \sum_{n=0}^{\infty} a_n \mathfrak{D}_z^\mu\{z^n; p, v\},\end{aligned}$$

which shows (2.12).  $\square$

**Theorem 2.3** *We have*

$$\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-z)^{-\beta}; p, v\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} {}_2F_{1; p, v}^{\lambda, \rho}(\beta, \eta; \mu; z), \quad (2.13)$$

where  $\operatorname{Re}(\mu) > \operatorname{Re}(\eta) > 0$  and  $|z| < 1$ .

**Proof** By direct calculation, we have

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-z)^{-\beta}; p, v\} &= \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(\mu-\eta)} \int_0^z t^{\eta-\frac{3}{2}} (1-t)^{-\beta} (z-t)^{\mu-\eta-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{t(z-t)} \right) dt \\ &= \sqrt{\frac{2pz^2}{\pi}} \frac{z^{\mu-\eta-\frac{3}{2}}}{\Gamma(\mu-\eta)} \int_0^z t^{\eta-\frac{3}{2}} (1-t)^{-\beta} \left(1-\frac{t}{z}\right)^{\mu-\eta-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{pz^2}{t(z-t)} \right) dt. \\ &\stackrel{(t=zu)}{=} \frac{z^{\mu-1}}{\Gamma(\mu-\eta)} \sqrt{\frac{2p}{\pi}} \int_0^1 u^{\eta-\frac{3}{2}} (1-uz)^{-\beta} (1-u)^{\mu-\eta-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{p}{u(1-u)} \right) du.\end{aligned}$$

Using (1.16) and after simplification, we get the required proof.  $\square$

**Theorem 2.4** *We have*

$$\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; p, v\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} F_1(\eta, \alpha, \beta; \mu; az, bz; p, v), \quad (2.14)$$

where  $\operatorname{Re}(\mu) > \operatorname{Re}(\eta) > 0$ ,  $\operatorname{Re}(\alpha) > 0$ ,  $\operatorname{Re}(\beta) > 0$ ,  $|az| < 1$ , and  $|bz| < 1$ .

**Proof** To prove (2.14), we use the power series expansion

$$(1-az)^{-\alpha}(1-bz)^{-\beta} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_m (\beta)_n \frac{(az)^m}{m!} \frac{(bz)^n}{n!}.$$

Now, applying Theorem 2.3, we obtain

$$\begin{aligned}\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; p, v\} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_m (\beta)_n \frac{(a)^m}{m!} \frac{(b)^n}{n!} \mathfrak{D}_z^{\eta-\mu}\{z^{\eta+m+n-1}; p, v\} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_m (\beta)_n \frac{(a)^m}{m!} \frac{(b)^n}{n!} \frac{\beta_{p, v}(\eta+m+n, \mu-\eta)}{\Gamma(\mu-\eta)} z^{\mu+m+n-1}.\end{aligned}$$



Now, applying (1.21), we get

$$\mathfrak{D}_z^{\eta-\mu}\{z^{\eta-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; p, v\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} z^{\mu-1} F_1(\eta, \alpha, \beta; \mu; az, bz; p, v),$$

which shows (2.14).  $\square$

**Theorem 2.5** *We have the Mellin transform formula*

$$M\{\mathfrak{D}_{z;p,v}^\mu(z^\eta); p \rightarrow r\} = \frac{z^{\eta-\mu} 2^{r-1} \Gamma(\frac{r-v}{2}) \Gamma(\frac{r+v+1}{2})}{\sqrt{\pi} \Gamma(-\mu)} \beta(\eta+r+1, r-\mu), \quad (2.15)$$

where  $\operatorname{Re}(\eta) > -1$ ,  $\operatorname{Re}(\mu) > 0$ ,  $\operatorname{Re}(r) > 0$ .

**Proof** Applying the Mellin transform to (2.9), we have

$$\begin{aligned} M\{\mathfrak{D}_{z;p,v}^\mu(z^\eta); p \rightarrow r\} &= \int_0^\infty p^{r-1} \mathfrak{D}_{z;p,v}^\mu(z^\eta) dp \\ &= \sqrt{\frac{2pz^2}{\pi}} \frac{1}{\Gamma(-\mu)} \int_0^\infty p^{r-1} \left\{ \int_0^z t^{-\frac{1}{2}} t^\eta (z-t)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt \right\} dp \\ &= \frac{z^{-\mu-\frac{3}{2}} \sqrt{\frac{2pz^2}{\pi}}}{\Gamma(-\mu)} \int_0^\infty p^{r-1} \left\{ \int_0^z t^{\eta-\frac{1}{2}} \left(1-\frac{t}{z}\right)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt \right\} dp \\ &= \frac{z^{\eta-\mu} \sqrt{\frac{2p}{\pi}}}{\Gamma(-\mu)} \int_0^\infty p^{r-1} \left\{ \int_0^1 u^{\eta-\frac{1}{2}} (1-u)^{-\mu-\frac{3}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right) du \right\} dp \\ &= \frac{z^{\eta-\mu} \sqrt{\frac{2}{\pi}}}{\Gamma(-\mu)} \int_0^1 u^{\eta-\frac{1}{2}} (1-u)^{-\mu-\frac{3}{2}} \left( \int_0^\infty p^{r-\frac{1}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right) dp \right) du. \end{aligned} \quad (2.16)$$

Next,

$$\begin{aligned} \int_0^\infty p^{r-\frac{1}{2}} K_{v+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right) dp &= u^{r+\frac{1}{2}} (1-u)^{r+\frac{1}{2}} \int_0^\infty w^{r-\frac{1}{2}} K_{v+\frac{1}{2}}(w) dw \\ &= u^{r+\frac{1}{2}} (1-u)^{r+\frac{1}{2}} 2^{r-\frac{3}{2}} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right), \end{aligned} \quad (2.17)$$

where  $w = \frac{p}{u(1-u)}$ ,  $\operatorname{Re}(r-v) > 0$ ,  $\operatorname{Re}(r+v) > -1$  (see [2, 13]). Using (2.17) in (2.16), we obtain

$$\begin{aligned} M\{\mathfrak{D}_{z;p,v}^\mu(z^\eta); p \rightarrow r\} &= \frac{z^{\eta-\mu} 2^{r-1} \Gamma(\frac{r-v}{2}) \Gamma(\frac{r+v+1}{2})}{\sqrt{\pi} \Gamma(-\mu)} \int_0^1 u^{\eta+r} (1-u)^{r-\mu-1} du \\ &= \frac{z^{\eta-\mu} 2^{r-1} \Gamma(\frac{r-v}{2}) \Gamma(\frac{r+v+1}{2})}{\sqrt{\pi} \Gamma(-\mu)} \beta(\eta+r+1, r-\mu), \end{aligned}$$

which is (2.15).  $\square$

**Theorem 2.6** We have the Mellin transform formula

$$M \left\{ \mathfrak{D}_{z;p,v}^{\mu}((1-z)^{-\alpha}); p \rightarrow r \right\} = \frac{2^{r-1} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right) z^{-\mu}}{\sqrt{\pi} \Gamma(-\mu)} \beta(r+1, r-\mu) {}_2F_1(\alpha, r+1; r-\mu+1; z), \quad (2.18)$$

where  $\operatorname{Re}(\alpha) > 0$ ,  $\operatorname{Re}(\mu) < 0$ ,  $\operatorname{Re}(r) > 0$ ,  $\operatorname{Re}(r) > 0$ , and  $|z| < 1$ .

**Proof** Using the power series for  $(1-z)^{-\alpha}$  and applying Theorem 2.5 with  $\eta = n$ , we find

$$\begin{aligned} M \left\{ \mathfrak{D}_{z;p,v}^{\mu}((1-z)^{-\alpha}); p \rightarrow r \right\} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} M \left\{ \mathfrak{D}_{z;p,v}^{\mu}(z^n); p \rightarrow r \right\} \\ &= \frac{2^{r-1} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right)}{\sqrt{\pi} \Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \beta(n+r+1, r-\mu) z^{n-\mu} \\ &= \frac{2^{r-1} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right) z^{-\mu}}{\sqrt{\pi} \Gamma(-\mu)} \sum_{n=0}^{\infty} \beta(n+r, r-\mu) \frac{(\alpha)_n z^n}{n!} \\ &= \frac{2^{r-1} \Gamma\left(\frac{r-v}{2}\right) \Gamma\left(\frac{r+v+1}{2}\right) z^{-\mu}}{\sqrt{\pi} \Gamma(-\mu)} \beta(r+1, r-\mu) {}_2F_1(\alpha, r+1; r-\mu+1; z), \end{aligned}$$

which shows (2.18). □

### 3. Generating relations and some further results

In this section, we derive some generating relations of linear and bilinear type for extended  $(p, v)$ -hypergeometric functions.

**Theorem 3.1** We have the generating relation

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_{1;p,v}(\alpha+n, \beta; \gamma; z) t^n = (1-t)^{-\alpha} {}_2F_{1;p,v}\left(\alpha, \beta; \gamma; \frac{z}{1-t}\right), \quad (3.1)$$

where  $|z| < \min(|1, 1-t|)$ ,  $\operatorname{Re}(\alpha) > 0$ ,  $\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0$ .

**Proof** Consider the series identity

$$[(1-z)-t]^{-\alpha} = (1-t)^{-\alpha} \left[1 - \frac{z}{1-t}\right]^{-\alpha}.$$

Thus, the power series expansion yields

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1-z)^{-\alpha} \left(\frac{t}{1-z}\right)^n = (1-t)^{-\alpha} \left[1 - \frac{z}{1-t}\right]^{-\alpha}. \quad (3.2)$$

Multiplying both sides of (3.2) by  $z^{\beta-1}$  and then applying the operator  $\mathfrak{D}_{z;p,v}^{\beta-\gamma}$  on both sides, we get

$$\mathfrak{D}_{z;p,v}^{\beta-\gamma} \left[ \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1-z)^{-\alpha} \left(\frac{t}{1-z}\right)^n z^{\beta-1} \right] = (1-t)^{-\alpha} \mathfrak{D}_{z;p,v}^{\beta-\gamma} \left[ z^{\beta-1} \left(1 - \frac{z}{1-t}\right)^{-\alpha} \right].$$

Interchanging the order of summation and the operator  $\mathfrak{D}_{z;p,v}^{\beta-\gamma}$ , we have

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \mathfrak{D}_{z;p,v}^{\beta-\gamma} [z^{\beta-1}(1-z)^{-\alpha-n}] t^n = (1-t)^{-\alpha} \mathfrak{D}_{z;p,v}^{\beta-\gamma} \left[ z^{\beta-1} \left( 1 - \frac{z}{1-t} \right)^{-\alpha} \right].$$

Thus, by applying Theorem 2.3, we obtain (3.1).  $\square$

**Theorem 3.2** *We have the generating relation*

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} {}_2F_{1;p,v}(\delta-n, \beta; \gamma; z) t^n = (1-t)^{-\beta} F_1 \left( \alpha, \delta, \beta; \gamma; -\frac{zt}{1-t}; p, v \right), \quad (3.3)$$

where  $|t| < \frac{1}{1+|t|}$ ,  $\operatorname{Re}(\delta) > 0$ ,  $\operatorname{Re}(\beta) > 0$ ,  $\operatorname{Re}(\gamma) > \operatorname{Re}(\alpha) > 0$ .

**Proof** Consider the series identity

$$[1 - (1-z)t]^{-\beta} = (1-t)^{-\beta} \left[ 1 + \frac{zt}{1-t} \right]^{-\beta}.$$

Using power series expansion on the left-hand side, we have

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} (1-z)^n t^n = (1-t)^{-\beta} \left[ 1 - \frac{zt}{1-t} \right]^{-\beta}. \quad (3.4)$$

Multiplying both sides of (3.4) by  $z^{\alpha-1}(1-z)^{-\delta}$  and then applying the operator  $\mathfrak{D}_{z;p,v}^{\alpha-\gamma}$  on both sides, we get

$$\mathfrak{D}_{z;p,v}^{\alpha-\gamma} \left[ \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} z^{\alpha-1}(1-z)^{-\delta+n} t^n \right] = (1-t)^{-\beta} \mathfrak{D}_{z;p,v}^{\alpha-\gamma} \left[ z^{\alpha-1}(1-z)^{-\delta} \left( 1 - \frac{zt}{1-t} \right)^{-\beta} \right],$$

where  $\operatorname{Re}(\alpha) > 0$  and  $|zt| < |1-t|$ . Thus, by Theorem 2.2, we obtain

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \mathfrak{D}_{z;p,v}^{\alpha-\gamma} [z^{\alpha-1}(1-z)^{-\delta+n}] t^n = (1-t)^{-\beta} \mathfrak{D}_{z;p,v}^{\alpha-\gamma} \left[ z^{\alpha-1}(1-z)^{-\delta} \left( 1 - \frac{zt}{1-t} \right)^{-\beta} \right].$$

Applying Theorem 2.4 on both sides, we get (3.3).  $\square$

**Theorem 3.3** *We have*

$$\mathfrak{D}_{z;p,v}^{\eta-\mu} \left[ z^{\eta-1} E_{\gamma,\delta}^{\mu}(z) \right] = \frac{z^{\mu-1}}{\Gamma(\mu-\eta)} \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\gamma n + \delta)} \beta_{p,v}(\eta+n, \mu-\eta) \frac{z^n}{n!}, \quad (3.5)$$

where  $\gamma, \delta, \mu \in \mathbb{C}$ ,  $\operatorname{Re}(p) > 0$ ,  $\operatorname{Re}(q) > 0$ ,  $\operatorname{Re}(\mu) > \operatorname{Re}(\eta) > 0$ ,  $\operatorname{Re}(\lambda) > 0$ ,  $\operatorname{Re}(\rho) > 0$ , and  $E_{\gamma,\delta}^{\mu}(z)$  is the Mittag-Leffler function (see [14]) defined as

$$E_{\gamma,\delta}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\gamma n + \delta)} \frac{z^n}{n!}. \quad (3.6)$$

**Proof** Using (3.6) in (3.5), we obtain

$$\mathfrak{D}_{z;p,v}^{\eta-\mu} \left[ z^{\eta-1} E_{\gamma,\delta}^{\mu}(z) \right] = \mathfrak{D}_{z;p,v}^{\eta-\mu} \left[ z^{\eta-1} \left\{ \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\gamma n + \delta)} \frac{z^n}{n!} \right\} \right].$$

By Theorem 2.2, we have

$$\mathfrak{D}_{z;p,v}^{\eta-\mu} \left[ z^{\eta-1} E_{\gamma,\delta}^{\mu}(z) \right] = \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\gamma n + \delta)} \left\{ \mathfrak{D}_{z;p,v}^{\eta-\mu} \left[ z^{\eta+n-1} \right] \right\}.$$

Applying Theorem 2.1, we get (3.5). □

**Theorem 3.4** We have

$$\begin{aligned} & \mathfrak{D}_{z;p,v}^{\eta-\mu} \left\{ z^{\eta-1} {}_m\Psi_n \left[ \begin{matrix} (\alpha_i, A_i)_{1,m}; \\ (\beta_j, B_j)_{1,n}; \end{matrix} \middle| z \right] \right\} \\ &= \frac{z^{\mu-1}}{\Gamma(\mu-\eta)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(\alpha_i + A_i k)}{\prod_{j=1}^n \Gamma(\beta_j + B_j k)} \beta_{p,v}(\eta+k, \mu-\eta) \frac{z^k}{k!}, \end{aligned} \quad (3.7)$$

where  $\operatorname{Re}(p) > 0$ ,  $\operatorname{Re}(q) > 0$ ,  $\operatorname{Re}(\mu) > \operatorname{Re}(\eta) > 0$ ,  $\operatorname{Re}(\lambda) > 0$ ,  $\operatorname{Re}(\rho) > 0$ , and  ${}_m\Psi_n(z)$  denotes the Fox–Wright function (see [6, pp. 56–58] defined by

$${}_m\Psi_n(z) = {}_m\Psi_n \left[ \begin{matrix} (\alpha_i, A_i)_{1,m}; \\ (\beta_j, B_j)_{1,n}; \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(\alpha_i + A_i k)}{\prod_{j=1}^n \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}. \quad (3.8)$$

**Proof** Applying Theorem 2.1 and following the same procedure used in the proof of Theorem 3.3, we obtain (3.7). □

#### 4. Concluding remarks

In this paper, we defined an extension of the extended fractional derivative operator. We conclude that when  $v = 0$  and using the fact that  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ , all results established in this paper reduce to the results related to the extended Riemann–Liouville fractional derivative operator defined in [12]. Also, when  $v = 0$  and  $p = 0$ , we get the results related to the classical Riemann–Liouville fractional derivative operator.

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