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Research Article

Decoupling the Stationary Navier-Stokes-Darcy System with the Beavers-Joseph-Saffman Interface Condition

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This paper proposes a domain decomposition method for the coupled stationary Navier-Stokes and Darcy equations with the Beavers-Joseph-Saffman interface condition in order to improve the efficiency of the finite element method. The physical interface conditions are directly utilized to construct the boundary conditions on the interface and then decouple the Navier-Stokes and Darcy equations. Newton iteration will be used to deal with the nonlinear systems. Numerical results are presented to illustrate the features of the proposed method.

1. Introduction

The Stokes-Darcy model has been extensively studied in the recent years due to its wide range of applications in many natural world problems and industrial settings, such as the subsurface flow in karst aquifers, oil flow in vuggy porous media, industrial filtrations, and the interaction between surface and subsurface flows [1–8]. Since the problem domain naturally consists of two different physical subdomains, several different numerical methods have been developed to decouple the Stokes and Darcy equations [6, 9–26]. For other works on the numerical methods and analysis of the Stokes-Darcy model, we refer the readers to [27–45].

Recently the more physically valid Navier-Stokes-Darcy model has attracted scientists' attention, and several coupled finite element methods have been studied for it [46–51]. On the other hand, the advantages of the domain decomposition methods (DDMs) in parallel computation and natural preconditioning have motivated the development of different DDMs for solving the Stokes-Darcy model [6, 10–18, 21, 22]. In this paper, we will develop a domain decomposition method for the Navier-Stokes-Darcy model based on Robin boundary conditions constructed from the interface conditions. This physics-based DDM is different from the traditional ones in the sense that they focus on decomposing different physical

domains by directly utilizing the given physical interface conditions.

The rest of paper is organized as follows. In Section 2, we introduce the Navier-Stokes-Darcy model with the Beavers-Joseph-Saffman interface condition. In Section 3, we recall the coupled weak formulation and the corresponding coupled finite element method for the Navier-Stokes-Darcy model. In Section 4, a parallel domain decomposition method and its finite element discretization are proposed to decouple the Navier-Stokes-Darcy system by using the Robin-type boundary conditions constructed from the physical interface conditions. Finally, in Section 5, we present a numerical example to illustrate the features of the proposed method.

2. Stationary Navier-Stokes-Darcy Model

In this section we introduce the following coupled Navier-Stokes-Darcy model on a bounded domain $\Omega = \Omega_m \cup \Omega_c \subset \mathbb{R}^d$, ($d = 2, 3$); see Figure 1. In the porous media region Ω_m , the flow is governed by the Darcy system

$$\begin{aligned}\vec{u}_m &= -\mathbb{K}\nabla\phi_m, \\ \nabla \cdot \vec{u}_m &= f_m.\end{aligned}\tag{1}$$

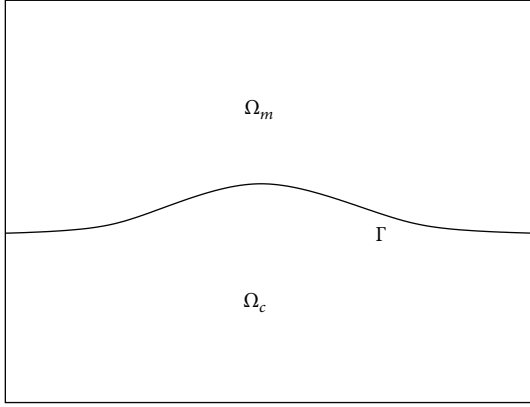


FIGURE 1: A sketch of the porous median domain Ω_m , fluid domain Ω_c , and the interface Γ .

Here, \vec{u}_m is the fluid discharge rate in the porous media, \mathbb{K} is the hydraulic conductivity tensor, f_m is a sink/source term, and ϕ_m is the hydraulic head defined as $z + (p_m/\rho g)$, where p_m denotes the dynamic pressure, z the height, ρ the density, and g the gravitational acceleration. We will consider the following second-order formulation, which eliminates \vec{u}_m in the Darcy system:

$$-\nabla \cdot (\mathbb{K} \nabla \phi_m) = f_m. \quad (2)$$

In the fluid region Ω_c , the fluid flow is assumed to be governed by the Navier-Stokes equations:

$$\vec{u}_c \cdot \nabla \vec{u}_c - \nabla \cdot \mathbb{T}(\vec{u}_c, p_c) = \vec{f}_c, \quad (3)$$

$$\nabla \cdot \vec{u}_c = 0, \quad (4)$$

where \vec{u}_c is the fluid velocity, p_c is the kinematic pressure, \vec{f}_c is the external body force, ν is the kinematic viscosity of the fluid, $\mathbb{T}(\vec{u}_c, p_c) = 2\nu \mathbb{D}(\vec{u}_c) - p_c \mathbb{I}$ is the stress tensor, and $\mathbb{D}(\vec{u}_c) = (\nabla \vec{u}_c + \nabla^T \vec{u}_c)/2$ is the deformation tensor.

Let $\Gamma = \overline{\Omega_m} \cap \overline{\Omega_c}$ denote the interface between the fluid and porous media regions. On the interface Γ , we impose the following three interface conditions:

$$\vec{u}_c \cdot \vec{n}_c = -\vec{u}_m \cdot \vec{n}_m, \quad (5)$$

$$-\vec{u}_c \cdot (\mathbb{T}(\vec{u}_c, p_c) \cdot \vec{n}_c) = g(\phi_m - z), \quad (6)$$

$$-\tau_j \cdot (\mathbb{T}(\vec{u}_c, p_c) \cdot \vec{n}_c) = \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\mathbb{I})}} \tau_j \cdot \vec{u}_c, \quad (7)$$

where \vec{n}_c and \vec{n}_m denote the unit outer normal to the fluid and the porous media regions at the interface Γ , respectively, τ_j ($j = 1, \dots, d-1$) denote mutually orthogonal unit tangential vectors to the interface Γ , and $\mathbb{I} = \mathbb{K}\nu/g$. The third condition (7) is referred to as the Beavers-Joseph-Saffman (BJS) interface condition [52–55].

In this paper, for simplification, we assume that the hydraulic head ϕ_m and the fluid velocity \vec{u}_c satisfy the

homogeneous Dirichlet boundary condition except on Γ , that is, $\phi_m = 0$ on the boundary $\partial\Omega_m/\Gamma$ and $\vec{u}_c = 0$ on the boundary $\partial\Omega_c/\Gamma$.

3. Coupled Weak Formulation and Finite Element Method

In this section we will recall the coupled weak formulation and the corresponding coupled finite element method for the Navier-Stokes-Darcy model with Beavers-Joseph-Saffman condition. Let $(\cdot, \cdot)_D$ denote the L^2 inner product on the domain D ($D = \Omega_c$ or Ω_m) and let $\langle \cdot, \cdot \rangle$ denote the L^2 inner product on the interface Γ or the duality pairing between $(H_0^{1/2}(\Gamma))'$ and $H_0^{1/2}(\Gamma)$ [5]. Define the spaces

$$X_c = \left\{ \vec{v} \in [H^1(\Omega_c)]^d \mid \vec{v} = 0 \text{ on } \frac{\partial\Omega_c}{\Gamma} \right\},$$

$$Q_c = L^2(\Omega_c), \quad (8)$$

$$X_m = \left\{ \psi \in H^1(\Omega_m) \mid \psi = 0 \text{ on } \frac{\partial\Omega_m}{\Gamma} \right\},$$

the bilinear forms

$$a_m(\phi_m, \psi) = (\mathbb{K} \nabla \phi_m, \nabla \psi)_{\Omega_m},$$

$$a_c(\vec{u}_c, \vec{v}) = 2\nu (\mathbb{D}(\vec{u}_c), \mathbb{D}(\vec{v}))_{\Omega_c}, \quad (9)$$

$$b_c(\vec{v}, q) = -(\nabla \cdot \vec{v}, q)_{\Omega_c},$$

and the projection onto the tangent space on Γ :

$$P_\tau \vec{u} = \sum_{j=1}^{d-1} (\vec{u} \cdot \tau_j) \tau_j. \quad (10)$$

With these notations, the weak formulation of the coupled Navier-Stokes-Darcy model with BJS interface condition is given as follows [46–51]: find $(\vec{u}_c, p_c, \phi_m) \in X_c \times Q_c \times X_m$ such that

$$\begin{aligned} & (\vec{u}_c \cdot \nabla \vec{u}_c, \vec{v})_{\Omega_c} + a_c(\vec{u}_c, \vec{v}) + b_c(\vec{v}, p_c) \\ & - b_c(\vec{u}_c, q) + a_m(\phi_m, \psi) \\ & + \langle g\phi_m, \vec{v} \cdot \vec{n}_c \rangle - \langle \vec{u}_c \cdot \vec{n}_c, \psi \rangle \\ & + \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\mathbb{I})}} \langle P_\tau \vec{u}_c, P_\tau \vec{v} \rangle \\ & = (f_m, \psi)_{\Omega_m} + (\vec{f}_c, \vec{v})_{\Omega_c} \\ & + \langle gz, \vec{v} \cdot \vec{n}_c \rangle, \quad \forall (\vec{v}, q, \psi) \in X_c \times Q_c \times X_m. \end{aligned} \quad (11)$$

Assume that we have in hand regular subdivisions of Ω_m and Ω_c into finite elements with mesh size h . Then one can define finite element spaces $X_{mh} \subset X_m$, $X_{ch} \subset X_c$ and

$Q_{ch} \subset Q_c$. We assume that X_{ch} and Q_{ch} satisfy the inf-sup condition [56, 57]

$$\inf_{0 \neq q_h \in Q_{ch}} \sup_{0 \neq \vec{v}_h \in X_{ch}} \frac{b_c(\vec{v}_h, q_h)}{\|\vec{v}_h\|_1 \|q_h\|_0} > \gamma, \quad (12)$$

where $\gamma > 0$ is a constant independent of h . This condition is needed in order to ensure that the spatial discretizations of the Navier-Stokes equations used here are stable. See [56, 57] for more details of finite element spaces X_{mh} , X_{ch} , and Q_{ch} that satisfy (12). One example is the Taylor-Hood element pair that we use in the numerical experiments; for that pair, X_{ch} consists of continuous piecewise quadratic polynomials and Q_{ch} consists of continuous piecewise linear polynomials.

Then a coupled finite element method with Newton iteration for the coupled Navier-Stokes-Darcy model is given as follows [46]: find $(\vec{u}_{c,h}, p_{c,h}, \phi_{m,h}) \in X_{ch} \times Q_{ch} \times X_{mh}$ in the following procedure.

- (1) The initial value $\vec{u}_{c,h}^0$ is chosen.
- (2) For $m = 0, 1, 2, \dots, M$, solve

$$\begin{aligned} & (\vec{u}_{c,h}^{m+1} \cdot \nabla \vec{u}_{c,h}^m, \vec{v}_h)_{\Omega_c} + (\vec{u}_{c,h}^m \cdot \nabla \vec{u}_{c,h}^{m+1}, \vec{v}_h)_{\Omega_c} \\ & + a_c(\vec{u}_{c,h}^{m+1}, \vec{v}_h) + b_c(\vec{v}_h, p_{c,h}^{m+1}) \\ & - b_c(\vec{u}_{c,h}^{m+1}, q_h) + a_m(\phi_{m,h}^{m+1}, \psi_h) \\ & + \langle g\phi_{m,h}^{m+1}, \vec{v}_h \cdot \vec{n}_c \rangle - \langle \vec{u}_{c,h}^{m+1} \cdot \vec{n}_c, \psi_h \rangle \\ & + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\mathbf{\Pi})}} \langle P_\tau \vec{u}_{c,h}^{m+1}, P_\tau \vec{v}_h \rangle \\ & = (\vec{u}_{c,h}^m \cdot \nabla \vec{u}_{c,h}^m, \vec{v}_h)_{\Omega_c} + (f_m, \psi_h)_{\Omega_m} \\ & + (\vec{f}_c, \vec{v}_h)_{\Omega_c} + \langle gz, \vec{v}_h \cdot \vec{n}_c \rangle, \\ & \forall (\vec{v}_h, q_h, \psi_h) \in X_{ch} \times Q_{ch} \times X_{mh}. \end{aligned} \quad (13)$$

- (3) Set $\vec{u}_{c,h} = \vec{u}_{c,h}^{m+1}$, $p_{c,h} = p_{c,h}^{m+1}$, and $\phi_{m,h} = \phi_{m,h}^{m+1}$.

4. Physics-Based Domain Decomposition Method

The coupled finite element method may end up with a huge algebraic system, which combines all parts from the Navier-Stokes equations, Darcy equation, and interface conditions together into one sparse matrix. Hence it is often impractical to directly apply this method to large-scale real world applications. In order to develop a more efficient numerical method in this section, we will directly utilize the three physical interface conditions to construct a physics-based parallel domain decomposition method to decouple the Navier-Stokes and Darcy equations.

Let us first consider the following Robin condition for the Darcy system: for a given constant $\gamma_p > 0$ and a given function η_p defined on Γ ,

$$\gamma_p \mathbb{K} \nabla \hat{\phi}_m \cdot \vec{n}_m + g \hat{\phi}_m = \eta_p, \quad \text{on } \Gamma. \quad (14)$$

Then, the corresponding weak formulation for the Darcy part is given by the following: for $\eta_p \in L^2(\Gamma)$, find $\hat{\phi}_m \in X_m$ such that

$$\begin{aligned} a_m(\hat{\phi}_m, \psi) + \left\langle \frac{g\hat{\phi}_m}{\gamma_p}, \psi \right\rangle \\ = (f_m, \psi)_{\Omega_m} + \left\langle \frac{\eta_p}{\gamma_p}, \psi \right\rangle, \quad \forall \psi \in X_m. \end{aligned} \quad (15)$$

Second, we can propose the following two Robin-type conditions for the Navier-Stokes equations: for a given constant $\gamma_f > 0$ and given functions η_f and $\vec{\eta}_{f\tau}$ defined on Γ ,

$$\begin{aligned} \vec{n}_c \cdot (\mathbb{T}(\vec{u}_c, \hat{p}_c) \cdot \vec{n}_c) + \gamma_f \vec{u}_c \cdot \vec{n}_c = \eta_f, \quad \text{on } \Gamma, \\ -P_\tau(\mathbb{T}(\vec{u}_c, \hat{p}_c) \cdot \vec{n}_c) = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_\tau \vec{u}_c, \quad \text{on } \Gamma. \end{aligned} \quad (16)$$

Then, the corresponding weak formulation for the Navier-Stokes equation is given by the following: for $\eta_f \in L^2(\Gamma)$, find $(\vec{u}_c, \hat{p}_c) \in X_c \times Q_c$ such that

$$\begin{aligned} & (\vec{u}_c \cdot \nabla \vec{u}_c, \vec{v})_{\Omega_c} + a_c(\vec{u}_c, \vec{v}) + b_c(\vec{v}, \hat{p}_c) \\ & - b_c(\vec{u}_c, q) + \gamma_f \langle \vec{u}_c \cdot \vec{n}_c, \vec{v} \cdot \vec{u}_c \rangle \\ & + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\mathbf{\Pi})}} \langle P_\tau \vec{u}_c, P_\tau \vec{v} \rangle \\ & = (\vec{f}_c, \vec{v})_{\Omega_c} + \langle \eta_f, \vec{v} \cdot \vec{n}_c \rangle, \quad \forall (\vec{v}, q) \in X_c \times Q_c. \end{aligned} \quad (17)$$

Our next step is to show that, for appropriate choices of γ_f , γ_p , η_f , and η_p , the solutions of the coupled system (11) are equivalent to the solutions of the decoupled equations (15) and (17), and hence we may solve the latter system instead of the former.

Lemma 1. *Let (ϕ_m, \vec{u}_c, p_c) be the solution of the coupled Navier-Stokes-Darcy system (11) and let $(\hat{\phi}_m, \vec{u}_c, \hat{p}_c)$ be the solution of the decoupled Navier-Stokes and Darcy equations (15) and (17) with Robin boundary conditions at the interface. Then, $(\hat{\phi}_m, \vec{u}_c, \hat{p}_c) = (\phi_m, \vec{u}_c, p_c)$ if and only if γ_f , γ_p , η_f , $\vec{\eta}_{f\tau}$ and η_p satisfy the following compatibility conditions:*

$$\eta_p = \gamma_p \vec{u}_c \cdot \vec{n}_c + g \hat{\phi}_m, \quad (18)$$

$$\eta_f = \gamma_f \vec{u}_c \cdot \vec{n}_c - g \hat{\phi}_m + gz. \quad (19)$$

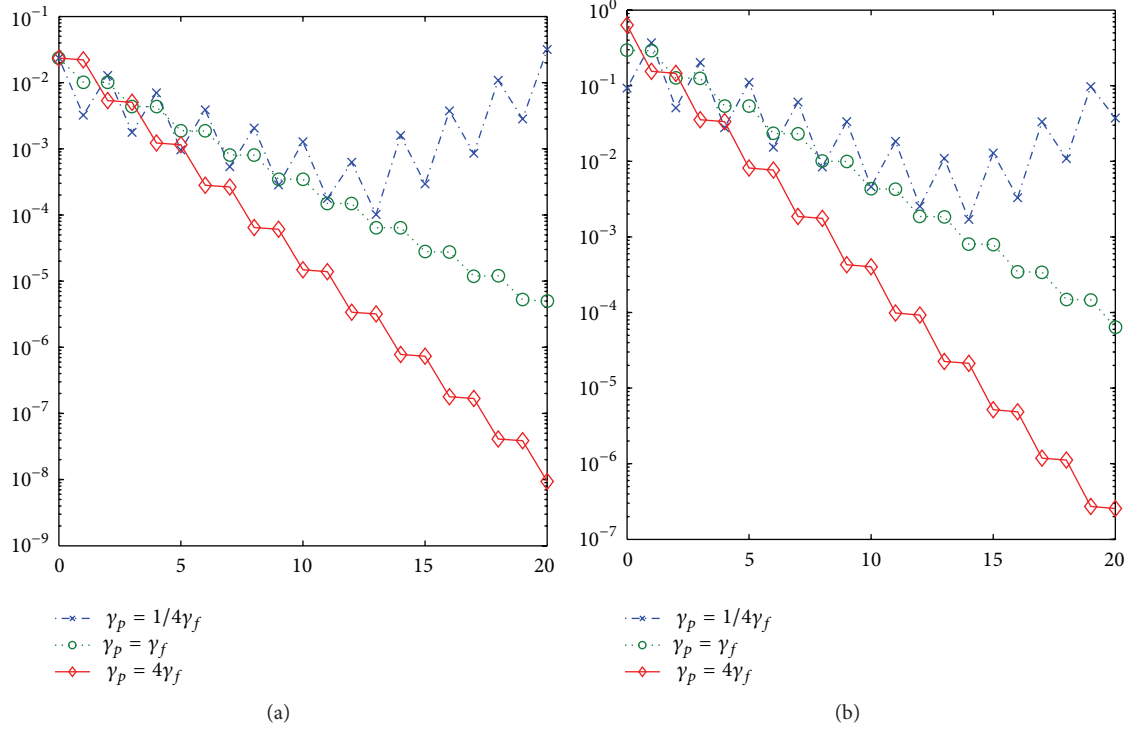


FIGURE 2: Convergence for the velocity of the free flow (a) and the hydraulic head of the porous medium flow (b) versus the iteration counter m for the parallel DDM with BJS interface condition.

TABLE 1: Errors of the finite element method for the steady Navier-Stokes-Darcy model with BJS interface condition.

h	$\ \tilde{u}_h - \tilde{u}\ _0$	$\ \tilde{u}_h - \tilde{u}\ _1$	$\ p_h - p\ _0$	$\ \phi_h - \phi\ _0$	$ \phi_h - \phi _1$
1/8	1.367×10^{-3}	6.147×10^{-2}	8.002×10^{-3}	6.940×10^{-4}	2.452×10^{-2}
1/16	1.687×10^{-4}	1.577×10^{-2}	8.559×10^{-4}	8.687×10^{-5}	6.187×10^{-3}
1/32	2.086×10^{-5}	3.978×10^{-3}	9.506×10^{-5}	1.089×10^{-5}	1.553×10^{-3}
1/64	2.594×10^{-6}	9.974×10^{-4}	1.121×10^{-5}	1.363×10^{-6}	3.890×10^{-4}
1/128	3.235×10^{-7}	2.496×10^{-4}	1.363×10^{-6}	1.705×10^{-7}	9.733×10^{-5}

Proof. Adding (15) and (17) together, we obtain the following: given $\eta_p, \eta_f \in L^2(\Gamma)$, find $(\hat{\phi}_m, \hat{u}_f, \hat{p}_c) \in X_m \times X_c \times Q_c$ such that

$$\begin{aligned}
& (\hat{u}_c \cdot \nabla \hat{u}_c, \tilde{v})_{\Omega_c} + a_c(\hat{u}_c, \tilde{v}) + b_c(\tilde{v}, \hat{p}_c) \\
& - b_c(\hat{u}_c, q) + a_m(\hat{\phi}_m, \psi) + \gamma_f \langle \hat{u}_c \cdot \tilde{n}_c, \tilde{v} \cdot \tilde{n}_c \rangle \\
& + \left\langle \frac{g\hat{\phi}_m}{\gamma_p}, \psi \right\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\mathbf{\Pi})}} \langle P_\tau \hat{u}_c, P_\tau \tilde{v} \rangle \\
& = (f_m, \psi)_{\Omega_m} + (\vec{f}_c, \tilde{v})_{\Omega_c} + \langle \eta_f, \tilde{v} \cdot \tilde{n}_c \rangle \\
& + \left\langle \frac{\eta_p}{\gamma_p}, \psi \right\rangle, \quad \forall (\tilde{v}, q, \psi) \in X_m \times X_c \times Q_c.
\end{aligned} \tag{20}$$

For the necessity of the lemma, we pick $\psi = 0$ and \tilde{v} such that $P_\tau \tilde{v} = 0$ in (11) and (20); then by subtracting (20) from (11), we get

$$\begin{aligned}
\langle \eta_f - \gamma_f \tilde{v}_f \cdot \tilde{n}_c + g\phi_m - gz, \tilde{v} \cdot \tilde{n}_c \rangle & = 0, \\
\forall \tilde{v} \in X_c \text{ with } P_\tau \tilde{v} & = 0,
\end{aligned} \tag{21}$$

which implies (19). The necessity of (18) can be derived in a similar fashion.

As for the sufficiency of the lemma, by substituting the compatibility conditions (18)-(19), we easily see that $(\hat{\phi}_m, \hat{u}_c, \hat{p}_c)$ solves the coupled Navier-Stokes-Darcy system (11), which completes the proof. \square

Now we use the decoupled weak formulation constructed above to propose a physics-based parallel domain decomposition method with Newton iteration as follows.

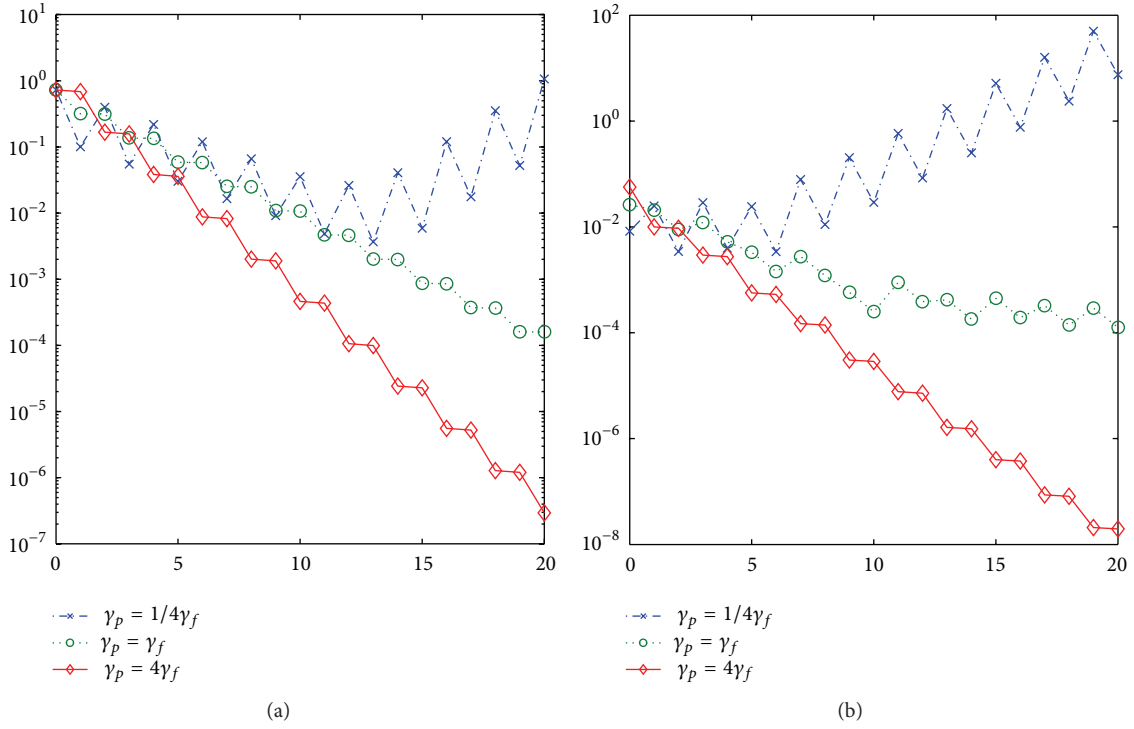


FIGURE 3: Convergence for the pressure of the free flow (a) and η_f (b) versus the iteration counter m for the parallel DDM with BJS interface condition.

- (1) Initial values η_p^0 and η_f^0 are guessed. They may be taken to be zero.
- (2) For $k = 0, 1, 2, \dots$, independently solve the Darcy and Navier-Stokes equations constructed above. More precisely, $\phi_m^k \in X_m$ is computed from

$$a_m(\phi_m^k, \psi) + \left\langle \frac{g\phi_m^k}{\gamma_p}, \psi \right\rangle = \left\langle \frac{\eta_p^k}{\gamma_p}, \psi \right\rangle + (f_m, \psi)_{\Omega_m}, \quad (22)$$

$$\forall \psi \in X_m,$$

and $\vec{u}_c^k \in X_c$ and $p_c^k \in Q_c$ are computed from the following Newton iteration.

- (i) Initial value $\vec{u}_c^{k,0}$ is chosen for the Newton iteration. For instance, it may be taken to be $\vec{u}_c^{0,0} = 0$ and $\vec{u}_c^{k,0} = \vec{u}_c^{k-1}$ for $k = 1, 2, \dots$
- (ii) For $m = 0, 1, 2, \dots, M$, solve

$$\begin{aligned} & (\vec{u}_c^{k,m+1} \cdot \nabla \vec{u}_c^{k,m}, \vec{v})_{\Omega_c} + (\vec{u}_c^{k,m} \cdot \nabla \vec{u}_c^{k,m+1}, \vec{v})_{\Omega_c} \\ & + a_c(\vec{u}_c^{k,m+1}, \vec{v}) + b_c(\vec{v}, p_c^{k,m+1}) \\ & - b_c(\vec{u}_c^{k,m+1}, q) + \gamma_f \langle \vec{u}_c^{k,m+1} \cdot \vec{n}_c, \vec{v} \cdot \vec{n}_c \rangle \\ & + \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \langle P_\tau \vec{u}_c^{k,m+1}, P_\tau \vec{v} \rangle \end{aligned}$$

$$= (\vec{u}_c^{k,m} \cdot \nabla \vec{u}_c^{k,m}, \vec{v})_{\Omega_c} + \langle \eta_f^k, \vec{v} \cdot \vec{n}_c \rangle + (\vec{f}_c, \vec{v})_{\Omega_c}, \quad (23)$$

$$\forall (\vec{v}, q, \psi) \in X_c \times Q_c \times X_m.$$

- (iii) Set $\vec{u}_c^k = \vec{u}_c^{k,M+1}$ and $p_c^k = p_c^{k,M+1}$.

- (3) η_p^{k+1} and η_f^{k+1} are updated in the following manner:

$$\eta_f^{k+1} = \frac{\gamma_f}{\gamma_p} \eta_p^k - \left(1 + \frac{\gamma_f}{\gamma_p} \right) g\phi_m^k + gz, \quad (24)$$

$$\eta_p^{k+1} = -\eta_f^k + (\gamma_f + \gamma_p) \vec{u}_c^k \cdot \vec{n}_c + gz.$$

Then the corresponding domain decomposition finite element method is proposed as follows.

- (1) Initial values $\eta_{p,h}^0$ and $\eta_{f,h}^0$ are guessed. They may be taken to be zero.
- (2) For $k = 0, 1, 2, \dots$, independently solve the Darcy and Navier-Stokes equations with the Robin boundary conditions on the interface, which are constructed previously. More precisely, $\phi_{m,h}^k \in X_{mh}$ is computed from

$$\begin{aligned} & a_m(\phi_{m,h}^k, \psi_h) + \left\langle \frac{g\phi_{m,h}^k}{\gamma_p}, \psi_h \right\rangle \\ & = \left\langle \frac{\eta_{p,h}^k}{\gamma_p}, \psi_h \right\rangle + (f_m, \psi_h)_{\Omega_m}, \quad \forall \psi_h \in X_{mh}, \end{aligned} \quad (25)$$

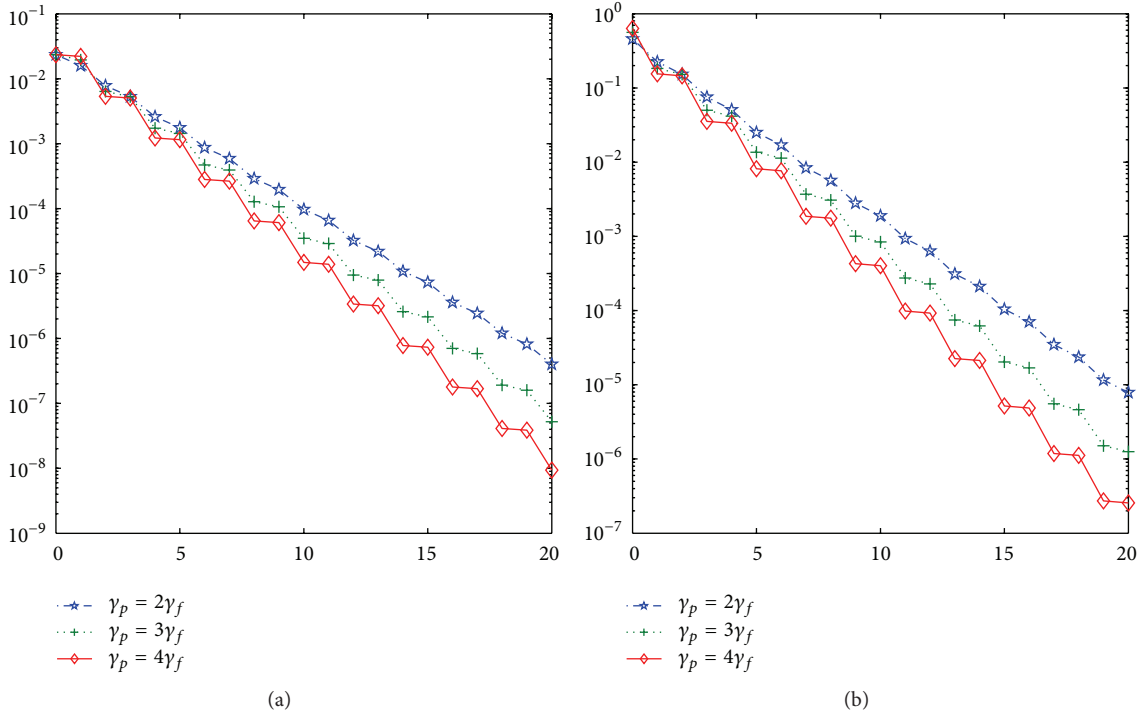


FIGURE 4: Geometric convergence rate of the velocity of the free flow (a) and the hydraulic head of the porous medium flow (b) for the parallel DDM with BJS interface condition.

and $\vec{u}_{c,h}^k \in X_{ch}$ and $p_{c,h}^k \in Q_{ch}$ are computed from the following Newton iteration.

- (i) Initial value $\vec{u}_{c,h}^{k,0}$ is chosen for the Newton iteration. For instance, it may be taken to be $\vec{u}_{c,h}^{0,0} = 0$ and $\vec{u}_{c,h}^{k,0} = \vec{u}_{c,h}^{k-1}$ for $k = 1, 2, \dots$

- (ii) For $m = 0, 1, 2, \dots, M$, solve

$$\begin{aligned}
 & \left(\vec{u}_{c,h}^{k,m+1} \cdot \nabla \vec{u}_{c,h}^{k,m}, \vec{v}_h \right)_{\Omega_c} + \left(\vec{u}_{c,h}^{k,m} \cdot \nabla \vec{u}_{c,h}^{k,m+1}, \vec{v}_h \right)_{\Omega_c} \\
 & + a_c \left(\vec{u}_{c,h}^{k,m+1}, \vec{v}_h \right) + b_c \left(\vec{v}_h, p_{c,h}^{k,m+1} \right) - b_c \left(\vec{u}_{c,h}^{k,m+1}, q_h \right) \\
 & + \gamma_f \left\langle \vec{u}_{c,h}^{k,m+1} \cdot \vec{n}_c, \vec{v}_h \cdot \vec{n}_c \right\rangle \\
 & + \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\mathbb{I})}} \left\langle P_\tau \vec{u}_{c,h}^{k,m+1}, P_\tau \vec{v}_h \right\rangle \\
 & = \left(\vec{u}_{c,h}^{k,m} \cdot \nabla \vec{u}_{c,h}^{k,m}, \vec{v}_h \right)_{\Omega_c} + \left\langle \eta_{f,h}^k, \vec{v}_h \cdot \vec{n}_c \right\rangle + \left(\vec{f}_c, \vec{v}_h \right)_{\Omega_c}, \\
 & \quad \forall (\vec{v}_h, q_h, \psi_h) \in X_{ch} \times Q_{ch} \times X_{mh}. \tag{26}
 \end{aligned}$$

- (iii) Set $\vec{u}_{c,h}^k = \vec{u}_{c,h}^{k,m+1}$ and $p_{c,h}^k = p_{c,h}^{k,m+1}$.

- (3) $\eta_{p,h}^{k+1}$ and $\eta_{f,h}^{k+1}$ are updated in the following manner:

$$\begin{aligned}
 \eta_{f,h}^{k+1} &= \frac{\gamma_f}{\gamma_p} \eta_{p,h}^k - \left(1 + \frac{\gamma_f}{\gamma_p} \right) g \phi_{m,h}^k + g z, \\
 \eta_{p,h}^{k+1} &= -\eta_{f,h}^k + \left(\gamma_f + \gamma_p \right) \vec{u}_{c,h}^k \cdot \vec{n}_c + g z. \tag{27}
 \end{aligned}$$

5. Numerical Example

Example 1. Consider the model problem (2)–(6) with the BJS interface condition (7) on $\Omega = [0, \pi] \times [-1, 1]$ with $\Omega_m = [0, \pi] \times [0, 1]$ and $\Omega_c = [0, \pi] \times [-1, 0]$. Choose $(\alpha \nu \sqrt{\mathbf{d}} / \sqrt{\text{trace}(\mathbb{I})}) = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = K \mathbb{I}$, where \mathbb{I} is the identity matrix and $K = 1$. The boundary condition data functions and the source terms are chosen such that the exact solution is given by

$$\begin{aligned}
 \phi_m &= (e^y - e^{-y}) \sin(x) e^t, \\
 \vec{u}_c &= \left[\frac{K}{\pi} \sin(2\pi y) \cos(x) e^t, \right. \\
 & \quad \left. \left(-2K + \frac{K}{\pi^2} \sin^2(\pi y) \right) \sin(x) e^t \right]^T, \tag{28} \\
 p_c &= 0.
 \end{aligned}$$

We divide Ω_m and Ω_c into rectangles of height $h = 1/N$ and width πh , where N is a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. The Taylor-Hood element pair is used for the Navier-Stokes equations, and the quadratic finite element is used for the second-order formulation of the Darcy equation.

For the coupled finite element method of the steady Navier-Stokes-Darcy model with BJS interface condition,

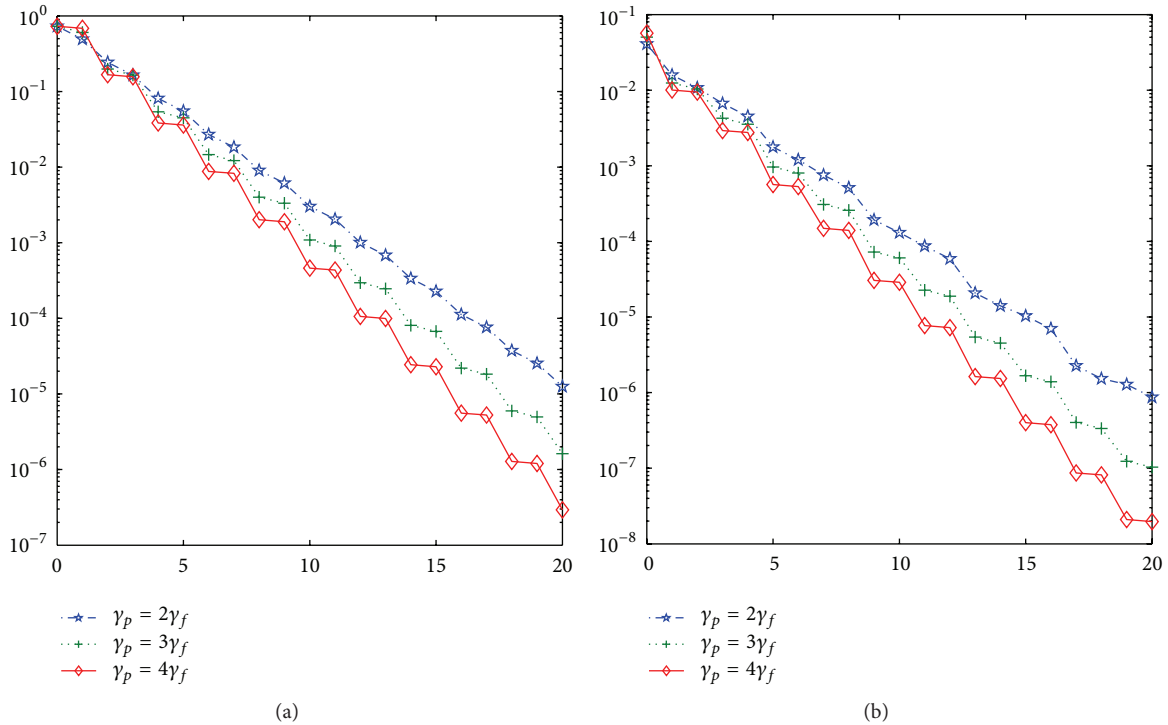


FIGURE 5: Geometric convergence rate of the pressure of the free flow (a) and η_f (b) versus the iteration counter m for the parallel DDM with BJS interface condition.

TABLE 2: L^2 errors in velocity and hydraulic head for the parallel DDM with BJS interface condition.

	L^2 velocity errors	$e(i)/e(i - 4)$	L^2 hydraulic head errors	$e(i)/e(i - 4)$
$e(0)$	2.342×10^{-2}		6.338×10^{-1}	
$e(4) (i = 4)$	1.225×10^{-3}	0.0523	3.337×10^{-2}	0.0527
$e(8) (i = 8)$	6.450×10^{-5}	0.0527	1.756×10^{-3}	0.0526
$e(12) (i = 12)$	3.395×10^{-6}	0.0526	9.246×10^{-5}	0.0527
$e(16) (i = 16)$	1.787×10^{-7}	0.0526	4.868×10^{-6}	0.0527
$e(20) (i = 20)$	9.409×10^{-9}	0.0527	2.562×10^{-7}	0.0526

Table 1 provides errors for different choices of h . Using linear regression, the errors in Table 1 satisfy

$$\begin{aligned} \|\bar{u}_{c,h} - \bar{u}_c\|_0 &\approx 0.714h^{3.011}, & |\bar{u}_{c,h} - \bar{u}_c|_1 &\approx 3.867h^{1.987}, \\ \|p_{c,h} - p_c\|_0 &\approx 5.123h^{3.129}, \\ \|\phi_{m,h} - \phi_m\|_0 &\approx 0.354h^{2.998}, & |\phi_{m,h} - \phi_m|_1 &\approx 1.556h^{1.995}. \end{aligned} \tag{29}$$

These rates of convergence are consistent with the approximation capability of the Taylor-Hood element and quadratic element, which is third order with respect to L^2 norm of \bar{u}_c and ϕ_m , second order with respect to H^1 norm of \bar{u}_c and ϕ_m , and second order with respect to L^2 norms of p_c . In particular, the third-order convergence rate of p_c observed above, which is better than the approximation capability of the linear element, is mainly due to the special analytic solution $p = 0$.

For the parallel DDM with $\nu = 1$, $K = 1$, $\gamma_f = 0.3$, and $h = 1/32$, Figures 2 and 3 show the L^2 errors of hydraulic head, velocity, pressure, and η_f . We can see that the parallel domain decomposition method is convergent for $\gamma_f \leq \gamma_p$. Moreover, Figures 4 and 5 show that a smaller γ_f/γ_p leads to faster convergence.

Then Tables 2 and 3 list some L^2 errors in velocity, hydraulic head, pressure, and η_f for the parallel domain decomposition method with $\gamma_f = 0.3$ and $\gamma_p = 1.2$. The data in these two tables indicate the geometric convergence rate $\sqrt{\gamma_f/\gamma_p}$ since all the error ratios are less than $(\sqrt{\gamma_f/\gamma_p})^4 = (\sqrt{1/4})^4 = 0.0625$.

Finally, for the preconditioning feature of the domain decomposition method, Table 4 shows the number of iterations M is independent of the grid size h . Here, we set $\gamma_S = 0.3$, $\gamma_D = 1.2$, $\nu = 1$, and $K = 1$. Let ϕ_h^k , \bar{u}_h^k , and p_h^k denote the finite element solutions of ϕ_D^k , \bar{u}_S^k , and p_S^k at the k th step

TABLE 3: L^2 errors in pressure and η_f for the parallel DDM with BJS interface condition.

	L^2 velocity errors	$e(i)/e(i-4)$	L^2 hydraulic head errors	$e(i)/e(i-4)$
$e(0)$	7.268×10^{-1}		5.668×10^{-2}	
$e(4)$ ($i = 4$)	3.826×10^{-2}	0.0526	2.752×10^{-3}	0.0486
$e(8)$ ($i = 8$)	2.014×10^{-3}	0.0526	1.399×10^{-4}	0.0508
$e(12)$ ($i = 12$)	1.060×10^{-4}	0.0526	7.233×10^{-6}	0.0517
$e(16)$ ($i = 16$)	5.579×10^{-6}	0.0526	3.767×10^{-7}	0.0521
$e(20)$ ($i = 20$)	2.937×10^{-7}	0.0526	1.969×10^{-8}	0.0523

TABLE 4: The iteration counter M versus the grid size h for both the parallel Robin-Robin domain decomposition method with BJS interface condition.

h	1/8	1/16	1/32	1/64
M	19	19	19	19

of the domain decomposition algorithm. The criterion used to stop the iteration, that is, to determine the value M , is $\|\vec{u}_h^k - \vec{u}_h^{k-1}\|_0 + \|\phi_h^k - \phi_h^{k-1}\|_0 + \|p_h^k - p_h^{k-1}\|_0 < \varepsilon$, where the tolerance $\varepsilon = 10^{-5}$.

6. Conclusions

In this paper, a parallel physics-based domain decomposition method is proposed for the stationary Navier-Stokes-Darcy model with the BJS interface condition. This method is based on the Robin boundary conditions constructed from the three physical interface conditions. Moreover, it is convergent with geometric convergence rates if the relaxation parameter is selected properly. The number of iteration steps is independent of the grid size due to the natural preconditioning advantage of the domain decomposition methods.

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