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Sub-supersolution method for quasilinear parabolic variational inequalities

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Abstract

This paper is about a systematic attempt to apply the sub-supersolution method to parabolic variational inequalities. We define appropriate concepts of sub-supersolutions and derive existence, comparison, and extremity results for such inequalities.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0,\tau)$ and $\Gamma = \partial\Omega \times (0,\tau)$, $\tau > 0$. In this paper we are concerned with existence and comparison results of the following parabolic variational inequality:

$$u \in Y_0 \cap K,\ u(\cdot\,,0) = 0: \quad \langle u_t + A(u) + F(u) - h, v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{1.1}$$

where K is a closed and convex subset of $X_0 := L^p(0,\tau; W_0^{1,p}(\Omega))$, $Y_0 = \{u \in X_0\colon u_t \in X_0^*\}$, $\langle \cdot\,,\cdot \rangle$ denotes the duality pairing between X_0^* and X_0, and $p \in [2,\infty)$. The operator $A : X_0 \to X_0^*$ is related with a nonlinear elliptic operator of Leray–Lions type in divergence form given by

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$$A(u)(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x,t,\nabla u(x,t)\big),$$

and F is the Nemytskij operator associated with the Carathéodory function $f : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$F(u)(x,t) = f\big(x,t,u(x,t),\nabla u(x,t)\big).$$

We assume that $h \in L^{p'}(Q) \subset X_0^*$, where p' is the Hölder conjugate of p.

Solutions of the variational inequality (1.1) are usually referred to as *strong* solutions, cf. [17]. There is a large number of papers dealing with parabolic inequalities under different structure and regularity hypotheses of the data such as, e.g., [7–12,17,19–21,23,24] and the recent survey paper [22].

The aim of this paper is to develop the method of sub-supersolutions for the parabolic variational inequality (1.1). While the sub-supersolution method is well established for parabolic equations that result from (1.1) in case that K is the entire space X_0, i.e., $K = X_0$, there are only a few papers dealing with sub-supersolutions for (1.1) and only for the case that K is given by an obstacle problem, see, e.g., [8–10,20]. To our knowledge, it seems that there has not been any publication that addresses the sub-supersolution method to the general case of a closed convex set K considered here. Also, as would be seen in the sequel, the arguments for parabolic variational inequalities do not follow straightforwardly from neither those for elliptic inequalities, nor those for parabolic equations.

One of our main ingredients here is a suitable notion of sub- and supersolution of (1.1) for general convex sets K, which in some sense is symmetric and which yields the notion of sub-supersolution in the special case of an obstacle. Moreover, the general notion of sub-supersolution introduced here is shown to be consistent with the usual notion of sub-supersolutions for equations, i.e., when $K = X_0$.

The plan of our paper is as follows. In Section 2 we introduce the basic notions and hypotheses, and in Section 3 we prove our main existence and comparison result. Finally, in Section 4 we demonstrate the applicability of the theory developed in the previous sections to a parabolic obstacle problem.

2. Notions and hypotheses

Let $W^{1,p}(\Omega)$ denote the usual Sobolev space and $(W^{1,p}(\Omega))^*$ its dual space with $2 \leqslant p < \infty$. Then $W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))^*$ forms an evolution triple with all the embeddings being continuous, dense and compact, cf. [25].

We set $X = L^p(0,\tau; W^{1,p}(\Omega))$, and its dual space $X^* = L^{p'}(0,\tau; (W^{1,p}(\Omega))^*)$, and we define a function space Y by

$$Y = \{u \in X\colon\ u_t \in X^*\},$$

where the derivative $\partial/\partial t$ is understood in the sense of vector-valued distributions, cf. [25], which is characterized by

$$\int_0^\tau u'(t)\phi(t)\,dt = -\int_0^\tau u(t)\phi'(t)\,dt, \quad \forall \phi \in C_0^\infty(0,\tau).$$

The space Y endowed with the graph norm

$$\|u\|_Y = \|u\|_X + \|u_t\|_{X^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of X and X^*, respectively. Furthermore it is well known that the embedding $Y \subset C([0,\tau], L^2(\Omega))$ is continuous, cf. [25]. Finally, because $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, we have by Aubin's lemma a compact embedding of $Y \subset L^p(Q)$, cf. [25].

By $W_0^{1,p}(\Omega)$ we denote the subspace of $W^{1,p}(\Omega)$ whose elements have generalized homogeneous boundary values. Let $W^{-1,p'}(\Omega)$ denote the dual space of $W_0^{1,p}(\Omega)$. Then obviously $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$ forms an evolution triple and all statements made above remain true also in this situation when setting $X_0 = L^p(0,\tau; W_0^{1,p}(\Omega))$, $X_0^* = L^{p'}(0,\tau; W^{-1,p'}(\Omega))$ and $Y_0 = \{u \in X_0\colon\ u_t \in X_0^*\}$. Let $\|\cdot\|_X$ and $\|\cdot\|_{X_0}$ be the usual norms defined on X and X_0 (and similarly on X^* and X_0^*),

$$\|u\|_X = \left(\int_0^\tau \big\|u(t)\big\|^p_{W^{1,p}(\Omega)}\,dt\right)^{1/p}, \qquad \|u\|_{X_0} = \left(\int_0^\tau \big\|u(t)\big\|^p_{W_0^{1,p}(\Omega)}\,dt\right)^{1/p}.$$

We use the notation $\langle\cdot,\cdot\rangle$ for any of the dual pairings between X and X^*, X_0 and X_0^*, $W^{1,p}(\Omega)$ and $[W^{1,p}(\Omega)]^*$, and $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. For example, with $f \in X^*$, $u \in X$,

$$\langle f, u\rangle = \int_0^\tau \big\langle f(t), u(t)\big\rangle\,dt.$$

Let $L := \partial/\partial t$ and its domain of definition $D(L)$ given by

$$D(L) = \big\{u \in X_0\colon\ u_t \in X_0^* \text{ and } u(0) = 0\big\}.$$

The linear operator $L : D(L) \to X_0^*$ can be shown to be closed, densely defined and maximal monotone, e.g., cf. [25, Chapter 32].

We assume that $a_i : Q \times \mathbb{R}^N \to \mathbb{R}$ and $f : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, where f has certain growth conditions to be specified later and a_i satisfies

$$\big|a_i(x,t,\xi)\big| \leqslant c_1|\xi|^{p-1} + c_2(x,t), \tag{2.1}$$

$$\sum_{i=1}^N \big[a_i(x,t,\xi) - a_i(x,t,\xi')\big](\xi_i - \xi_i') > 0, \tag{2.2}$$

$$\sum_{i=1}^N a_i(x,t,\xi)\xi_i \geqslant c_3|\xi|^p - c_4(x,t) \tag{2.3}$$

for almost all $(x,t) \in Q$, all $\xi, \xi' \in \mathbb{R}^N$ with $\xi' \neq \xi$, where $c_1, c_3 \in (0,\infty)$, $c_2 \in L^{p'}(Q)$, and $c_4 \in L^1(Q)$.

The operators $A : X \to X^* \subset X_0^*$ related with the quasilinear elliptic operator, and $F : X \to X^* \subset X_0^*$, as well as $h \in L^{p'}(Q) \subset X_0^*$, are defined as follows:

$$\langle A(u), v\rangle = \sum_{i=1}^{N} \int_Q a_i(x,t,\nabla u) v_{x_i}\, dx\, dt,$$

$$\langle F(u), v\rangle = \int_Q f(x,t,u,\nabla u) v\, dx\, dt,$$

$$\langle h, v\rangle = \int_Q h(x,t) v(x,t)\, dx\, dt \tag{2.4}$$

for all $v, u \in X$. Thus the variational inequality (1.1) may be rewritten as

$$u \in D(L) \cap K: \quad \langle Lu + A(u) - F(u) - h, v - u\rangle \geqslant 0, \quad \forall v \in K. \tag{2.5}$$

A partial ordering in $L^p(Q)$ is defined by $u \leqslant w$ if and only if $w - u$ belongs to the positive cone $L^p_+(Q)$ of all nonnegative elements of $L^p(Q)$. This induces a corresponding partial ordering also in the subspace Y of $L^p(Q)$, and if $\underline{u}, \bar{u} \in Y$ with $\underline{u} \leqslant \bar{u}$ then

$$[\underline{u}, \bar{u}] = \{u \in Y: \ \underline{u} \leqslant u \leqslant \bar{u}\}$$

denotes the order interval formed by $\underline{u}$ and $\bar{u}$. Further, for $u, v \in X$, $U, V \subset X$, we use the notation $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $U * V = \{u * v: \ u \in U,\ v \in V\}$, $u * U = \{u\} * U$ with $* \in \{\wedge, \vee\}$.

Our basic notion of sub- and supersolution of (1.1) is defined as follows.

Definition 2.1. A function $\underline{u} \in Y$ is called a *subsolution* of (1.1) if

(i) $F\underline{u} \in L^{p'}(Q)$,

(ii) $\underline{u}(\cdot, 0) \leqslant 0$ a.e. in Ω, $\quad \underline{u} \leqslant 0$ on Γ,

(iii) $\langle \underline{u}_t, v - \underline{u}\rangle + \langle A(\underline{u}), v - \underline{u}\rangle + \langle F(\underline{u}), v - \underline{u}\rangle \geqslant \langle h, v - \underline{u}\rangle$,
$\forall v \in \underline{u} \wedge K.$ (2.6)

We have a similar definition for supersolutions of (1.1).

Definition 2.2. A function $\bar{u} \in Y$ is called a *supersolution* of (1.1) if

(i) $F\bar{u} \in L^{p'}(Q)$,

(ii) $\bar{u}(\cdot, 0) \geqslant 0$ a.e. in Ω, $\quad \bar{u} \geqslant 0$ on Γ,

(iii) $\langle \bar{u}_t, v - \bar{u}\rangle + \langle A(\bar{u}), v - \bar{u}\rangle + \langle F(\bar{u}), v - \bar{u}\rangle \geqslant \langle h, v - \bar{u}\rangle$,
$\forall v \in \bar{u} \vee K.$ (2.7)

Remark 2.1. The above definition of sub- and supersolutions of (1.1) is in some sense symmetric, since, e.g., the notion of supersolution is obtained from the notion of subsolution by reversing the inequalities in (2.6)(ii) and replacing $\underline{u} \wedge K$ in (2.6)(iii) by $\bar{u} \vee K$.

Definition 2.3. Let $C \neq \emptyset$ be a closed and convex subset of a reflexive Banach space X. A bounded, hemicontinuous and monotone operator $P : X \to X^*$ is called a *penalty operator* associated with $C \subset X$ if

$$P(u) = 0 \quad \Leftrightarrow \quad u \in C.$$

We assume that there exists a pair of sub-supersolutions $\underline{u}$ and $\bar{u}$ of (1.1) such that $\underline{u} \leqslant \bar{u}$ a.e. in Q and that f has the following growth between $\underline{u}$ and $\bar{u}$:

(H1) $$\left|f(x,t,u,\xi)\right| \leqslant c_5(x,t) + c_6|\xi|^{p-1} \tag{2.8}$$

for some $c_5 \in L^{p'}(Q)$, for a.e. $(x,t) \in Q$, all $\xi \in \mathbb{R}^N$, and all $u \in [\underline{u}(x,t), \bar{u}(x,t)]$.

Moreover, suppose that there exists a penalty operator $P : X_0 \to X_0^*$ associated with $K \subset X_0$ with the following properties:

(H2) For each $u \in D(L)$, there exists $w = w(u) \in X_0$ such that

$$\begin{aligned} &\text{(i)} \quad \langle u_t + Au, w\rangle \geqslant 0, \\ &\text{(ii)} \quad \langle Pu, w\rangle \geqslant D\|Pu\|_{X_0^*}\|w\|_{L^p(Q)} \end{aligned} \tag{2.9}$$

for some constant $D > 0$ independent of u and w.

Remark 2.2. In Section 4 we shall see that hypothesis (H2) can easily be satisfied for obstacle problems if the obstacle function ψ basically satisfies $\psi_t + A\psi \geqslant 0$ in X_0^*.

3. Main result

In this section we prove our main existence and comparison result which reads as follows.

Theorem 3.1. *Assume* (1.1) *has an ordered pair of sub- and supersolutions* $\underline{u}$ *and* $\bar{u}$ *and that* (2.1)–(2.3) *and* (H1)–(H2) *are satisfied. Suppose furthermore that* $D(L) \cap K \neq \emptyset$ *and*

$$\underline{u} \vee K \subset K, \qquad \bar{u} \wedge K \subset K. \tag{3.1}$$

Then, (1.1) *has a solution* u *such that* $\underline{u} \leqslant u \leqslant \bar{u}$ *a.e. in* Q.

Proof. The proof is a combination of arguments for parabolic equations in [4] with those for elliptic variational inequalities in [18]. We define the following cut-off function b and truncation function T:

$$b(x,t,u) = \begin{cases} [u - \bar{u}(x,t)]^{p-1} & \text{if } u > \bar{u}(x,t), \\ 0 & \text{if } \underline{u}(x,t) \leqslant u \leqslant \bar{u}(x,t), \\ -[\underline{u}(x,t) - u]^{p-1} & \text{if } u < \underline{u}(x,t), \end{cases}$$

for $(x,t,u) \in \Omega \times (0,\tau) \times \mathbb{R}$ and

$$(Tu)(x,t) = \begin{cases} \bar{u}(x,t) & \text{if } u(x,t) > \bar{u}(x,t), \\ u(x,t) & \text{if } \underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t), \\ \underline{u}(x,t) & \text{if } u(x,t) < \underline{u}(x,t), \end{cases}$$

for $(x,t) \in Q$, $u \in X$. It is easy to check that b is a Carathéodory function with the growth condition

$$\bigl|b(x,t,u)\bigr| \leqslant c_7(x,t) + c_8|u|^{p-1} \quad \text{for a.e. } (x,t) \in Q, \text{ all } u \in \mathbb{R}, \tag{3.2}$$

with $c_7 \in L^{p'}(Q)$, $c_8 > 0$. Hence the operator $B : X_0 \to X_0^*$ given by

$$\langle Bu, v\rangle = \int_Q b(x,t,u)v\,dx\,dt \quad (u, v \in X), \tag{3.3}$$

is well defined. Moreover, there are $c_9, c_{10} > 0$ such that

$$\int_Q b(x,t,u)u\,dx\,dt \geqslant c_9\|u\|^p_{L^p(Q)} - c_{10}, \quad \forall u \in X_0. \tag{3.4}$$

We define the operator $\mathcal{C}$ from X_0 to X_0^* by

$$\mathcal{C}(u) = \gamma Bu + F \circ T(u), \quad u \in X_0 \tag{3.5}$$

(γ is a positive constant to be determined later and $F \circ T$ denotes the composition of F and T), and consider the following auxiliary variational inequality in X_0:

$$u \in D(L) \cap K: \quad \bigl\langle Lu + A(u) + \mathcal{C}(u) - h, v - u\bigr\rangle \geqslant 0, \quad \forall v \in K. \tag{3.6}$$

Using usual arguments, one readily verifies that $A + \mathcal{C}$ is pseudomonotone with respect to $D(L)$. Let us check that $A + \mathcal{C}$ is coercive on X_0 in the following sense:

$$\lim_{\|u\|_{X_0} \to \infty} \frac{\langle (A + \mathcal{C})(u), u - \varphi\rangle}{\|u\|_{X_0}} = \infty \tag{3.7}$$

for any $\varphi \in X_0$. In fact, from (2.3), we have

$$\langle Au, u\rangle \geqslant c_3\bigl\||\nabla u|\bigr\|^p_{L^p(Q)} - c_{11}, \quad \forall u \in X_0, \tag{3.8}$$

with some constant $c_{11} > 0$. Using Stampacchia's theorem (cf. [13,16]) and Young's inequality together with (2.8), we have for each $\varepsilon > 0$ constants $c_{12} = c_{12}(\varepsilon)$, $c_{13} = c_{13}(\varepsilon) > 0$ such that for all $u \in X_0$,

$$\begin{aligned} \bigl|\bigl\langle F \circ T(u), u\bigr\rangle\bigr| &= \left|\int_Q (F \circ T)(u)u\,dx\,dt\right| \\ &\leqslant \|c_5\|_{L^{p'}(Q)}\|u\|_{L^p(Q)} + c_6\bigl\||\nabla u|\bigr\|^{p-1}_{L^p(Q)}\|u\|_{L^p(Q)} \\ &\leqslant \varepsilon\bigl\||\nabla u|\bigr\|^p_{L^p(Q)} + c_{12}\|u\|^p_{L^p(Q)} + c_{13}. \end{aligned} \tag{3.9}$$

Combining (3.4) with (3.8) and (3.9), one gets

$$\big\langle (A+\mathcal{C})(u),u\big\rangle \geqslant (c_3-\varepsilon)\big\| |\nabla u| \big\|^p_{L^p(Q)} + (\gamma c_9 - c_{12})\|u\|^p_{L^p(Q)} \\ - (c_{11}+\gamma c_{10}+c_{13}), \quad \forall u \in X_0.$$

Choosing $\varepsilon = c_3/2$ and $\gamma = c_{12}c_9^{-1}$, we have $c_{14}, c_{15} > 0$ such that

$$\big\langle (A+\mathcal{C})(u),u\big\rangle \geqslant c_{14}\|u\|^p_{X_0} - c_{15}, \quad \forall u \in X_0. \tag{3.10}$$

For any $\varphi \in X_0$ fixed, it is inferred from (2.1), (3.2), and (2.8) that

$$\big|\big\langle (A+\mathcal{C})(u),\varphi\big\rangle\big| \leqslant c_{16}\big(\|u\|^{p-1}_{X_0}+1\big), \quad \forall u \in X_0, \tag{3.11}$$

for some constant $c_{16} = c_{16}(\varphi) > 0$. From (3.10) and (3.11), we obtain (3.7).

It follows from the pseudomonotonicity and coercivity of $A+\mathcal{C}$ with respect to $D(L)$ that the variational inequality (3.6) has a solution u. The proof of this claim is given in Lemma 3.1 below. Now, let us show that any solution u of (3.6) satisfies $\underline{u} \leqslant u \leqslant \bar{u}$ a.e. in Q. We verify that $\underline{u} \leqslant u$, the second inequality is proved in the same way. Because $u \in K$, it follows from (3.1) that

$$u + (\underline{u}-u)^+ = \underline{u} \vee u \in K.$$

Letting $v = u + (\underline{u}-u)^+$ into (3.6), one gets

$$\big\langle u_t, (\underline{u}-u)^+\big\rangle + \big\langle Au + \gamma Bu + F(Tu), (\underline{u}-u)^+\big\rangle \geqslant \big\langle h, (\underline{u}-u)^+\big\rangle. \tag{3.12}$$

On the other hand, since $\underline{u}$ is a subsolution, it follows from (2.6)(iii), with

$$v = \underline{u} - (\underline{u}-u)^+ = \underline{u} \wedge u \in \underline{u} \wedge K,$$

that

$$-\big\langle \underline{u}_t, (\underline{u}-u)^+\big\rangle - \big\langle A\underline{u}, (\underline{u}-u)^+\big\rangle - \big\langle F(\underline{u}), (\underline{u}-u)^+\big\rangle \geqslant -\big\langle h, (\underline{u}-u)^+\big\rangle. \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\big\langle (u-\underline{u})_t, (\underline{u}-u)^+\big\rangle + \big\langle Au - A\underline{u} + \gamma Bu, (\underline{u}-u)^+\big\rangle \\ + \big\langle F(Tu) - F(\underline{u}), (\underline{u}-u)^+\big\rangle \geqslant 0. \tag{3.14}$$

We have $\underline{u} - u \in Y$ and $(\underline{u}-u)^+(\cdot,0) = 0$, and thus

$$\big\langle (\underline{u}-u)_t, (\underline{u}-u)^+\big\rangle = \frac{1}{2}\big\|(\underline{u}-u)^+(\cdot,\tau)\big\|^2_{L^2(\Omega)} \geqslant 0. \tag{3.15}$$

On the other hand, it is easy to check from (2.2) that

$$\big\langle A\underline{u} - Au, (\underline{u}-u)^+\big\rangle \geqslant 0. \tag{3.16}$$

Moreover,

$$\big\langle F(Tu) - F(\underline{u}), (\underline{u}-u)^+\big\rangle \\ = \int_{Q^+} \big[f\big(\cdot,\cdot,Tu,\nabla(Tu)\big) - f(\cdot,\cdot,\underline{u},\nabla\underline{u})\big](\underline{u}-u)\,dx\,dt,$$

where $Q^+ = \{(x,t) \in Q\colon \underline{u}(x,t) \geqslant u(x,t)\}$. But because of

$$Tu = \underline{u} \quad \text{and} \quad \nabla(Tu) = \nabla\underline{u} \quad \text{a.e. on } Q^+,$$

one has

$$\big\langle F(Tu) - F(\underline{u}), (\underline{u} - u)^+\big\rangle = 0. \tag{3.17}$$

Combining (3.15)–(3.17) with (3.14), we obtain

$$0 \leqslant \gamma\big\langle Bu, (\underline{u} - u)^+\big\rangle = -\int_{Q^+} (\underline{u} - u)^p \, dx \, dt \leqslant 0.$$

This proves that $\underline{u} - u = 0$ a.e. on Q^+ and thus $\underline{u} \leqslant u$ a.e. on Q. A similar proof shows that $u \leqslant \bar{u}$. From $\underline{u} \leqslant u \leqslant \bar{u}$, we have $Bu = 0$ and $Tu = u$. Consequently, u is also a solution of (2.5). □

To complete the proof of the Theorem 3.1, we need to show the solvability of the inequality (3.6), which is given in the following lemma.

Lemma 3.1. *Under the assumptions of Theorem* 3.1*, the variational inequality* (3.6) *has solutions.*

Proof. The penalty arguments we use here are motivated by Deuel and Hess' paper [10]. For $\varepsilon > 0$, let us consider the following penalized equation:

$$u \in D(L): \quad \langle u_t, v\rangle + \big\langle (A + \mathcal{C})(u), v\big\rangle + \frac{1}{\varepsilon}\langle Pu, v\rangle = \langle h, v\rangle, \quad \forall v \in X_0, \tag{3.18}$$

where P is a penalty operator (associated to K) that satisfies (2.9).

Because $A + \mathcal{C}$ is pseudomonotone with respect to $D(L)$ and $\varepsilon^{-1}P$ is monotone, $A + \mathcal{C} + \varepsilon^{-1}P$ is also pseudomonotone with respect to $D(L)$. Moreover, it is bounded and hemicontinuous on X_0. From the coercivity of $A + \mathcal{C}$, see (3.7), and the monotonicity of $\varepsilon^{-1}P$, it is easy to see that $A + \mathcal{C} + \varepsilon^{-1}P$ is coercive on X_0,

$$\lim_{\|u\|_{X_0} \to \infty} \frac{\langle (A + \mathcal{C} + \varepsilon^{-1}P)(u), u - \varphi\rangle}{\|u\|_{X_0}} = \infty \tag{3.19}$$

for any $\varphi \in X_0$ (fixed). According to existence results for solutions of parabolic variational equalities (cf., e.g., [2,3,17]), for each $\varepsilon > 0$, (3.18) has solutions. Let u_ε be a solution of (3.18). We show that the family $\{u_\varepsilon\colon\ \varepsilon > 0,\ \text{small}\}$ is bounded with respect to the graph norm of $D(L)$. In fact, let u_0 be a (fixed) element of $D(L) \cap K$. Putting $v = u_\varepsilon - u_0$ into (3.18) (with u_ε) and noting the monotonicity of L and that $Pu_0 = 0$, one gets

$$\begin{aligned}
&\langle h - u_{0t}, u_\varepsilon - u_0\rangle \\
&\quad = \langle u_{\varepsilon t} - u_{0t}, u_\varepsilon - u_0\rangle + \big\langle (A + \mathcal{C})(u_\varepsilon), u_\varepsilon - u_0\big\rangle + \frac{1}{\varepsilon}\langle Pu_\varepsilon - Pu_0, u_\varepsilon - u_0\rangle \\
&\quad \geqslant \big\langle (A + \mathcal{C})(u_\varepsilon), u_\varepsilon - u_0\big\rangle.
\end{aligned}$$

Thus,

$$\frac{\langle (A + \mathcal{C})(u_\varepsilon), u_\varepsilon - u_0\rangle}{\|u_\varepsilon - u_0\|_{X_0}} \leqslant \|h - u_{0t}\|_{X_0^*}$$

for all $\varepsilon > 0$. From (3.7), we have that $\|u_\varepsilon\|_{X_0}$ is bounded. As a consequence, we see that $\{Au_\varepsilon\}$ and $\{\mathcal{C}u_\varepsilon\}$ are bounded sequences in X_0^*. Moreover, from the growth conditions of b and F and the definition of T, we can also prove that $\{\mathcal{C}u_\varepsilon\}$ is a bounded sequence in $L^{p'}(Q)$.

Next, we check that the sequence $\{\varepsilon^{-1}Pu_\varepsilon\}$ is also bounded in X_0^*. To see this, for each ε, we choose $w = w_\varepsilon$ to be an element satisfying (2.9) with $u = u_\varepsilon$. From (3.18), we have

$$\langle u_{\varepsilon t}, w_\varepsilon\rangle + \big\langle (A+\mathcal{C})(u_\varepsilon), w_\varepsilon\big\rangle + \frac{1}{\varepsilon}\langle Pu_\varepsilon, w_\varepsilon\rangle = \langle h, w_\varepsilon\rangle.$$

From (2.9)(i), $\langle u_{\varepsilon t}, w_\varepsilon\rangle + \langle Au_\varepsilon, w_\varepsilon\rangle \geqslant 0$. Therefore,

$$\frac{1}{\varepsilon}\langle Pu_\varepsilon, w_\varepsilon\rangle \leqslant \big\langle h - \mathcal{C}(u_\varepsilon), w_\varepsilon\big\rangle. \tag{3.20}$$

Since $\{\|\mathcal{C}u_\varepsilon\|_{L^{p'}(Q)}\}$ is bounded, there exists a constant $c > 0$ such that

$$\big|\langle h - \mathcal{C}u_\varepsilon, w_\varepsilon\rangle\big| \leqslant c\|w_\varepsilon\|_{L^p(Q)}, \quad \forall\varepsilon.$$

This and (2.9)(ii) imply that

$$\frac{1}{\varepsilon}\|Pu_\varepsilon\|_{X_0^*} \leqslant \frac{c}{D}, \quad \forall\varepsilon.$$

On the other hand, since

$$u_{\varepsilon t} = h - (A + \mathcal{C} + \varepsilon^{-1}P)(u_\varepsilon)$$

in X_0^*, the above estimate implies that $\{u_{\varepsilon t}\}$ is also bounded in X_0^*. We have shown that $\{u_\varepsilon\}$ is bounded with respect the graph norm of $D(L)$. As a consequence, there exist $u \in X_0$ (with $u_t \in X_0^*$) and a subsequence of $\{u_\varepsilon\}$, still denoted by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } X_0, \qquad u_{\varepsilon t} \rightharpoonup u_t \quad \text{in } X_0^* \ (\varepsilon \to 0^+). \tag{3.21}$$

Since $D(L)$ is closed in Y and convex, it is weakly closed in Y, and thus $u \in D(L)$.

Now, we prove that u is a solution of the variational inequality (3.6). First, note that $Pu = 0$. In fact, we have $Pu_\varepsilon \to 0$ in X_0^*. It follows from the monotonicity of P that

$$\langle Pv, v - u\rangle \geqslant 0, \quad \forall v \in X_0.$$

As in the proof of Minty's lemma (cf. [16]), one obtains from this inequality that

$$\langle Pu, v\rangle \geqslant 0, \quad \forall v \in X_0.$$

Hence, $Pu = 0$ in X_0^*, that is, $u \in K$. On the other hand, (3.21) and Aubin's lemma (see [17]) imply that

$$u_\varepsilon \to u \quad \text{in } L^p(Q). \tag{3.22}$$

As a consequence, we get

$$\langle \mathcal{C}u_\varepsilon, u_\varepsilon - u\rangle \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{3.23}$$

For $w \in K$, letting $v = w - u_\varepsilon$ in (3.18) (with $u = u_\varepsilon$), one gets

$$\langle u_{\varepsilon t}, w - u_\varepsilon\rangle + \big\langle (A + \mathcal{C})(u_\varepsilon), w - u_\varepsilon\big\rangle - \langle h, w - u_\varepsilon\rangle$$
$$= \frac{1}{\varepsilon}\langle -Pu_\varepsilon, w - u_\varepsilon\rangle \geqslant 0. \tag{3.24}$$

By choosing $w = u$ in (3.24), we have

$$\langle Au_\varepsilon, u - u_\varepsilon\rangle \geqslant \langle h, u - u_\varepsilon\rangle - \langle \mathcal{C}u_\varepsilon, u - u_\varepsilon\rangle - \langle u_t, u - u_\varepsilon\rangle + \langle u_t - u_{\varepsilon t}, u - u_\varepsilon\rangle$$
$$\geqslant \langle h, u - u_\varepsilon\rangle - \langle \mathcal{C}u_\varepsilon, u - u_\varepsilon\rangle - \langle u_t, u - u_\varepsilon\rangle.$$

As a consequence, one gets

$$\liminf_{\varepsilon\to 0^+}\langle Au_\varepsilon, u - u_\varepsilon\rangle \geqslant 0.$$

Because A is of class (S_+) with respect to $D(L)$ (cf., e.g., [2,3] or [5]), we infer from (3.21) and this limit that

$$u_\varepsilon \to u \quad \text{in } X_0. \tag{3.25}$$

Letting $\varepsilon \to 0$ in (3.24) and taking (3.21) and (3.25) into account, we obtain

$$\langle u_t, w - u\rangle + \big\langle (A + \mathcal{C})(u), w - u\big\rangle - \langle h, w - u\rangle \geqslant 0.$$

This holds for all $w \in K$, proving that u is in fact a solution of (3.6). □

Remark 3.2. (a) Theorem 3.1 can be extended to the case where $\underline{u}$ is the maximum of some subsolutions and $\bar{u}$ is the minimum of some supersolutions. In fact, assume that

$$\underline{u} := \max\{\underline{u}_1, \dots, \underline{u}_k\} \leqslant \bar{u} := \min\{\bar{u}_1, \dots, \bar{u}_m\},$$

where $\underline{u}_1, \dots, \underline{u}_k$ (respectively, $\bar{u}_1, \dots, \bar{u}_m$) are subsolutions (respectively, supersolutions) of (1.1). If f has the growth condition (2.8) for a.e. $(x,t) \in Q$, all $\xi \in \mathbb{R}^N$, all u in the interval

$$\big[\min\{\underline{u}_1, \dots, \underline{u}_k\}(x,t), \max\{\bar{u}_1, \dots, \bar{u}_m\}(x,t)\big],$$

then (1.1) has a solution within the interval $[\underline{u}, \bar{u}]$.

The proof of this more general result follows the same lines as that of Theorem 3.1 with the following modified operator $\mathcal{C}$:

$$\mathcal{C}(u) = \gamma Bu + F\circ T(u) + \sum_{i=1}^{k}\sum_{j=1}^{m}\big|F\circ T_{ij}(u) - F\circ T(u)\big| \quad (u \in X),$$

where,

$$T_{ij}u(x,t) = \begin{cases} \underline{u}_i(x,t) & \text{if } u(x,t) < \underline{u}_i(x,t),\\ u(x,t) & \text{if } \underline{u}_i(x,t) \leqslant u(x,t) \leqslant \bar{u}_j(x,t),\\ \bar{u}_j(x,t) & \text{if } u(x,t) > \bar{u}_j(x,t), \end{cases}$$

for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, $(x,t) \in Q$, and

$$\big\langle \big|F\circ T_{ij}(u) - F\circ T(u)\big|, v\big\rangle = \int_Q \big|f(\cdot,\cdot,T_{ij}u, \nabla T_{ij}u) - f(\cdot,\cdot,Tu,\nabla Tu)\big| v\,dx\,dt$$

for all $u, v \in X$.

(b) If K satisfies $K \wedge K \subset K$ (respectively, $K \vee K \subset K$) then any solution of (1.1) is also a subsolution (respectively, supersolution).

In the following result, we show the existence of extremal (i.e., greatest and smallest) solutions of (1.1).

Corollary 3.1. *Let the hypotheses of Theorem* 3.1 *be satisfied, and assume, in addition, that K satisfies $K \wedge K \subset K$ (respectively, $K \vee K \subset K$). Let $\mathcal{S}$ denote the set of all solutions of* (1.1) *in $[\underline{u}, \bar{u}]$. If $\mathcal{S}$ is bounded in Y_0, then the variational inequality* (1.1) *possesses extremal solutions within $[\underline{u}, \bar{u}]$, i.e., there exists the greatest solution u^* and the smallest solution u_* of* (1.1) *in $[\underline{u}, \bar{u}]$ such that for any other solution u of* (1.1) *in $[\underline{u}, \bar{u}]$ one has $u_* \leqslant u \leqslant u^*$.*

Proof. Due to Remarks 3.2(a), (b) the set $\mathcal{S}$ is *directed*, i.e., $\mathcal{S}$ is both *upward directed* and *downward directed* (see, e.g., [5] for detailed definitions). Let us focus on the existence of the greatest element in $\mathcal{S}$, i.e., the existence of a greatest solution of (1.1) in $[\underline{u}, \bar{u}]$, since the proof of the smallest solution can be done analogously.

The space Y_0 is separable and thus $\mathcal{S} \subset Y_0$ is separable too. Let $Z = \{z_n\colon n \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{S}$. Since $\mathcal{S}$ is upward directed, we can construct an increasing sequence $\{u_n\} \subset \mathcal{S}$ in the following way. Let $u_1 := z_1$. Select $u_{n+1} \in \mathcal{S}$ such that $\max\{z_n, u_n\} \leqslant u_{n+1} \leqslant \bar{u}$. The existence of such $u_{n+1} \in \mathcal{S}$ follows from the upward directness of $\mathcal{S}$. By induction we get an increasing sequence $\{u_n\} \subset \mathcal{S}$, which is bounded in Y_0 by hypothesis.

Due to the monotonicity of $\{u_n\}$ and the compact embedding $Y_0 \subset L^p(Q)$ this sequence converges weakly in Y_0 and strongly in $L^p(Q)$ to $u = \sup_n u_n$. Next we are going to show that u belongs to $\mathcal{S}$. The weak convergence of $\{u_n\}$ in Y_0 implies that $\{u_n\}$ is weakly convergent in X_0. Since $u_n \in K$ and K is a closed and convex subset of X_0, it follows that K is weakly closed in X_0, and thus $u \in K$. Since $u_n \in D(L)$ and $D(L)$ is a closed and convex subset of Y_0, it is weakly closed, and thus also $u \in D(L)$, which shows $u \in D(L) \cap K$. By definition u_n are solutions of (1.1) within $[\underline{u}, \bar{u}]$, i.e., they satisfy

$$u_n \in D(L) \cap K\colon \quad \left\langle Lu_n + A(u_n) + F(u_n) - h, v - u_n \right\rangle \geqslant 0, \quad \forall v \in K. \tag{3.26}$$

Taking $v = u$ in (3.26) we obtain in view of the monotonicity of L the inequality

$$\left\langle A(u_n), u_n - u \right\rangle \leqslant \langle Lu, u - u_n \rangle + \left\langle F(u_n), u - u_n \right\rangle - \langle h, u - u_n \rangle. \tag{3.27}$$

The boundedness of (u_n) in Y_0, the weak convergence of (u_n) in Y_0 and its strong convergence in $L^p(Q)$ imply that the right-hand side of (3.27) tends to zero as $n \to \infty$, and thus we obtain

$$\limsup_n \left\langle A(u_n), u_n - u \right\rangle \leqslant 0, \tag{3.28}$$

which implies the strong convergence $u_n \to u$ in X_0, since A satisfies the (S_+)-property with respect to $D(L)$, see, e.g., [5, Theorem E.3.2]. The weak convergence of (u_n) in Y_0 and its strong convergence in X_0 allow to pass to the limit in (3.26), which proves that $u \in \mathcal{S}$.

By construction we have $\max\{z_1, \dots, z_n\} \leqslant u_{n+1} \leqslant u$ for all $n \in \mathbb{N}$, and thus u is an upper bound for Z, which implies $Z \subset [\underline{u}, u]$. Since the latter interval is closed in Y and Z is dense in $\mathcal{S}$ we get

$$\mathcal{S} \subset \bar{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u]. \tag{3.29}$$

From (3.29) it follows that the limit u, which is an element of $\mathcal{S}$, is also an upper bound of $\mathcal{S}$, and thus $u \in \mathcal{S}$ must be greatest element of $\mathcal{S}$. This completes the proof. □

Remark 3.3. The boundedness of $\mathcal{S}$ in X_0 follows from the monotonicity of the operator $L : D(L) \to X_0^*$ and the coercivity of $A + F$.

Some obvious cases where one has the boundedness of $\mathcal{S}$ in Y_0 are when K is bounded in Y_0 or $\mathcal{S} \subset \operatorname{int}(K)$.

4. Obstacle problem

As an example of the applicability of the general results of the preceding sections we consider an obstacle problem, where the convex set K is given by

$$K = \{u \in X_0 \colon u \leqslant \psi \text{ a.e. on } Q\},$$

with ψ a given function in Y such that $\psi(\cdot, 0) \geqslant 0$ on Ω, $\psi \geqslant 0$ on Γ, and $\psi_t + A\psi \geqslant 0$ in X_0^*, i.e.,

$$\langle \psi_t + A\psi, v\rangle \geqslant 0, \quad \forall v \in X_0 \cap L_+^p(Q).$$

The penalty function P can be chosen as

$$\langle Pu, v\rangle = \int_Q \bigl[(u - \psi)^+\bigr]^{p-1} v \, dx \, dt \tag{4.1}$$

for all $u, v \in X_0$. It is easy to verify that P satisfies (2.9). To check (2.9), for each $u \in D(L)$, we choose $w = (u - \psi)^+$. Then, $w \in X_0$ and (2.9)(i) is satisfied. In fact, since $(u - \psi)^+(\cdot, 0) = 0$, we have

$$\bigl\langle u_t - \psi_t, (u - \psi)^+\bigr\rangle = \frac{1}{2} \bigl\| (u - \psi)^+(\cdot, \tau)\bigr\|_{L^2(\Omega)}^2 \geqslant 0.$$

On the other hand, as above, one infers easily from (2.2) that

$$\bigl\langle Au - A\psi, (u - \psi)^+\bigr\rangle \geqslant 0.$$

These inequalities imply that

$$\bigl\langle u_t + Au, (u - \psi)^+\bigr\rangle \geqslant \bigl\langle \psi_t + A\psi, (u - \psi)^+\bigr\rangle \geqslant 0,$$

since $(u - \psi)^+ \in X_0 \cap L_+^p(Q)$. We have checked (i) of (2.9). To verify (2.9)(ii), we note that

$$\langle Pu, w\rangle = \int_Q \bigl[(u - \psi)^+\bigr]^p \, dx = \bigl\| (u - \psi)^+\bigr\|_{L^p(Q)}^p. \tag{4.2}$$

From (4.1) and Hölder's inequality, we have

$$\left|\langle Pu, v\rangle\right| \leqslant \left\|(u-\psi)^+\right\|_{L^p(Q)}^{p-1} \|v\|_{L^p(Q)}$$

for all $v \in X_0$. Hence,

$$\|Pu\|_{X_0^*} \leqslant c\left\|(u-\psi)^+\right\|_{L^p(Q)}^{p-1}, \quad \forall u \in X_0,$$

for some constant $c > 0$. This, together with (4.2), implies (2.9)(ii).

For our example of K, $\bar{u} \wedge K \subset K$ for every $\bar{u} \in Y$ and $\underline{u} \vee K \subset K$ if $\underline{u} \leqslant \psi$ on Q. Moreover, the conditions $K \wedge K \subset K$ (respectively, $K \vee K \subset K$) are satisfied which allows to apply Theorem 3.1. As far as the existence of extremal solutions are concerned Corollary 3.1 cannot be applied directly. However, if we assume the existence of special sub- and supersolutions we are able to prove the existence of extremal solutions by a penalty approach which has been recently applied by the authors for elliptic variational inequalities. The special sub- and supersolutions $\underline{u}$ and $\bar{u}$, respectively, are assumed to satisfy

Definition 4.1. $\underline{u} \in Y$,

$$\begin{aligned}
&\text{(i)} \quad F\underline{u} \in L^{p'}(Q),\\
&\text{(ii)} \quad \underline{u}(\cdot,0) \leqslant 0 \quad \text{a.e. in } \Omega, \qquad \underline{u} \leqslant 0 \quad \text{on } \Gamma, \quad \text{and} \quad \underline{u} \leqslant \psi,\\
&\text{(iii)} \quad \left\langle \underline{u}_t + A(\underline{u}) + F(\underline{u}), v\right\rangle \leqslant \langle h, v\rangle, \quad \forall v \in X_0 \cap L_+^p(\Omega).
\end{aligned} \tag{4.3}$$

Definition 4.2. $\bar{u} \in Y$,

$$\begin{aligned}
&\text{(i)} \quad F\bar{u} \in L^{p'}(Q),\\
&\text{(ii)} \quad \bar{u}(\cdot,0) \geqslant 0 \quad \text{a.e. in } \Omega, \qquad \bar{u} \geqslant 0 \quad \text{on } \Gamma,\\
&\text{(iii)} \quad \left\langle \bar{u}_t + A(\bar{u}) + F(\bar{u}), v\right\rangle \geqslant \langle h, v\rangle, \quad \forall v \in X_0 \cap L_+^p(\Omega).
\end{aligned} \tag{4.4}$$

Note, functions $\underline{u}$ and $\bar{u}$ satisfying Definitions 4.1 and 4.2, respectively, are, in particular, sub- and supersolutions according to Definitions 2.1 and 2.2. For more details in this direction, we refer the interested reader to [6].

Let us conclude this paper with some more remarks on the concepts and results presented above.

Remark 4.1. As a byproduct of the arguments used in Lemma 3.1, we have an alternative approach to an obstacle problem considered in [1], where A is the p-Laplacian,

$$\langle Au, v\rangle = \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx\, dt,$$

$f = 0$, and $K = \{u \in X_0\colon\ u \geqslant 0 \text{ a.e. on } Q\}$. A topological degree approach, together with an appropriate reformulation of the inequality, is used to show the existence of solutions. The approach here could be applied to the problem in [1, Theorem 5.2], with more general (nonzero) obstacles.

Variational inequalities similar to (1.1) (or (3.6)) were studied in [19] by Rothe's method (see also [15]). In addition to coercivity, smoothness conditions are usually required for the coefficients of the principal operators and lower order terms. These smoothness conditions are relaxed here, and if sub-supersolutions exist, we also have existence in noncoercive cases.

Finally, we are going to verify that the notion of sub-supersolutions of the parabolic variational inequality (1.1) introduced in Section 2 is consistent with the usual notion of (weak) sub-supersolutions of the corresponding nonlinear parabolic boundary value problem, i.e., the case of variational equalities, that is when $K = X_0$. We show that in this case, the definitions given above agree with those in [4] for sub- and supersolutions of equations [4, Definition 2.2]. It is enough to show that if $\underline{u}$ satisfies (2.6) (with $K = X_0$) then it satisfies the inequality

$$\langle \underline{u}_t, v\rangle + \langle A\underline{u}, v\rangle + \langle F\underline{u}, v\rangle \leqslant \langle h, v\rangle \tag{4.5}$$

for all $v \in X_0 \cap L^p_+(Q)$. Note that, since $\underline{u} \wedge w = \underline{u} - (\underline{u} - w)^+$, the inequality in (2.6)(iii) is equivalent to that in (4.5) for all $v \in M$, where

$$\begin{aligned} M &= \big\{(\underline{u} + w)^+ \colon\ w \in X_0\big\} \\ &= \big\{v^+ \colon\ v \in X \text{ and } v(t)|_{\partial\Omega} = \underline{u}(t)|_{\partial\Omega} \text{ for a.e. } t \in (0,\tau)\big\}. \end{aligned} \tag{4.6}$$

Since $\underline{u}(t) \leqslant 0$ for a.e. $t \in (0,\tau)$, we have $M \subset X_0 \cap L^p_+(Q)$. To show that (2.6)(iii) is equivalent to (4.5), one only needs to verify that

$$\bar{M}^{X_0} = X_0 \cap L^p_+(Q). \tag{4.7}$$

First, we observe that if $v \in X_0 \cap L^p_+(Q)$ and there is a compact subset κ of Ω (independent of t) such that

$$\operatorname{supp} v(t) \subset \kappa \quad \text{for a.e. } t \in (0,\tau), \tag{4.8}$$

then $v \in M$. In fact, one can choose $\varphi \in C^\infty_c(\Omega)$ such that $\varphi(x) \in [0,1]$, $\forall x \in \Omega$, and $\varphi(x) = 1$, $\forall x \in \kappa$ (cf. [14]). We define

$$\tilde{v}(x,t) = v(x,t) + \big[1 - \varphi(x)\big] \min\big\{\underline{u}(x,t), 0\big\} \quad \big((x,t) \in Q\big).$$

Since $\min\{\underline{u}, 0\} \in X$ and $1 - \varphi$ is smooth on $\bar{\Omega}$, we have $\tilde{v} \in X$. Moreover, because $1 - \varphi(x) \in [0,1]$, $\forall x$, and $1 - \varphi(x) = 0$, $\forall x \in \kappa$, and

$$v(x,t) = 0 \quad \text{for a.e. } x \in \Omega \setminus \kappa,\ \text{a.e. } t \in (0,\tau),$$

one has

$$\tilde{v}^+(x,t) = v(x,t) \quad \text{for a.e. } (x,t) \in Q.$$

On the other hand, for almost all $t \in (0,\tau)$, because

$$\tilde{v}(t) = (1 - \varphi) \min\big\{\underline{u}(t), 0\big\} = \min\big\{\underline{u}(t), 0\big\},$$

a.e. on $\Omega \setminus \operatorname{supp}\varphi$, we have

$$\tilde{v}(t)|_{\partial\Omega} = \min\big\{\underline{u}(t), 0\big\}\big|_{\partial\Omega} = \underline{u}(t)|_{\partial\Omega}.$$

This shows that $v = \tilde{v}^+ \in M$.

Now, let $v \in X_0 \cap L^p_+(Q)$. Then v can be approximated (in X_0) by polynomials of the form

$$v_n(x,t) = \sum_{i=0}^{m_n} a_{in}(x) t^i, \tag{4.9}$$

where $a_{in} \in W_0^{1,p}(\Omega)$ (cf., e.g., [25, Chapter 23]). Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ (with respect to the norm topology), one can choose the functions a_{in} above to be in $C_c^\infty(\Omega)$. Because the truncation operator

$$v \mapsto \max\{v, 0\}$$

is continuous from $W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$ and thus from X_0 to X_0 (cf. [5]), we have

$$w_n := \max\{v_n, 0\} \to \max\{v, 0\} = v \quad \text{in } X_0. \tag{4.10}$$

It is clear that $w_n \in X_0 \cap L^p_+(\Omega)$. Moreover, for almost all $t \in (0, \tau)$,

$$\operatorname{supp} w_n(t) \subset \operatorname{supp} v_n(t) \subset \bigcup_{i=0}^{m_n} \operatorname{supp} a_{in},$$

where $\bigcup_{i=0}^{m_n} \operatorname{supp} a_{in}$ is a compact subset of Ω. This means that w_n satisfies (4.8). From the above arguments, $w_n \in M$. This and (4.10) show (4.7).

References

[1] A. Addou, B. Mermri, Topological degree and application to a parabolic variational inequality problem, Internat. J. Math. Math. Sci. 25 (2001) 273–287.

[2] J. Berkovits, V. Mustonen, Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems, Rend. Mat. Appl. (7) 12 (1992) 597–621.

[3] J. Berkovits, V. Mustonen, Monotonicity methods for nonlinear evolution equations, Nonlinear Anal. 27 (1996) 1397–1405.

[4] S. Carl, On the existence of extremal weak solutions for a class of quasilinear parabolic problems, Differential Integral Equations 6 (1993) 1493–1505.

[5] S. Carl, S. Heikkilä, Nonlinear Differential Equations in Ordered Spaces, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[6] S. Carl, V.K. Le, Monotone penalty approximation of extremal solutions for quasilinear noncoercive variational inequalities, preprint.

[7] P. Charrier, G.M. Troianiello, On strong solutions to parabolic unilateral problems with obstacle dependent on time, J. Math. Anal. Appl. 65 (1978) 110–125.

[8] M. Chipot, J.-F. Rodriques, Comparison and stability of solutions to a class of quasilinear parabolic problems, Proc. Roy. Soc. Edinburgh Sect. A 110 (1988) 275–285.

[9] J. Deuel, Nichtlineare parabolische Randwertprobleme mit Unter- und Oberlösungen, ETH Diss. Nr. 5750, Zürich, 1976.

[10] J. Deuel, P. Hess, Nonlinear parabolic boundary value problems with upper and lower solutions, Israel J. Math. 29 (1978) 92–106.

[11] G. Duvaut, J.L. Lions, Les inequations en mechanique et en physique, Dunod, Paris, 1972.

[12] A. Friedman, Variational Principles and Free Boundary Value Problems, Wiley–Interscience, New York, 1983.

[13] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1983.

[14] L. Hörmander, Linear Partial Differential Operators, Springer, Berlin, 1976.

[15] J. Kacur, Method of Rothe and Evolution Equations, in: Teubner-Texte zur Mathematik, Teubner, Leipzig, 1985.

[16] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[17] J.L. Lions, Quelques méthodes de résolution des problèmes aus limites non linéaires, Dunod, Gauthier–Villars, Paris, 1969.

[18] V.K. Le, Subsolution–supersolution method in variational inequalities, Nonlinear Anal. 45 (2001) 775–800.

[19] H. Nagase, On an application of Rothe's method to nonlinear parabolic variational inequalities, Funkcial. Ekvac. 32 (1989) 273–299.

[20] N.S. Papageorgiou, F. Papalini, S. Vercillo, Minimal solutions of nonlinear parabolic problems with unilateral constraints, Houston J. Math. 23 (1997) 189–201.

[21] J.-P. Puel, Existence, comportement a l'infini et stabilité dans certains problemes quasilinéaires elliptiques et paraboliques d'ordre 2, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 3 (1976) 89–119.

[22] M. Rudd, K. Schmitt, Variational inequalities of elliptic and parabolic type, Taiwanese J. Math. 6 (2002) 287–322.

[23] G.M. Troianiello, Bilateral constraints and invariant sets for semilinear parabolic systems, Indiana Univ. Math. J. 32 (1983) 563–577.

[24] M.G. Vivaldi, Existence of strong solutions for nonlinear parabolic variational inequalities, Nonlinear Anal. 11 (1987) 285–295.

[25] E. Zeidler, Nonlinear Functional Analysis and Its Applications, vols. II A/B, Springer, Berlin, 1990.