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New Superconvergent Dispersion Relations for the Forward πN Crossing-Even Amplitude*

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A Liu-Okubo-type dispersion relation is derived for the crossing-even pion-nucleon forward elastic scattering amplitude $T^{(+)}$. Two subtractions are made, one at the physical scattering threshold and the other at a previously determined zero of $T^{(+)}$ on the imaginary axis of the complex ω (pion lab energy) plane. The dispersion relation is well satisfied over the whole allowed range of the Liu-Okubo parameter. Moreover, it is nearly saturated by low-energy scattering for a considerable range of the parameter. It should thus serve as an extremely sensitive test of the low-energy scattering data when such data become more accurately known.

IN a recent letter,¹ Liu and Okubo used a generalization of the method of Gilbert² to derive a new πN superconvergent relation. They considered the case of the forward crossing-odd amplitude $T^{(-)}$. In this paper, we use a generalization of their technique to derive several interesting results for the πN forward crossing-even amplitude $T^{(+)}$.

$T^{(+)}(\omega)$ has zeros at $\omega = \pm ia$, where ω is the pion laboratory energy^{3,4} and a^2 has the value $0.103 \mu^2$ (μ is the pion mass).^{5,6} $T^{(+)}(\omega)$ has nucleon poles at $\omega = \pm \omega_0 = \pm \mu^2/2M$ (M is the nucleon mass) and is assumed to have a high-energy behavior

$$T^{(+)}(\omega) \sim \omega \quad (1)$$

corresponding to constant infinite-energy total cross sections, $\sigma_{\pi^{\pm}p}(\infty)$. We normalize $T^{(+)}(\omega)$ so that the optical theorem has the form,

$$\begin{aligned} \text{Im}T^{(+)}(\omega) &= (\omega^2 - \mu^2)^{1/2} \sigma(\omega), \\ \sigma(\omega) &= \frac{1}{2} [\sigma_{\pi^+p}(\omega) + \sigma_{\pi^-p}(\omega)]. \end{aligned} \quad (2)$$

The value of $T^{(+)}(\mu)$ is subject to large errors⁷:

$$T^{(+)}(\mu) = -0.010 \pm 0.040 \mu^{-1}.$$

$$\frac{2\omega_0 f^2}{(\omega_0^2 + a^2)(\mu^2 - \omega_0^2)^\beta} = -\frac{2}{\pi} \int_{\mu}^{\infty} \frac{\omega d\omega \{ \cos\pi\beta \text{Im}T^{(+)}(\omega) + \sin\pi\beta \text{Re}[T^{(+)}(\omega) - T^{(+)}(\mu)] \}}{(\omega^2 + a^2)(\omega^2 - \mu^2)^\beta} - \frac{T^{(+)}(\mu)}{(\mu^2 + a^2)^\beta}. \quad (7)$$

In the limit, $\beta = \frac{1}{2}$, we have an extra contribution from the contour integration at infinity which is simply $-\sigma(\infty)$. Therefore we derive

$$\begin{aligned} \sigma(\infty) &= \frac{T^{(+)}(\mu)}{(\mu^2 + a^2)^{1/2}} + \frac{2\omega_0 f^2}{(\omega_0^2 + a^2)(\mu^2 - \omega_0^2)^{1/2}} \\ &+ \frac{2}{\pi} \int_{\mu}^{\infty} \frac{\omega d\omega \text{Re}[T^{(+)}(\omega) - T(\mu)]}{(\omega^2 + a^2)(\omega^2 - \mu^2)^{1/2}}. \end{aligned} \quad (8)$$

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¹ Y. C. Liu and S. Okubo, *Phys. Rev. Letters* **19**, 190 (1967).
² W. Gilbert, *Phys. Rev.* **108**, 1078 (1957).
³ M. Sugawara and A. Tubis, *Phys. Rev.* **130**, 2127 (1963).
⁴ M. Sugawara and A. Tubis, *Phys. Rev.* **138**, B242 (1965).
⁵ Natural units $\hbar = c = 1$ are used throughout this work.
⁶ The uncertainty in a^2 is estimated to be about 3% mainly because of the experimental uncertainty in $T^{(+)}(\mu)$.
⁷ J. Hamilton [*Phys. Letters* **20**, 687 (1966)] gives for the s -wave scattering length combination $a_1 + 2a_3 = (-0.002 \pm 0.008) \mu^{-1}$ so that $T^{(+)}(\mu) = \frac{2}{3} \pi (1 + \mu/m)(a_1 + 2a_3) = (-0.010 \pm 0.040) \mu^{-1}$; $\sigma(\mu) = -\frac{2}{3} \pi (a_1^2 + 2a_3^2) = 3.4 \pm 0.2$ mb.

Following Liu and Okubo,¹ we consider

$$t(\omega) = \frac{\omega [T^{(+)}(\omega) - T^{(+)}(\mu)] e^{\pi i \beta}}{(\omega^2 + a^2)(\omega^2 - \mu^2)^\beta}, \quad (3)$$

where the function $(\omega^2 - \mu^2)^\beta$ is defined as in Ref. 1. For large ω ,

$$\text{Im}t(\omega) \sim \omega^{-2\beta}, \quad (4)$$

so for $\beta > \frac{1}{2}$, $t(\omega)$ is superconvergent,⁸

$$\int_{-\infty}^{\infty} \text{Im}t(\omega) d\omega = -\pi T^{(+)}(\mu) / (\mu^2 + a^2)^\beta. \quad (5)$$

The right side of (5) comes from the poles of $t(\omega)$ at $\pm ia$.

Near $\omega = \mu$, $T^{(+)}$ has the expansion

$$T^{(+)}(\omega) - T^{(+)}(\mu) = C(\omega^2 - \mu^2) + i\sigma(\mu)(\omega^2 - \mu^2)^{1/2}, \quad (6)$$

where C is a real constant related to the scattering lengths and effective ranges. Hence $t(\omega)$ has no pole at the physical scattering threshold if $\beta < \frac{3}{2}$. The only δ -function contributions to (5) are from the nucleon poles at $\omega = \pm \omega_0$. Thus for $\frac{1}{2} < \beta < \frac{3}{2}$,

⁸ V. de Alfaro *et al.*, *Phys. Letters* **21**, 576 (1966).

obtain,

$$\frac{-\sigma(\mu)}{\mu^2+a^2} = \frac{-T^{(+)}(\mu)}{(\mu^2+a^2)^{3/2}} - \frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^{3/2}} + \frac{2}{\pi} \int_{\mu}^{\infty} \frac{\omega d\omega \operatorname{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{3/2}}. \quad (9)$$

We have performed the numerical evaluation of (7)–(9) by taking the values of Hohler *et al.*⁹ and doing a careful numerical integration which emphasizes the low-energy region. Above 30 BeV we took a Regge form for $T^{(+)}(\omega)$ containing the P (Pomeranchon) and P' (f^0 meson) terms,

$$T^{(+)}(\omega) = -\gamma_P \frac{e^{-i\pi\alpha_P/2}}{\sin(\pi\alpha_P/2)} \omega^{\alpha_P} - \gamma_{P'} \frac{e^{-i\pi\alpha_{P'}/2}}{\sin(\pi\alpha_{P'}/2)} \omega^{\alpha_{P'}}, \quad (10)$$

with the parameter values¹⁰

$$\begin{aligned} \gamma_P &= 1.11\mu^{-2}, & \alpha_P &= 1; \\ \gamma_{P'} &= 1.82\mu^{-1}, & \alpha_{P'} &= 0.39. \end{aligned}$$

The high-energy integration is then

$$\begin{aligned} & \int_{\Lambda}^{\infty} \frac{\omega d\omega \operatorname{Im}[T^{(+)}(\omega)-T^{(+)}(\mu)] e^{\pi i\beta}}{(\omega^2+a^2)(\omega^2-\mu^2)^{\beta}} \\ &= -\gamma_P \frac{\sin[(2\beta-\alpha_P)\pi/2]}{\sin(\pi\alpha_P/2)\Lambda^{2\beta-\alpha_P}(2\beta-\alpha_P)} \\ & \quad - \frac{\gamma_{P'} \sin[(2\beta-\alpha_{P'})\pi/2]}{\sin(\pi\alpha_{P'}/2)\Lambda^{2\beta-\alpha_{P'}}(2\beta-\alpha_{P'})}. \quad (11) \end{aligned}$$

Because of the factors $\Lambda^{2\beta-\alpha_P}$ and $\Lambda^{2\beta-\alpha_{P'}}$ these contributions rapidly become negligible as β increases from $\frac{1}{2}$.

The results from Eq. (7) are shown in Table I, and it can be seen that the right- and left-hand sides differ by at most 1%. The convergence is very rapid for $\beta > 1$, but for $\beta < 1$ the high-energy contribution is quite important. For β close to its limiting value of $\frac{3}{2}$, the integral is almost saturated by the low-energy region $\omega < 5$ BeV.

The evaluation of the terms in Eq. (8) yields

$$\frac{T^{(+)}(\mu)}{(\mu^2+a^2)^{1/2}} = 0 \text{ mb (Ref. 9)}, \quad (12)$$

$$\frac{2\omega_0 f^2}{(\omega_0^2+a^2)(\mu^2-\omega_0^2)^{1/2}} = 28.2 \text{ mb}, \quad (13)$$

TABLE I. The comparison of the left- and right-hand sides of Eq. (7) for various values of the cutoff Λ of the integral.

β	Right-hand side (mb $\times \mu^{1-2\beta}$)				∞
	Left-hand side (mb $\times \mu^{1-2\beta}$)	$\Lambda=5$ BeV	15 BeV	30 BeV	
0.501	28.19	-1.27	2.00	3.37	27.65
0.6	28.12	14.22	18.06	19.76	27.84
0.7	28.22	21.81	24.40	25.40	28.00
0.8	28.24	25.38	26.80	27.29	28.10
0.9	28.25	27.03	27.73	27.94	28.18
1.0	28.27	27.83	28.14	28.23	28.30
1.1	28.29	28.29	28.41	28.45	28.47
1.2	28.30	28.61	28.65	28.67	28.67
1.3	28.32	28.87	28.88	28.89	28.89
1.4	28.33	28.77	28.78	28.78	28.78
1.499	28.35	28.67	28.67	28.67	28.67

$$\frac{2}{\pi} \int_{\mu}^{30 \text{ BeV}} \frac{\omega d\omega \operatorname{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{1/2}} = -3.1 \text{ mb}, \quad (14)$$

$$\frac{2}{\pi} \int_{30 \text{ BeV}}^{\infty} \frac{\omega d\omega \operatorname{Re}[T(\omega)-T(\mu)]}{(\omega^2+a^2)(\omega^2-\mu^2)^{1/2}} = -2.3 \text{ mb}, \quad (15)$$

or

$$\sigma(\infty) = 22.8 \text{ mb}, \quad (16)$$

compared with the experimental value of 22.1 ± 0.9 mb.¹¹ Note that in this case the Pomeranchon term in (10) does not contribute because it is pure imaginary.

When we evaluate Eq. (9) we find

$$\sigma(\mu) = 2.2 \pm 0.76 \text{ mb} \quad (17)$$

compared with the "experimental" values of 3.8 ± 0.2 mb⁷ and 3.4 ± 0.2 mb.⁹ In (17), only the uncertainty in $T^{(+)}(\mu)$ in the first term of (9) is accounted for.

It can easily be shown that all the sum rules become identities if $\operatorname{Re}T^{(+)}(\omega)$ is calculated from the ordinary dispersion relations so we should not be surprised by the good results. Therefore our results are most useful when accurate experimental values of $\operatorname{Re}T^{(+)}(\omega)$ become available. Then they will provide a good test of low-energy values of $\operatorname{Re}T^{(+)}(\omega)$ because of the rapid convergence of our integrals in the case $1 < \beta < \frac{3}{2}$.

The formalism in this paper is very convenient for deriving finite-energy sum rules¹² from which Regge parameters may be estimated. This application will be discussed in a separate paper.

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⁹ G. Hohler, G. Ebel, and J. Gieseke, *Z. Physik* **180**, 430 (1964). These authors use $a_1 = (0.192 \pm 0.004)\mu^{-1}$, $a_2 = (-0.096 \pm 0.002)\mu^{-1}$ so that $T^{(+)}(\mu) = (0.00 \pm 0.04)\mu^{-1}$; $\sigma(\mu) = 3.8 \pm 0.2$ mb.

¹⁰ These values are taken from [Y.-C. Liu and S. Okubo, *Phys. Rev.* **168**, 1712 (1968)] a report which was received while this work was in progress. The treatment of the $T^{(+)}(\omega)$ amplitude in their paper differs somewhat from ours.

¹¹ K. J. Foley *et al.*, *Phys. Rev. Letters*, **19**, 193 (1967).

¹² K. Igi, *Phys. Rev. Letters* **9**, 76 (1962); K. Igi, *Phys. Rev.* **130**, 820 (1963); K. Igi and S. Matsuda, *Phys. Rev. Letters* **18**, 625 (1967); A. Logunov, L. D. Soloviev, and A. Tavkelidze, *Phys. Letters* **24B**, 181 (1967); D. Horn and C. Schmid, California Institute of Technology Report No., CALT-68-127, 1967 (unpublished); M. G. Olsson, *Phys. Rev. Letters* **19**, 550 (1967).