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Modified Phase Representation and Effects of Inelasticity in N/D Calculation of p-Wave Pion-Pion Scattering*

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An N/D formalism based on a modified phase representation is used to study the effects of inelasticity on the p-wave pion-pion amplitude. The effects of high-energy inelasticity are introduced in terms of the assumed behavior of the high-energy phase (not phase shift) of the partial-wave amplitude. Using a ρ-exchange input force with the experimental ρ mass and a ρ width of about 100 MeV, and the assumption that the average phase is $\frac{1}{2}\pi$, for total c.m. energies greater than about $8M_\pi$, we find that there is no appreciable reduction in the width of the calculated p-wave resonance. We also investigate the effects of low-energy inelastic channels that may contribute through the inelasticity parameter η for $E\leqslant E_i$, where E_i is the energy above which the phase assumption is made. None of the forms for η that were used resulted in an output width less than about 280 MeV.

I. INTRODUCTION

DIVERGENCE problems generally plague N/D calculations,[1] in which use is made of forces corresponding to the exchange of spin ≥ 1 particles. Cutoffs or Reggeized exchange forces[2] are commonly used to remedy these difficulties. In addition, little is known about the effects of high- (and sometimes low-) energy inelasticity; often one simply assumes elastic unitarity over an extremely large energy range. In this paper, we present a formalism that might, in some cases, provide a useful alternative approach to both problems.

The formalism is developed in Sec. II. Using the phase representation[3,4] of the partial-wave amplitude A_l, a modified partial-wave amplitude a_l is formed by dividing A_l by a factor that (a) has the same phase as A_l for c.m. energy $E>E_i$ and (b) is real for $E<E_i$. Thus the modified amplitude has a finite right-hand cut and has the same cuts as A_l for $E<E_i$. High-energy ($E>E_i$) inelasticity is estimated through an assumption about the average phase of A_l ($E>E_i$). The formalism is most useful if E_i is assumed to be somewhere in the region of the first or second inelastic threshold. This provides a short right-hand cut—and hence a small energy range over which the "effective generalized potential" must be well approximated. Low-energy ($E<E_i$) inelasticity may be introduced into the formalism as usual via the inelasticity parameter η.

In Sec. III, the formalism is applied to the p-wave pion-pion amplitude A_1^1, using a simple ρ exchange as an input force. The simplifying assumption that the average phase of A_1^1 ($E>E_i\approx 8M_\pi c^2$) is $\frac{1}{2}\pi$ is made and is found to lead to output p-wave resonances that are quite similar (characteristically wide and asymmetric) to those given by previous N/D calculations[2,5] in which only elastic unitarity is assumed. In Sec. IV, the effects of low-energy inelasticity are investigated. Concluding remarks are contained in Sec. V.

II. MODIFIED PARTIAL-WAVE AMPLITUDE USING THE PHASE REPRESENTATION

We consider the partial-wave amplitude for the elastic scattering of two spinless, equal-mass particles, $A_l(\nu(\equiv q^2))$, where q is the magnitude of the c.m. three-momentum of one of the particles.[6] Following Ref. 4, a real phase $\Phi_l(\nu)$ of $A_l(\nu)$ is defined for real ν as follows: (i) $A_l(\nu+i\epsilon)=\pm|A_l(\nu+i\epsilon)|\exp[i\Phi_l(\nu)]$; (ii) $\Phi_l(\nu)=0$ on the real ν axis where no cuts in $A_l(\nu)$ occur; and (iii) $\Phi_l(\nu)$ is continuous.

The condition (iii) can be satisfied when $A_l(\nu)$ passes through zero and changes sign by changing the sign in (i). If $A_l(\nu+i\epsilon)$ is continuous, the sign in (i) is uniquely given for all real ν once it is given for a single (real) value of ν.

In Ref. 4, it was shown that if (a) $A_l(z)$ is analytic everywhere in the complex z plane except for cuts on the real axis and a finite number of poles, (b) $A_l(z)$ is real analytic in the sense that $A_l^*(z)=A_l(z^*)$, (c) $A_l(z)$ is bounded at $|z|=\infty$ by a finite polynomial in z, and (d) $\Phi_l(\nu)$ has finite limits, $\Phi_l(\pm\infty)$, as $\nu\to\pm\infty$, then $A_l(z)$ may be represented as

$$A_l(z)=\frac{P_1(z)}{P_2(z)}\exp\left(\frac{z}{\pi}\int\frac{\Phi_l(\nu')d\nu'}{(\nu'-z)\nu'}\right), \qquad (2.1)$$

where $P_1(z)$ and $P_2(z)$ are finite polynomials in z [accounting for zeros and poles, respectively, of $A_l(z)$] and the integral is along the cuts of $A_l(z)$.

Assuming that (a)–(d) are satisfied by the partial-wave amplitude $A_l(z)$ and by its defined phase $\Phi_l(\nu)$,

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] See, for example, D. Wong, Phys. Rev. **126**, 1220 (1962); M. Bander and G. L. Shaw, *ibid.* **135**, B267 (1964).

[3] N. Muskelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Gronigen, The Netherlands, 1953), p. 126 ff.; R. Omnès, Nuovo Cimento **8**, 316 (1958); G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962); Y. S. Jin and S. W. MacDowell, Phys. Rev. **138**, B1279 (1965).

[4] M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).

[5] See, for example, the single-channel calculation of J. R. Fulco, G. L. Shaw, and D. Wong, Phys. Rev. **137**, B1242 (1965), and the elastic-unitarity calculation of P. W. Coulter and G. L. Shaw, *ibid.* **138**, B1273 (1965).

[6] The system of natural units $\hbar=c=1$ is used. M is the mass of the particle.

we rewrite (2.1) as follows:

$$A_l(z)=\left[\frac{P_1(z)}{P_2(z)}\exp\left(\frac{z}{\pi}\int_{\text{left-hand cut}}\frac{\Phi_l(\nu')d\nu'}{(\nu'-z)\nu'}+\frac{z}{\pi}\int_0^{\nu_i}\frac{\Phi_l(\nu')d\nu'}{(\nu'-z)\nu'}\right)\right]\exp\left(\frac{z}{\pi}\int_{\nu_i}^{\infty}\frac{\Phi_l(\nu')}{(\nu'-z)\nu'}d\nu'\right)$$
$$\equiv a_l(z)\exp[\Delta(z)], \qquad (2.2)$$

where

$$\Delta_l(z)\equiv\frac{z}{\pi}\int_{\nu_i}^{\infty}\frac{\Phi_l(\nu')}{(\nu'-z)\nu'}d\nu'. \qquad (2.3)$$

$a_l(z)$ as defined by (2.2) and (2.3) will be referred to as the "modified partial-wave amplitude." For real $\nu>\nu_i$,

$$a_l(\nu+i\epsilon)=\pm|A_l(\nu+i\epsilon)|\exp[-\text{Re}\Delta_l(\nu)], \qquad (2.4)$$

where we have used (2.2), (2.3), and definition (i). Thus $a_l(\nu)$ is purely real for real $\nu>\nu_i$ and has a finite right-hand cut. For ν real and $<\nu_i$, $\exp[\Delta(\nu)]$ is purely real and

$$\text{Im}a_l(\nu+i\epsilon)=\exp[-\Delta_l(\nu)]\,\text{Im}A_l(\nu+i\epsilon). \qquad (2.5)$$

Thus the cuts of $a_l(z)$ coincide with those of $A_l(z)$ for z real and $<\nu_i$.

To examine the behavior of $\text{Im}a_l(\nu)$ near $\nu=\nu_i$, we note that

$$\exp[-\Delta_l(\nu)]=\bar{c}_l(\nu)[(\nu_i-\nu)/\nu_i]^{\Phi_l(\nu)/\pi}, \qquad (2.6)$$

where

$$\bar{c}_l(\nu)\equiv\exp\left(\frac{-\nu}{\pi}\text{P.V.}\int_{\nu_i}^{\infty}\frac{\Phi_l(\nu')-\Phi_l(\nu)}{(\nu'-\nu)\nu'}d\nu'\right) \qquad (2.7)$$

(where P.V. means the principal value); $\bar{c}_l(\nu_i)$ is finite, since $\Phi_l(+\infty)$ is finite and $\Phi_l(\nu)$ is continuous. Thus for $\nu\lesssim\nu_i$,

$$\exp[-\Delta_l(\nu)]\approx\bar{c}_l(\nu_i)[(\nu_i-\nu)/\nu_i]^{\Phi_l(\nu_i)/\pi} \qquad (2.8)$$

and

$$\text{Im}a_l(\nu)\approx\text{Im}A_l(\nu)\bar{c}_l(\nu_i)[(\nu_i-\nu)/\nu_i]^{\Phi_l(\nu_i)/\pi}. \qquad (2.9)$$

$a_l(\nu)$ has the same threshold behavior as $A_l(\nu)$, since $\exp[-\Delta(\nu)]\rightarrow 1$ as $\nu\rightarrow 0^+$.

Assuming that $\text{Im}A_l$(left-hand cut), $\text{Im}[A_l(0\leq\nu\leq\nu_i)]^{-1}$, and $\Phi_l(\nu>\nu_i)$ are given, the amplitude $a_l(\nu)$ can be conveniently used instead of $A_l(\nu)$ in an N/D calculation.

The motivation for considering such an amplitude is rather clear. With a reasonable approximation for the average value of the phase of the partial-wave amplitude above, say, the first or second inelastic threshold, the amplitude $a_l(\nu)$ will have a short right-hand cut. In this case the effective generalized potential—essentially the integral over the left-hand cut—would need to be approximated only over a small energy range above the elastic threshold. Since our present knowledge of the generalized potential is least uncertain at low energies, this could be an important

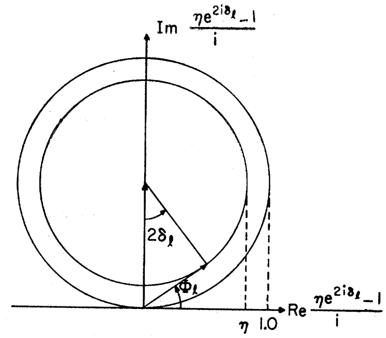

FIG. 1. Phase diagram for the partial-wave amplitude.

practical advantage of working with $a_l(\nu)$. To illustrate these points we note the following form for $\exp[-\Delta(\nu)]$:

$$\exp[-\Delta_l(\nu)]=[(\nu_i-\nu)/\nu_i]^{\frac{1}{2}(1+\langle\beta_l(\nu)\rangle_{\text{av}})}, \qquad (2.10)$$

where

$$\langle\beta_l(\nu)\rangle_{\text{av}}\equiv\frac{2}{\pi}\int_{\nu_i}^{\infty}\frac{\Phi_l(\nu')-\frac{1}{2}\pi}{(\nu'-\nu)\nu'}d\nu'\Big/\int_{\nu_i}^{\infty}\frac{d\nu'}{(\nu'-\nu)\nu'}. \qquad (2.11)$$

$\langle\beta_l(\nu)\rangle_{\text{av}}$ is a measure of the average deviation of the phase of $A_l(\nu)$ from $\frac{1}{2}\pi$ for $\nu>\nu_i$. Using the definition of $\Phi_l(\nu)$ and the unitary form for $A_l(\nu)$

$$A_l(\nu)\equiv\frac{\eta e^{2i\delta_l}-1}{2i}\left(\frac{\nu+M^2}{\nu}\right)^{1/2}, \qquad (2.12)$$

where δ_l is the real part of the complex phase shift $\tilde{\delta}_l$ and

$$\eta\equiv\exp[-2\,\text{Im}\tilde{\delta}_l], \qquad (2.13)$$

we have that

$$\Phi_l(\nu\geq 0)=\text{phase}(i-\eta e^{i(\pi/2+2\delta_l)}). \qquad (2.14)$$

With the convention $\delta_l(\nu=0)=0$ $[=\Phi_l(\nu=0)]$ and $\eta(\nu=0)=1$ it can be seen readily from Fig. 1 that for all ν such that

$$\eta(\nu)<1 \qquad (2.15)$$

or

$$\delta_l(\nu)\neq m\pi,\quad m=0,1,2,\cdots$$

when

$$\eta(\nu)=1\quad\text{and}\quad\nu\neq 0, \qquad (2.16)$$

$\Phi_l(\nu)$ is continuous and[7]

$$0<\Phi_l(\nu)<\pi. \qquad (2.17)$$

Thus in the case where there are no zeros of $A_l(\nu)$ for $\nu>0$ [i.e., condition (2.15) or (2.16) is satisfied for all $\nu>0$],

$$|\langle\beta_l(\nu)\rangle_{\text{av}}|<1 \qquad (2.18)$$

for ν real and in the interval $0<\nu<\nu_i$.

Assuming that $|\langle\beta_l(\nu)\rangle_{\text{av}}|<1$ (so that $\exp[\Delta(\nu)]$ does not have a pole at $\nu=\nu_i$), we can apply the N/D

[7] If $\eta(\nu_c)=1$ and $\delta_l(\nu_c)=\pi$ (with $d\delta_l/d\nu|_{\nu=\nu_c}\neq 0$), the sign in the definition (i) would have to be changed for $\nu>\nu_c$; the total phase Φ_l would then be continuous at ν_c but for $\nu>\nu_c$ would be greater than π.

method[1] to $a_l(\nu)$ as follows:

$$a_l(\nu)=N_l(\nu)/D_l(\nu)\,, \tag{2.19}$$

$$\mathrm{Im}N_l(\nu)=0\,,\quad -M^2<\nu<0\,,\quad \nu>\nu_i\,, \tag{2.20}$$

$$\mathrm{Im}N_l(\nu<-M^2)=\mathrm{Im}A_l(\nu)e^{-\Delta_l(\nu)}D_l(\nu)\,, \tag{2.21}$$

$$\mathrm{Im}D_l(0\leq\nu<\nu_i)=\mathrm{Im}[A_l(\nu)]^{-1}e^{\Delta_l(\nu)}N_l(\nu)\equiv\rho(\nu)N_l(\nu)\,, \tag{2.22}$$

$$N_l(\nu)=\frac{\nu^l}{\pi}\int_{-\infty}^{-M^2}\frac{\mathrm{Im}A_l(\nu')e^{-\Delta_l(\nu')}D_l(\nu')}{(\nu'-\nu)\nu'^l}d\nu'\,, \tag{2.23}$$

$$D_l(\nu)=1+\frac{\nu-\nu_0}{\pi}\int_0^{\nu_i}\frac{N_l(\nu')\,\mathrm{Im}A_l^{-1}(\nu')e^{\Delta_l(\nu')}}{(\nu'-\nu)(\nu'-\nu_0)}d\nu'\,. \tag{2.24}$$

In the above, we have assumed that the amplitude A_l has no CDD [8] poles. Inserting D_l into N_l and interchanging the order of integration, N_l becomes[9]

$$N_l(\nu)\equiv\bar{B}_l(\nu)+\frac{\nu^l}{\pi}\int_0^{\nu_i}\frac{\rho(\nu')N_l(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}\times\left(\bar{B}_l(\nu)\frac{\nu-\nu_0}{\nu^l}-\bar{B}_l(\nu')\frac{\nu'-\nu_0}{\nu'^l}\right), \tag{2.25}$$

where

$$\bar{B}_l(\nu)\equiv\frac{\nu^l}{\pi}\int_{-\infty}^{-M^2}\frac{\mathrm{Im}A_l(\nu')e^{-\Delta(\nu')}}{(\nu'-\nu)\nu'^l}d\nu'\,. \tag{2.26}$$

Thus, if $\Phi_l(\nu>\nu_i)$ can be well approximated when ν_i is near the first or second inelastic threshold, the effective generalized potential $\bar{B}_l(\nu)$ is only required over a small range $(0\leq\nu\leq\nu_i)$. In the usual cutoff procedure, where the N/D method is applied directly to A_l, the corresponding left-hand cut contribution must often be approximated over a much larger energy range. For example, in the N/D calculation of the p-wave pion-pion amplitude assuming elastic unitarity, a cutoff of $\nu\approx72M_\pi^2$ is used.[5] In the formalism presented here, the right-hand cut can be reduced to about $\frac{1}{9}$ this value if the phase (or the average value of the phase) of the p-wave pion-pion amplitude can be approximated for energies above the six-pion inelastic threshold $(\nu=8M_\pi^2)$. Also, the physical meaning of ν_i—the value of the momentum squared above which a reasonable assumption concerning the phase may be made—seems much clearer than that of the usual cutoff parameter.

It can be shown that the phase of A_l, as calculated from the expressions (2.2) and (2.19)–(2.24), is continuous at $\nu=\nu_i$ (see Appendix).

III. APPLICATION TO THE p-WAVE PION-PION AMPLITUDE

In the isospin-one, $l=1$ amplitude for pion-pion scattering $A_1^1(\nu)$, we assume that there are no zeros for $\nu>0$, so that $|\langle\beta(\nu')\rangle_{\mathrm{av}}|<1$ for all ν'. We also assume that A_1^1 has no CDD poles. Equations (2.19)–(2.26) can then be used.

We further make the simplifying assumption that the scattering amplitude is, on the average, purely imaginary for $\nu>\nu_i$. That is, we assume that $\langle\beta(\nu)\rangle_{\mathrm{av}}\equiv0$; hence

$$e^{-\Delta(\nu)}=[(\nu_i-\nu)/\nu_i]^{1/2}\,. \tag{3.1}$$

The motivations for this approximation, aside from its simplicity, are as follows:

(a) With the increase in the number of energetically available inelastic channels as energy increases, it is at least plausible that $\eta\to0$ for large energies. However, η need not approach zero for the assumption $\langle\beta(\nu)\rangle_{\mathrm{av}}\approx0$ to be valid.

(b) If the generalized Levinson theorem[10] is valid for $A_1^1(\nu)$, then $\delta_1^1(\infty)=m\pi$, where m is an integer. Then $\Phi_1^1(\nu\to\infty)=\frac{1}{2}\pi$ if $\eta(+\infty)<1$ and, at least asymptotically, our assumption is justified.

(c) $\langle\beta(\nu)\rangle_{\mathrm{av}}=0$ would follow from the assumption that $A_1^1(\nu)$ is purely imaginary for $\nu>\nu_i$; it is interesting to observe what effects this assumption of "maximum inelasticity" will have.

It is assumed that the left-hand cut contribution is dominated by the ρ exchange and that the imaginary part of $A_i(\nu<-M_\pi^2)$ is given by[11]

$$\mathrm{Im}A_1^1(\nu<-M_\pi^2)=3\Gamma\left(\frac{M_\rho^2+8\nu+4M_\pi^2}{2\nu}\right)\left(1+\frac{M_\rho^2}{2\nu}\right)\tfrac{1}{2}\pi\,. \tag{3.2}$$

The values $M_\rho=760$ MeV and $\Gamma=0.15$ (corresponding to about a 100-MeV width) are used.[12]

Elastic unitarity for $\nu\leq\nu_i$ and $\langle\beta(\nu)\rangle_{\mathrm{av}}=0$ gives

$$\mathrm{Im}[a_1^1(0\leq\nu<\nu_i)]^{-1}=-\left(\frac{\nu}{\nu+M_\pi^2}\right)^{1/2}\left(\frac{\nu_i}{\nu_i-\nu}\right)^{1/2}\equiv\bar{\rho}(\nu)\,. \tag{3.3}$$

For ν_i in the range 13–37M_π^2, the quantity $\bar{B}_1^1(0\leq\nu\leq\nu_i)$ [as obtained from (2.26) using (3.2) and $\langle\beta(\nu)\rangle_{\mathrm{av}}=0$] can be well approximated by one term of the form $f\nu/(\nu+b)$. [In fact, $\bar{B}_i^1(0\leq\nu\leq\nu_i)\sim\nu$.] In this case, N_1^1 can be written[13]

$$N_1^1(\nu)=c\bar{B}_1^1(\nu)\,, \tag{3.4}$$

[8] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[9] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[10] R. Warnock, Phys. Rev. **131**, 1320 (1961); J. M. Charap, Nuovo Cimento **36**, 414 (1965); C. R. Hagen, *ibid.* **43**, 597 (1966).

[11] This left-hand cut is obtained by assuming a Breit-Wigner (ρ)-resonance form for A_1^1 in the crossed channels and using the narrow-width approximation. (The s and $l>1$ partial-wave amplitudes in the crossed channels are neglected.) Expression (3.2) can also be obtained from the $I=J=1$ projection of the (first Born approximation) Feynman amplitude for ρ exchange.

[12] For the experimental values of the ρ mass and width, see A. H. Rosenfeld, Rev. Mod. Phys. **40**, 77 (1967). We use a value of about 100 MeV for the input width for purposes of comparison with previous calculations. Some results for an input width of about 150 MeV are also given.

[13] A. W. Martin, Phys. Rev. **135**, B967 (1964).

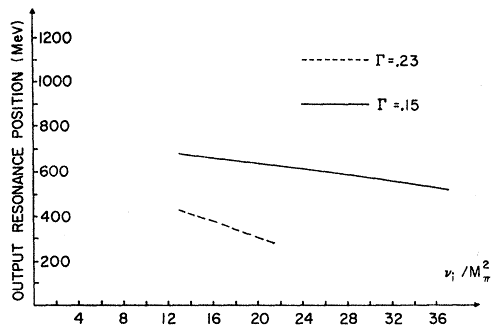

FIG. 2. Variation of the output-resonance position with ν_i.

where c is independent of ν. Thus[14]

$$\frac{\mathrm{Re}D_1{}^1(\nu)}{c}=1+\frac{\nu}{\bar{B}_i(\nu)}\mathrm{P.V.}\int_0^{\nu_i}\frac{\bar{\rho}(\nu')\bar{B}_1{}^1(\nu')^2}{(\nu'-\nu)\nu'}d\nu' \quad (3.5)$$

and

$$A_1{}^1(0\leq\nu<\nu_i)=\frac{\bar{B}_1{}^1(\nu)[\nu_i/(\nu_i-\nu)]^{1/2}}{\mathrm{Re}D_1{}^1(\nu)/c+i\bar{\rho}(\nu)\bar{B}_1{}^1(\nu)}. \quad (3.6)$$

Figure 2 shows the variation of the output-resonance position with ν_i. For ν_i in the range 13–37$M_\pi{}^2$, output resonances are obtained at c.m. energies of from 680 to 530 MeV[15]; the corresponding output widths range from 500 to 285 MeV. Figure 3 shows a typical output cross section for $\nu_i=17M_\pi{}^2$.

For $\nu_i<13M_\pi{}^2$, $\mathrm{Re}D(0\leq\nu<\nu_i)$ does not develop a zero. Hence there is no output resonance for $\nu_i<13M_\pi{}^2$ (and $\Gamma=0.15$). For a larger value of Γ (a stronger attractive force) there may be resonances for $\nu_i<13M_\pi{}^2$.

The solution appears to be quite similar to previous p-wave pion-pion N/D calculations that use only elastic unitarity.[2,5] If, for example, in the present formalism ν_i is fixed at $17M_\pi{}^2$, and the input-width parameter Γ is reduced to $\simeq0.13$,[16] then an output resonance is obtained at 760 MeV with a width of about 610 MeV (Fig. 4). Other values of ν_i in the range 13–37$M_\pi{}^2$ were found to give similarly large widths when the strength of the input force was decreased to produce a peak in the cross section at 760 MeV.

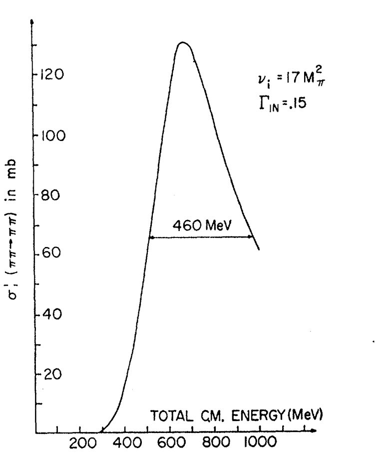

FIG. 3. The p-wave pion-pion elastic cross section for $\nu_i=17M_\pi{}^2$, $M_{\rho\ \mathrm{in}}=760$ MeV, and $\Gamma=0.15$.

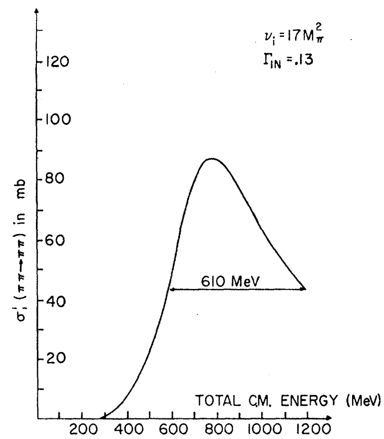

FIG. 4. The p-wave pion-pion elastic cross section for $\nu_i=17M_\pi{}^2$, $M_{\rho\ \mathrm{in}}=760$ MeV, and $\Gamma=0.13$.

It is also of interest that $\langle\beta(\nu)\rangle_{\mathrm{av}}=0$ follows from the assumption that $A_1{}^1(\nu>\nu_i)$ is purely imaginary. The results indicate that this assumption leads to no noticeable reduction of the output-resonance width when a simple ρ-exchange input force is used.

IV. ESTIMATING THE EFFECTS OF LOW-ENERGY ($\nu<\nu_i$) INELASTICITY ON THE p-WAVE PION-PION AMPLITUDE USING THE PHASE REPRESENTATION

Since the assumption of a purely imaginary p-wave amplitude for $\nu>\nu_i>13M_\pi{}^2$ appears to have little effect on the output width, and since the integral equations reduce to a convenient form, we use the formalism to investigate the effect of assuming some low-energy ($\nu<\nu_i$) inelasticity.

The nearby inelasticity is introduced through the η parameter. $\langle\beta(\nu)\rangle_{\mathrm{av}}=0$ is assumed and $\bar{\rho}(\nu)$ is as defined by (3.3). The Frye-Warnock[17] method is applied to $a_1{}^1\equiv\bar{N}_1{}^1/\bar{D}_1{}^1=A_1{}^1(\nu)[(\nu_i-\nu)/\nu_i]^{1/2}$ ($A_1{}^1$ is assumed to have no CDD poles):

$$\mathrm{Im}\bar{N}_i{}^1(\nu\leq-M_\pi{}^2)=\mathrm{Im}a_1{}^1(\nu)D_1{}^1(\nu)\,, \quad (4.1)$$

$$\mathrm{Im}\bar{N}_i{}^1(0\leq\nu\leq\nu_i)=-(1-\eta)\,\mathrm{Re}\bar{D}_1{}^1(\nu)/2\bar{\rho}(\nu)\,, \quad (4.2)$$

$$\mathrm{Im}\bar{D}_i{}^1(\nu<0)=\mathrm{Im}\bar{D}_1{}^1(\nu>\nu_i)=\mathrm{Im}\bar{N}_1{}^1(\nu>\nu_i)=0\,, \quad (4.3)$$

$$\mathrm{Im}\bar{D}_i{}^1(0\leq\nu\leq\nu_i)=2\bar{\rho}(\nu)\,\mathrm{Re}\bar{N}_1{}^1/(1+\eta)\,, \quad (4.4)$$

[14] J. Reinfelds and J. Smith, Phys. Rev. **146**, 1091 (1966).
[15] These energies correspond to the positions of the peaks in the cross section.
[16] $\Gamma\simeq0.13$ corresponds to about an 80-MeV input width.
[17] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

$$\frac{2\eta}{1+\eta}\mathrm{Re}\bar{N}_1^1(\nu)=\tilde{B}_1^1(\nu)+\frac{\nu}{\pi}\times\int_0^{\nu_i}\frac{2\bar{\rho}(\nu')}{1+\eta(\nu')}\frac{\mathrm{Re}\bar{N}_1^1(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}\times\left(\frac{\tilde{B}_1^1(\nu)}{\nu}(\nu-\nu_0)-\frac{\tilde{B}_1^1(\nu')}{\nu'}(\nu'-\nu_0)\right)d\nu', \quad (4.5)$$

$$\mathrm{Re}D_1^1(\nu)=1+\frac{\nu-\nu_0}{\pi}\mathrm{P.V.}\int_0^{\nu_i}\frac{2\bar{\rho}(\nu')}{1+\eta(\nu')}\times\frac{\mathrm{Re}\bar{N}_1^1(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}d\nu', \quad (4.6)$$

$$\tilde{B}_1^1(\nu)\equiv\bar{B}_1^1(\nu)-\frac{\nu}{\pi}\times\mathrm{P.V.}\int_0^{\nu_i}\frac{[1-\eta(\nu')]d\nu'}{2\bar{\rho}(\nu')(\nu'-\nu)\nu'}. \quad (4.7)$$

For η small over an appreciable region, $\tilde{B}_i(0\leq\nu\leq\nu_i)$ cannot be well approximated by a simple pole and (4.5) cannot be solved by the simple method of Sec. III. Further, the kernel of (4.5) is not square-integrable because of the square-root singularity in $\bar{\rho}(\nu)$ at $\nu=\nu_i$. Equation (4.5) can, however, be reduced to two integral equations of the Fredholm type. To show this, we make the following definitions:

$$F(\nu)\equiv\frac{2\bar{\rho}(\nu)}{1+\eta(\nu)}\frac{1}{\nu-\nu_0}, \quad (4.8)$$

$$g(\nu,\nu')\equiv\frac{1+\eta(\nu)}{2\eta(\nu)}\frac{\nu}{\pi}\left(\frac{\tilde{B}_1^1(\nu)(\nu-\nu_0)}{\nu}-\frac{\tilde{B}_1^1(\nu')(\nu'-\nu_0)}{\nu'}\right)\frac{1}{\nu'-\nu}, \quad (4.9)$$

$$\tilde{K}(\nu,\nu')\equiv F(\nu')[g(\nu,\nu')-g(\nu,\nu_i)]. \quad (4.10)$$

Equation (4.5) can now be written

$$\mathrm{Re}\bar{N}_1^1(\nu)=N_1(\nu)+c'N_2(\nu), \quad (4.11)$$

where c' is independent of ν;

$$N_1(\nu)=\frac{1+\eta(\nu)}{2\eta(\nu)}\tilde{B}_1^1(\nu)+\int_0^{\nu_i}\tilde{K}(\nu,\nu')N_1(\nu')d\nu' \quad (4.12)$$

and

$$N_2(\nu)=g(\nu,\nu_i)+\int_0^{\nu_i}\tilde{K}(\nu,\nu')N_2(\nu')d\nu'. \quad (4.13)$$

For $\nu_0\neq0$ and for η such that $\eta(0\leq\nu\leq\nu_i)>0$,

$$\int_0^{\nu_i}|\tilde{K}(\nu,\nu')|^2d\nu'<\infty \quad (4.14)$$

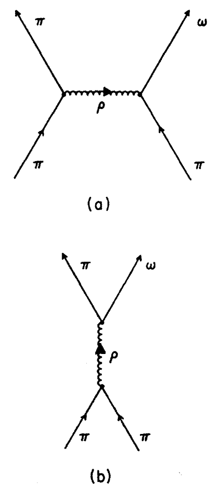

FIG. 5. Direct and exchange diagrams for the $\pi\pi\rightarrow\pi\omega$ reaction.

and (4.12) and (4.13) can be solved by the usual methods applied to Fredholm integral equations. Once N_1 and N_2 are known, c' can be obtained from

$$c'=\int_0^{\nu_i}F(\nu')N_1(\nu')d\nu'\bigg/\left(1-\int_0^{\nu_i}F(\nu')N_2(\nu')d\nu'\right). \quad (4.15)$$

Following Coulter and Shaw,[18] we first assume that the nearby inelasticity $\eta(\nu<\nu_i)$ is dominated by the $\pi\pi\rightarrow\pi\omega$ reaction. η is calculated from

$$\tfrac{1}{4}|1-\eta^2|=|A_{\pi\pi\rightarrow\pi\omega}{}^D+A_{\pi\pi\rightarrow\pi\omega}{}^E|^2\times\theta(S-(M_\omega+M_\pi)^2), \quad (4.16)$$

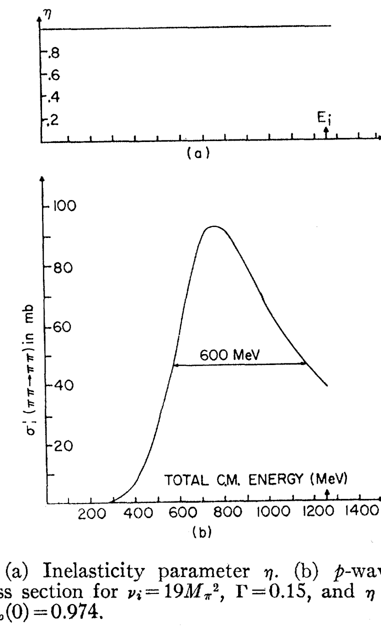

FIG. 6. (a) Inelasticity parameter η. (b) p-wave pion-pion elastic cross section for $\nu_i=19M_\pi{}^2$, $\Gamma=0.15$, and η as shown in part (a); $\alpha_\rho(0)=0.974$.

[18] P. W. Coulter and G. L. Shaw, Phys. Rev. **138**, B1273 (1965).

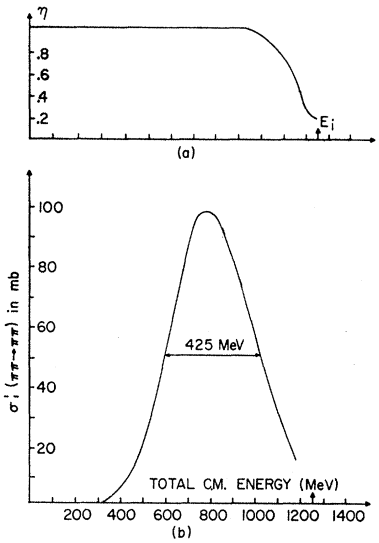

FIG. 7. (a) Inelasticity parameter η as calculated from the $\pi\pi\rightarrow\pi\omega$ reaction. (b) *p*-wave pion-pion elastic cross section for $\nu_i=19M_\pi^2$, $\Gamma=0.15$, and η as shown in part (a); $\alpha_\rho(0)=0.940$.

where A^E and A^D are the Feynman-graph amplitudes of Figs. 5(a) and 5(b), respectively:

$$A^D(s)=\frac{\gamma_{\rho\pi\pi}\gamma_{\rho\pi\omega}}{4\pi}\frac{(qq')^{3/2}}{M_\pi}\tfrac{1}{3}(\sqrt{8})\frac{1}{M_\rho^2-S}, \tag{4.17}$$

$$A^E(s)=\frac{\gamma_{\rho\pi\pi}\gamma_{\rho\pi\omega}}{4\pi}\frac{(qq')^{1/2}}{M_\pi}\tfrac{1}{3}\sqrt{2}[Q_0(R)-Q_2(R)], \tag{4.18}$$

$$s=4(\nu+M_\pi^2), \tag{4.19}$$

$$q=\tfrac{1}{2}(s-4M_\pi^2), \tag{4.20}$$

$$q'=\{[s-(M_\pi+M_\omega)^2][S-(M_\pi-M_\omega)^2]/4s\}^{1/2}, \tag{4.21}$$

$$R\equiv[M_\rho^2+\tfrac{1}{2}(s-M_\omega^2-3M_\pi^2)]/2qq'. \tag{4.22}$$

The coupling constant $\gamma_{\rho\pi\pi}$ is related to Γ by

$$\gamma_{\rho\pi\pi}^2/4\pi=3\Gamma. \tag{4.23}$$

$\gamma_{\rho\pi\omega}$ is estimated from the following expression for the $\omega\rightarrow 3\pi$ width[19]:

$$\Gamma(\omega\rightarrow 3\pi)=\frac{(M_\omega-3M_\pi)^4}{(M_\rho^2-4M_\pi^2)^2}\frac{M_\omega}{3^{3/2}}\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\frac{\gamma_{\rho\pi\omega}^2}{4\pi}W(M_\omega). \tag{4.24}$$

$W(M_\omega)=3.56$ for $M_\omega=787$ MeV. Using $\gamma_{\rho\pi\pi}^2/4\pi=0.5$ and $\gamma_{\rho\pi\omega}^2/4\pi=0.35$, the above expression gives a width of about 7 MeV for $\omega\rightarrow 3\pi$. We use these values of M_ω, $\gamma_{\rho\pi\pi}$, and $\gamma_{\rho\pi\omega}$ in all calculations.

$\eta(\nu)$ as calculated from (4.16) passes through zero at about $\nu=18M_\pi^2$. For $\nu\gtrsim 17M_\pi^2$, η is assumed to fall off smoothly to a finite value (≈ 0.2) at $\nu=\nu_i$ [see Fig. 7(a)].

The integral equations for N_1 and N_2 are solved numerically by matrix inversion and the resulting $\mathrm{Re}\bar{N}_1^1(\nu)$ is checked by recalculating $\mathrm{Re}\bar{N}_1^1$ from (4.5). The subtraction point for $\mathrm{Re}\bar{D}_i^1(\nu)$ is taken at $\nu_0=-0.1$.

The results of Sec. III indicate that with a simple ρ-exchange force as the dominant contribution to the left-hand cut (with $\Gamma=0.15$) a wide range of values of ν_i (13–$33M_\pi^2$) serve equally well to estimate the

[19] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1961).

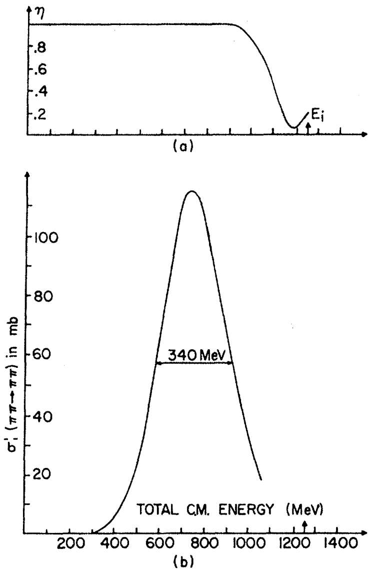

FIG. 8. (a) An arbitrary form for the inelasticity parameter η. (b) *p*-wave pion-pion elastic cross section for $\nu_i=19M_\pi^2$, $\Gamma=0.15$, and η as shown in part (a); $\alpha_\rho(0)=0.925$.

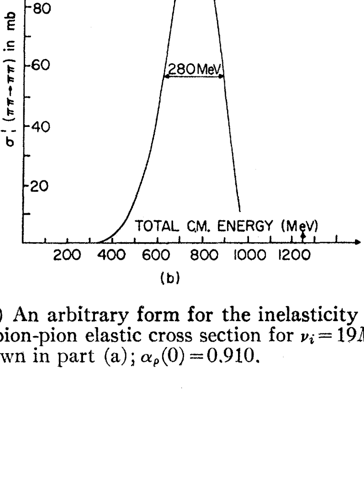

FIG. 9. (a) An arbitrary form for the inelasticity parameter η. (b) *p*-wave pion-pion elastic cross section for $\nu_i=19M_\pi^2$, $\Gamma=0.15$, and η as shown in part (a); $\alpha_\rho(0)=0.910$.

(apparently negligible) effect of maximum inelasticity for $\nu > \nu_i$ on the output width. To investigate the effects of low-energy inelasticity, we arbitrarily pick $\nu_i = 19M_\pi^2$ and fix the output-resonance width by using a Reggeized ρ-exchange force. That is, we assume the following form for the left-hand cut of the partial-wave amplitude[20]:

$$\mathrm{Im}A_i(\nu \leq -M_\pi^2) = \frac{3\Gamma}{2\nu}(M_\rho^2 + 8\nu + 4M_\pi^2) \times \left(1 + \frac{M_\rho^2}{2\nu}\right)\tfrac{1}{2}\pi\left(\frac{\nu + M_\pi^2}{M_\pi^2}\right)^{\alpha_\rho(0)-1}, \quad (4.25)$$

with $M_\rho = 760$ MeV and $\Gamma = 0.15$. The ρ Regge-trajectory intercept $\alpha_\rho(0)$ is adjusted to produce an output resonance at 760 MeV.

With the low-energy inelastic unitarity calculated from the $\pi\pi \to \pi\omega$ reaction, a symmetrical output resonance is obtained at 760 MeV with a full width of about 425 MeV; $\alpha_\rho(0) = 0.940$. A comparison of Fig. 7 with Fig. 6 shows that the nearby inelasticity in our formalism results in about a 175-MeV reduction in the width. Coulter and Shaw,[18] using the same expression for $\eta(\nu)$ for ν near the $\pi\omega$ threshold, obtain a reduction in the width of from 340 to 210 MeV—depending on their choice for the high-energy behavior for η. In Ref. 18, an additional term in the left-hand cut contribution was used to remove the divergence in the Frye-Warnock method when $\eta(\infty) < 1$. Our formalism, with its finite right-hand cut, avoids the divergence difficulty.

To observe the effects of an η that falls off faster near the $\pi\omega$ threshold than that prescribed by (4.16), we arbitrarily choose more drastic forms for $\eta(\nu \leq \nu_i)$ [see Figs. 8(a) and 9(a)]. In all cases, η is assumed to approach a finite value (≈ 0.2) at $\nu = \nu_i$. This is done for convenience in the numerical solution of the integral equation (4.5). For η small near $\nu = \nu_i$ (and, in fact, for $\eta \gtrsim 0.07$ for $0 \leq \nu \leq \nu_i$), the matrix inversion method applied to (4.12) and (4.13) requires a considerably larger number of mesh points than the 65 that were used in this calculation.

The results indicate that a rapidly decreasing η near the $\pi\omega$ threshold can reduce the width substantially (see Figs. 8 and 9). However, even the exaggerated form for η, as shown in Fig. 9(a), results in only a 280-MeV width. The cross section in this case is also asymmetric —falling off too rapidly on the high-energy side. Forms for η that decrease to smaller values than those shown in Fig. 9(a) were not used for the reasons mentioned above.

[20] We use the expression for the left-hand cut that produces the Reggeized ρ-exchange force proposed by Bander and Shaw (Ref. 2).

The values of $\alpha_\rho(0)$ used in these calculations are similar to those generally required to fix the output resonance at about 760 MeV in a single-channel N/D calculation of the p-wave pion-pion amplitude.[2,18] The ρ Regge-trajectory intercept $\alpha_\rho(0)$ is a parameter in the formalism presented here. Actually, in all cases the output-resonance position could have been fixed at 760 MeV by decreasing the input-width parameter Γ. Decreasing Γ and decreasing $\alpha_\rho(0)$ have the same effect in the calculations of Secs. III and IV because $\bar{B}_i$ is used only over a small region above threshold. For $\nu_i < 33M_\pi^2$, decreasing $\alpha_\rho(0)$ alters essentially only the slope of $\bar{B}_i(0 \leq \nu \leq \nu_i)$.

V. CONCLUSIONS

The phase representation and the approximations as described here can considerably simplify the solution of the integral equations for a partial-wave amplitude by reducing the range of integration on the right-hand cut. They also provide a method of estimating the effects of a purely imaginary partial-wave amplitude for energies above a given energy—for a specified model of the left-hand cut.

In our application to the p-wave pion-pion amplitude, we found that the assumption of a purely imaginary p-wave amplitude for total c.m. energies greater than about $6M_\pi$ did not produce any appreciable reduction in the output p-wave resonance width. (A simple ρ-exchange force with a ρ mass equal to 760 MeV and a width of about 100 MeV was assumed as the input force.) When, in addition to this maximum inelasticity for high energies, the effects of low-energy inelasticity from the $\pi\pi \to \pi\omega$ reaction were included, we found some reduction in the output width. However, even when the low-energy inelasticity from the $\pi\pi \to \pi\omega$ reaction was greatly exaggerated, we found that the width reduced to only about 280 MeV. These results are all subject to the assumption that the p-wave pion-pion amplitude has no CDD poles. As has been discussed by many authors,[21] the single-channel calculation with inelasticity and no CDD poles may not be equivalent to a multichannel calculation. The results of this paper seem to give another indication that a multichannel calculation is needed if the narrow width of the ρ resonance is to be obtained from a dispersion-relation calculation.

[21] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 270 (1965); E. J. Squires, Nuovo Cimento 34, 1751 (1964); J. Finkelstein, Phys. Rev. **140**, B175 (1965); D. Atkinson and M. B. Halpern, *ibid.* **150**, 1377 (1966); D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966); J. B. Hartle and C. E. Jones, Phys. Rev. **140**, B90 (1965).

APPENDIX: REMARKS ON THE CONTINUITY OF THE TOTAL PHASE Φ_l

Continuity of the Phase of $A_l(\nu)$ at $\nu=\nu_i$

By definition, the phase of $A_l(\nu)$ for $\nu>\nu_i$ is $\Phi_l(\nu)$. The phase of the calculated $A_l(\nu)$ for $\nu<\nu_i$ will be the phase of a $a_l(\nu)e^{\Delta(\nu)}$ as derived from the integral equations (2.23) and (2.24).[22] To ensure continuity of the phase of $A_l(\nu)$ at $\nu=\nu_i$, the following must be satisfied:

$$\lim_{(\nu_i-\nu)\to 0^+}[\text{phase}(a_l(\nu)e^{\Delta(\nu)})]=\Phi_l(\nu_i). \tag{A1}$$

Since $D_l(\nu)$ contains the entire right-hand cut of $a_l(\nu)$, and $e^{\Delta(\nu)}$ is purely real for $\nu<\nu_i$,

$$\text{phase}(a_l(\nu)e^{\Delta(\nu)})=-\text{phase}(D_l(\nu)),\quad 0<\nu<\nu_i. \tag{A2}$$

The phase of $D_l(\nu)$ can be written

$$-\text{phase}(D_l(\nu))=\cot^{-1}\left(\frac{-\text{Re}D_l(\nu)}{\text{Im}D_l(\nu)}\right). \tag{A3}$$

From Eqs. (2.22) and (2.24) we have[23]

$$\lim_{\nu\to\nu_i^-}\left(\frac{\text{Re}D_l(\nu)}{-\text{Im}D_l(\nu)}\right)$$

$$=\lim_{\nu\to\nu_i^-}\left[[-\rho(\nu)N_l(\nu)e^{\Delta(\nu)}]^{-1}\left(1+\frac{\nu-\nu_0}{\pi}\text{P.V.}\int_0^{\nu_i}\frac{\rho(\nu')N_l(\nu')e^{\Delta(\nu')}}{(\nu'-\nu)(\nu'-\nu_0)}d\nu'\right)\right] \tag{A4}$$

$$=\lim_{\nu\to\nu_i^-}\left[\frac{-\bar{c}(\nu_i)^{-1}[(\nu_i-\nu)/\nu_i]^{\Phi_l(\nu_i)/\pi}}{f(\nu)}\left(\frac{1}{\pi}\text{P.V.}\int_0^{\nu_i}\frac{[f(\nu')-f(\nu)]e^{\Delta(\nu')}}{\nu'-\nu}d\nu'\right.\right.$$
$$\left.\left.+\frac{f(\nu)}{\pi}\text{P.V.}\int_0^{\nu_i}\frac{e^{\Delta(\nu')}-\bar{c}(\nu_i)[\nu_i/(\nu_i-\nu')]^{\Phi_l(\nu_i)/\pi}}{\nu'-\nu}d\nu'+\bar{c}(\nu_i)f(\nu)\frac{1}{\pi}\text{P.V.}\int_0^{\nu_i}\frac{[\nu_i/(\nu_i-\nu')]^{\Phi_l(\nu_i)/\pi}}{\nu'-\nu}d\nu'\right)\right], \tag{A5}$$

where

$$f(\nu)\equiv\rho(\nu)N_l(\nu)/(\nu-\nu_0), \tag{A6}$$

$$\rho(\nu)=\text{Im}[A_l(\nu)]^{-1}, \tag{A7}$$

$$\bar{c}(\nu_i)\equiv\lim_{\nu\to\nu_i^-}\left[\exp\left(\frac{\nu}{\pi}\int_{\nu_i}^{\infty}\frac{\Phi_l(\nu')-\Phi_l(\nu)}{(\nu'-\nu)\nu'}d\nu'\right)\right], \tag{A8}$$

and

$$\lim_{\nu\to\nu_i^-}e^{\Delta(\nu)}=\bar{c}(\nu_i)\lim_{\nu\to\nu_i^-}\left(\frac{\nu_i}{\nu_i-\nu}\right)^{\Phi_l(\nu_i)/\pi}. \tag{A9}$$

$\bar{c}(\nu_i)$ is finite, since $\Phi_l(+\infty)$ is finite and $\Phi_l(\nu)$ is continuous.

For $\Phi_l(\nu_i)<\pi$, the first two integrals in (A5) converge. Thus, in the limit as $\nu\to\nu_i$ from below, only the third integral in (A5) contributes. That is,

$$\lim_{\nu\to\nu_i^-}\left(\frac{\text{Re}D_l(\nu)}{-\text{Im}D_l(\nu)}\right)=\lim_{\nu\to\nu_i^-}\left(-\frac{\text{P.V.}}{\pi}\int_0^{\nu_i}\frac{[(\nu_i-\nu)/(\nu_i-\nu')]^{\Phi_l(\nu_i)/\pi}}{\nu'-\nu}d\nu'\right). \tag{A10}$$

With $x\equiv(\nu_i-\nu)/(\nu_i-\nu')$, (A10) becomes

$$\lim_{\nu\to\nu_i^-}\left(-\frac{\text{P.V.}}{\pi}\int_{(\nu_i-\nu)/\nu_i}^{\infty}\frac{x^{\Phi_l(\nu_i)/\pi-1}}{x-1}dx\right)=-\frac{\text{P.V.}}{\pi}\int_0^{\infty}\frac{x^{\Phi_l(\nu_i)/\pi-1}}{x-1}dx. \tag{A11}$$

Thus for $0<\Phi_l(\nu_i)<\pi$,[24]

$$\lim_{\nu\to\nu_i^-}\left(\frac{\text{Re}D_l(\nu)}{-\text{Im}D_l(\nu)}\right)=\cot[\Phi_l(\nu_i)] \tag{A12}$$

and (A1) is satisfied. For $\Phi_l(\nu_i)=0$, (A10) gives

$$\lim_{\nu\to\nu_i^-}\left(\frac{\text{Re}D_l(\nu)}{-\text{Im}D_l(\nu)}\right)\to+\infty. \tag{A13}$$

Thus the calculated phase $\to 0^+$ as $\nu\to\nu_i^-$.

[22] Conditions (2.20)–(2.22) are assumed in this discussion.
[23] We let $D_l(\nu)$ be normalized to 1 at $\nu=0$.
[24] W. Gröbner and N. Hofreiter, *Integraltafeln, Bestimmte Integral* (Springer-Verlag, Austria, 1950), p. 178, Formula 20.