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Computer Graphics As An Aid To Teaching Geometric Transformations

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COMPUTER GRAPHICS as an AID to TEACHING GEOMETRIC TRANSFORMATIONS

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During the past several years, there has been much discussion and controversy over what should be taught in high school mathematics, in general, and in high school geometry, in particular. Numerous mathematicians have encouraged the teaching of transformations as part of the standard high school mathematics curriculum[1-4,6-9]. The results of a recent survey of high school teachers of mathematics indicate that 19% have taught geometric transformations, 26% feel adequately prepared to teach such a topic, and 50% would like to teach the topic if materials were available for the average college prep student[5].

The topic of transformations is important because transformations are a unifying factor in algebra and geometry. Algebra and geometry are essentially the same material taught from different approaches. In particular, the abstract algebraic concept of a group can be conveyed in purely geometric terms by groups of transformations.

Since the introduction of the new math in the late fifties and early sixties, students have been exposed to the concepts of group properties from the primary grades on through high school. Very often, a concept is presented without its formal mathematical title. The exact vocabulary is usually introduced in junior high, with the formal definition of a group being made in senior high - if at all. This puts the student in the position of having to understand the group-related concepts of closure, associativity, identity, inverses, and commutativity without understanding the underlying rationale of these properties. This requires a level of abstraction of which most students are not capable.

In most cases, numeric examples are used to explain the above concepts. Since most of these numeric examples involve infinite sets (integers, rationals, reals, etc.), this gives rise to further problems. The properties of closure and commutativity are not fully appreciated when explained by way of infinite sets. The intuitive meaning of closure is much better illustrated by means of some particular finite set for which it can be exhaustively demonstrated that, in fact, the given elements of the set are the only possible results obtained under a particular operation. The intuitive meaning of commutativity is best explained by the standard non-mathematical example of putting on shoes and socks - reversing the procedures drastically affects the outcome.

The above concepts - closure, associativity, identity, inverses, and commutativity - are also applicable to the following set of eight geometric transformations:

1. Identity
2. Rotation of 90° counterclockwise about the origin
3. Rotation of 180° "
4. Rotation of 270° "
5. Reflection in the x-axis
6. Reflection in the y-axis
7. Reflection in the line $x = y$
8. Reflection in the line $x = -y$

Although the sets of numbers in algebra have defined on them the four binary operations of addition, subtraction, multiplication, and division, the above set of geometric transformations has defined on it just one binary operation. We will call this binary operation 'followed by', and thus be able to speak of a rotation of 90° 'followed by' a reflection in the x-axis. This means that a figure is transformed by a rotation of 90° to yield a new figure. The new figure is transformed by a reflection in the x-axis to give the final figure. This final figure is the result of a rotation of 90° 'followed by' a reflection in the x-axis.

The concepts of closure and commutativity, as well as the other concepts, are much more clearly illustrated mathematically by using the above group of rotations and reflections in the plane, the main reason being that the students can "see" what is happening with the transformations. Using this approach, the concept of a non-commutative group can be (and has been) taught to average and below average students as early as the second semester of a plane geometry course. One advantage of using the group of rotations and reflections to illustrate non-commutativity is that even slower students catch on quickly to the fact that a rotation of 90° 'followed by' a reflection

BRIEF REVIEW OF PROPERTIES OF GROUPS

FOR SEVERAL YEARS NOW YOU HAVE BEEN STUDYING THE CONCEPTS OF
CLOSURE   ASSOCIATIVITY   IDENTITY   INVERSE   COMMUTATIVITY
OF SETS WITH RESPECT TO SOME BINARY OPERATION, SUCH AS ADDITION OR MULTIPLICATION OR SUBTRACTION OR DIVISION

CONSIDER THE FOLLOWING EXAMPLES:

THE SET OF INTEGERS IS CLOSED UNDER ADDITION BECAUSE THE SUM OF ANY TWO INTEGERS IS AN INTEGER.
THE SET OF NEGATIVE INTEGERS IS NOT CLOSED UNDER MULTIPLICATION BECAUSE THE PRODUCT OF TWO NEGATIVE NUMBERS IS NOT NEGATIVE.

THE SET OF INTEGERS IS ASSOCIATIVE FOR EITHER MULTIPLICATION OR ADDITION. THE INTEGERS ARE ASSOCIATIVE FOR NEITHER DIVISION NOR SUBTRACTION. THIS IS ILLUSTRATED BY THE FOLLOWING:

ADDITION:	20+(5+4) IS EQUAL TO (20+5)+4
SUBTRACTION:	20-(5-4) IS NOT EQUAL TO (20-5)-4
MULTIPLICATION:	20*(5*4) IS EQUAL TO (20*5)*4
DIVISION:	20/(5/4) IS NOT EQUAL TO (20/5)/4

IN ORDINARY ARITHMETIC ZERO IS CONSIDERED THE IDENTITY FOR ADDITION BECAUSE ZERO ADDED TO ANY NUMBER DOES NOT CHANGE THE NUMBER. LIKEWISE, ONE IS THE IDENTITY ELEMENT FOR MULTIPLICATION BECAUSE ANY NUMBER MULTIPLIED BY ONE IS NOT CHANGED.

CONSIDER INVERSES: A BINARY OPERATION PERFORMED ON AN ELEMENT AND ITS INVERSE RESULTS IN THE IDENTITY ELEMENT FOR THAT PARTICULAR BINARY OPERATION. THIS IS ILLUSTRATED BY:

-4 IS THE ADDITIVE INVERSE OF 4 BECAUSE 4+(-4)=0
1/2 IS THE MULTIPLICATIVE INVERSE OF 2 BECAUSE 2*(1/2)=1
-2/3 IS THE MULT. INVERSE OF -3/2 BECAUSE (-2/3)*(-3/2)=1
-4/7 IS THE ADD. INVERSE OF 4/7 BECAUSE (-4/7)+(4/7)=0

THE SET OF INTEGERS CONTAINS THE IDENTITY ELEMENT FOR ADDITION (0) AND ALSO CONTAINS ALL ADDITIVE INVERSES.
THE SET OF INTEGERS ALSO CONTAINS THE IDENTITY ELEMENT FOR MULTIPLICATION(1), BUT DOES NOT CONTAIN ALL INVERSES.
THE SET OF RATIONAL NUMBERS WITH ZERO EXCLUDED HAS THE IDENTITY FOR MULTIPLICATION AND ALL MULTIPLICATIVE INVERSES. (ZERO MUST BE EXCLUDED FROM THE SET BECAUSE THERE IS NO UNIQUE RATIONAL # SUCH THAT MULTIPLICATION OF ZERO BY THAT NUMBER YIELDS ONE.

ANY SET HAVING THE 4 PROPERTIES JUST DESCRIBED (CLOSED, ASSOCIATIVE, IDENTITY, INVERSE) IS CALLED A GROUP. SOME SETS HAVE THE ADDITIONAL PROPERTY THAT THE ORDER OF OPERATION DOES NOT MATTER. THIS IS THE PROPERTY OF COMMUTATIVITY. FOR EXAMPLE:

INTEGERS ARE COMMUTATIVE UNDER:

ADDITION:	5 + 4 = 4 + 5
MULTIPLICATION:	5 * 4 = 4 * 5

INTEGERS ARE NOT COMMUTATIVE UNDER:

SUBTRACTION:	5 - 4 IS NOT EQUAL TO 4 - 5
DIVISION:	5 / 4 IS NOT EQUAL TO 4 /5

THE SET OF INTEGERS IS A COMMUTATIVE GROUP UNDER ADDITION.
THE SET OF RATIONALS WITH ZERO EXCLUDED IS A COMMUTATIVE GROUP UNDER MULTIPLICATION.
THE SET (-1, 1) IS A COMMUTATIVE GROUP UNDER MULTIPLICATION.

Figure 1

GROUPS OF TRANSFORMATIONS

WE ARE NOW GOING TO CONSIDER A GROUP THAT DOES NOT INVOLVE A SET OF NUMBERS. INSTEAD, THE SET CONSISTS OF THE MOTIONS OR MOVEMENTS OR TRANSFORMATIONS OF A GEOMETRIC FIGURE ON THE 2-DIMENSIONAL PLANE AS DEFINED BY CARTESIAN COORDINATES.

THE TRANSFORMATIONS WE WANT TO CONSIDER ARE:

ROTATION OF 90 DEGREES (COUNTERCLOCKWISE)
ROTATION OF 180 DEGREES (COUNTERCLOCKWISE)
ROTATION OF 270 DEGREES (COUNTERCLOCKWISE)
REFLECTION IN THE X-AXIS
REFLECTION IN THE Y-AXIS
REFLECTION IN THE LINE X=Y
REFLECTION IN THE LINE X=-Y

ROTATIONS ARE CONSIDERED TO BE ROTATIONS ABOUT THE ORIGIN. REFLECTIONS MAY BE CONSIDERED AS THE MIRROR IMAGE OF THE FIGURE IN THE LINE OF REFLECTION.

THE IDENTITY ELEMENT WILL BE 'NO MOTION'. THAT IS, IF THE FINAL RESULT IS THE SAME AS THE ORIGINAL FIGURE, THE TRANSFORMATION (OR SERIES OF TRANSFORMATIONS) IS EQUIVALENT TO THE IDENTITY.

CONSIDER A RIGHT TRIANGLE IN THE FIRST QUADRANT WITH VERTICES
(5, 2)   (12, 2)   (12, 16)

YOU WILL BE GIVEN THE OPPORTUNITY TO OBSERVE WHAT EACH TRANSFORMATION DOES TO THIS TRIANGLE. YOU WILL SEE THE ORIGINAL FIGURE AND THE TRANSFORMED FIGURE FOR EACH TRANSFORMATION.

YOU WILL THEN BE GIVEN THE OPPORTUNITY TO INPUT COORDINATES FOR A FIGURE OF YOUR OWN CHOOSING, AND THEN OBSERVE THE EFFECTS OF THE TRANSFORMATIONS ON YOUR FIGURE.

YOU MAY REPEAT THE PROCESS AS OFTEN AS YOU WISH. BE SURE TO TRY SOME FIGURES IN QUADRANTS OTHER THAN THE FIRST, AS WELL AS FIGURES THAT OVERLAP QUADRANTS.

YOU WILL CONTROL THE SPEED WITH WHICH THE TRANSFORMATIONS ARE SHOWN. DO NOT RESPOND UNTIL YOU HAVE GIVEN YOURSELF SUFFICIENT TIME TO CONSIDER EACH FIGURE CAREFULLY.

Figure 2

in the x-axis does not yield the same result as a reflection in the x-axis 'followed by' a rotation of 90°. This approach has the additional advantage of reinforcing the skills of matrix multiplication and plotting in the Cartesian plane. However, in the process of doing all the multiplication and plotting, the students sometimes lose sight of what they are supposed to be learning. In order to "see" what happens to a figure under one transformation, a student is required to:

1) plot the points for the original figure,
2) calculate the new points by means of matrix multiplication, and
3) plot the new set of points.

All of this is very time consuming. Obviously, the work is increased greatly with each added transformation. It usually requires ten to twelve class periods to successfully present the lesson in a traditional classroom setting.

It seems that a computer-driven graphics screen is eminently suitable as a medium for presenting a geometry lesson on the subject of rotations and reflections. Such a program was developed as a semester project for Graphics and Artificial Intelligence. The programming was done on a Data General NOVA 840 in the BASIC language. A Tektronix 4010 graphics terminal is used as the primary input and output device, and a Centronics 100 Printer serves as a secondary output unit. The program is designed to be used by students from high school through graduate school (faculty may also benefit from it). Geometry is a prerequisite, as well as a standard first year high school course in modern algebra.

The program consists of four sections, the first of which is a review of the basic concepts related to groups, both commutative and non-commutative. This information is directed to the line printer rather than to the graphics screen so that the student will have a permanent copy for future reference (see figure 1). After studying the material, the student may indicate readiness for a quiz of ten questions generated at random from a large bank of items. On failing to attain 80% correct, the student is asked to restudy the material and then take another randomly generated ten question quiz. A record is kept of the total number correct out of the total number attempted. Understandably, some students will use more questions than others. These quizzes are presented on the graphics screen to take advantage of the capability of indicating an answer by positioning the crosshairs over the desired choice, rather than by typing in a number or letter. This cuts down on the number of incorrect responses caused by typing errors. Only after scoring at least 80% on a part one quiz will the student be able to proceed to the next section.

The purpose of part two is to explain and illustrate the group of rotations and reflections in the plane that is under consideration. Verbal information is directed to the line printer as in part one (see figure 2). When the student indicates that the material has been assimilated, the eight members of the set of rotations and reflections in the plane are presented one by one pictorially on the graphics screen. For each transformation, a first quadrant scalene triangle and the resulting figure from the transformation are shown. The student may observe the given figure for as long and as often as desired. The student is then encouraged to change the figure to one of his or her own choosing (see figure 3).

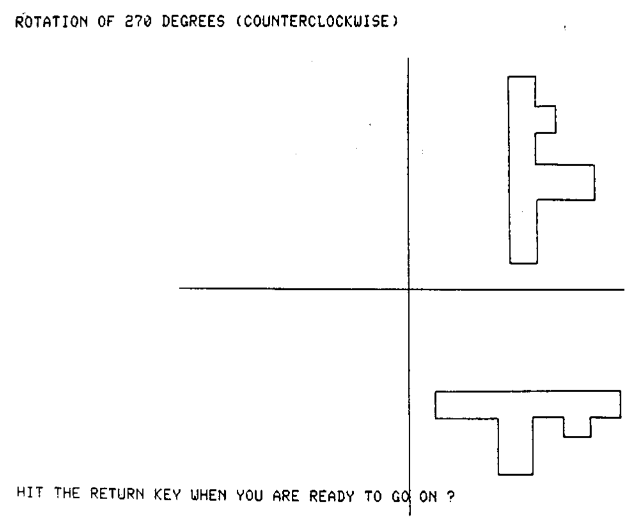

Figure 3

Directions are given on the screen to enable the student to do this easily. The ability to create a new figure gives the opportunity to observe the transformations working on a figure that originates in a quadrant other than the first, as well as figures that overlap quadrants. The lesson is self-paced in that the student may stay on this section until satisfied that the concepts presented are completely understood. At the student's indication of readiness, a quiz on the part two material is presented in the same manner as in part one.

The purpose of part three, the most important section of the program, is to permit the student to observe what happens when one transformation 'followed by' another operates on a given figure. An operation table for the set, together with an explanation of its use, is sent to the line printer (see figures 4 and 5). The student may draw any figure on the graphics terminal, and then may try all possible combinations of one transformation 'followed by' another, paying particular attention to reverses in order to check commutativity. The student fills in the table as each combination is observed.

After the student chooses a figure, the screen presents that figure labeled "origi-

CLOSURE

SO FAR, WE HAVE STARTED WITH A FIGURE AND PERFORMED ONE TRANSFORMATION ON IT. THIS IS EQUIVALENT TO TAKING AN INTEGER AND MULTIPLYING IT BY ONE(THE IDENTITY). WE NOW WANT TO SEE WHAT HAPPENS WHEN WE COMBINE TWO OR MORE TRANSFORMATIONS. YOU CAN CONSIDER THE BINARY OPERATION TO BE 'FOLLOWED BY'. FOR EXAMPLE, A ROTATION OF 90 DEGREES FOLLOWED BY A ROTATION OF 180 DEGREES YIELDS A ROTATION OF 270 DEGREES. A ROTATION OF 270 IS A TRANSFORMATION IN THE SET UNDER CONSIDERATION, SO CLOSURE IS PRESERVED. IT IS EASY TO DETERMINE WHAT HAPPENS WITH ONE ROTATION FOLLOWED BY ANOTHER. IT BECOMES MORE DIFFICULT TO RECOGNIZE WHICH TRANSFORMATION RESULTS WHEN A ROTATION IS FOLLOWED BY A REFLECTION.

TO HELP YOU TO OBSERVE ALL POSSIBLE PERMUTATIONS OF THE TRANSFORMATIONS TAKEN TWO AT A TIME, YOU WILL BE PERMITTED TO ENTER ANY TWO TRANSFORMATIONS. YOU WILL THEN SEE ON THE SCREEN:

A. THE ORIGINAL FIGURE
B. THE FIRST TRANSFORMATION
C. THE FINAL FIGURE (RESULT OF FIRST TRANSFORMATION FOLLOWED BY THE SECOND)

THE PICTURES WILL BE LABELED AS TO ORIGINAL FIGURE AND FINAL FIGURE.

TO HELP YOU KEEP TRACK OF THE PERMUTATIONS YOU HAVE TRIED, FILL IN THE TABLE BELOW. THE RESULTS OF ANY TRANSFORMATION COMBINED WITH THE IDENTITY IS ALREADY SHOWN. THE ENTRY FOR A ROTATION OF 90 FOLLOWED BY A ROTATION OF 180 IS ALSO SHOWN. NOTE THAT THE FIRST TRANSFORMATION (90) IS READ FROM THE LEFT HAND COLUMN. MOVE ACROSS THE CORRESPONDING ROW UNTIL YOU ARE UNDER THE COLUMN HEADED BY THE SECOND TRANSFORMATION (180). THE ANSWER(270) IS FILLED IN AT THAT POSITION.

TRY A FIRST QUADRANT SCALENE TRIANGLE FOR ALL COMBINATIONS, AND THEN TRY FIGURES IN VARIOUS OTHER POSITIONS.

Figure 4

	I	90	180	270	X-AXIS	Y-AXIS	X=Y	X=-Y
I	I	90	180	270	X-AXIS	Y-AXIS	X=Y	X=-Y
90	90		270					
180	180							
270	270							
X-AXIS	X-AXIS							
Y-AXIS	Y-AXIS							
X=Y	X=Y							
X=-Y	X=-Y							

Figure 5

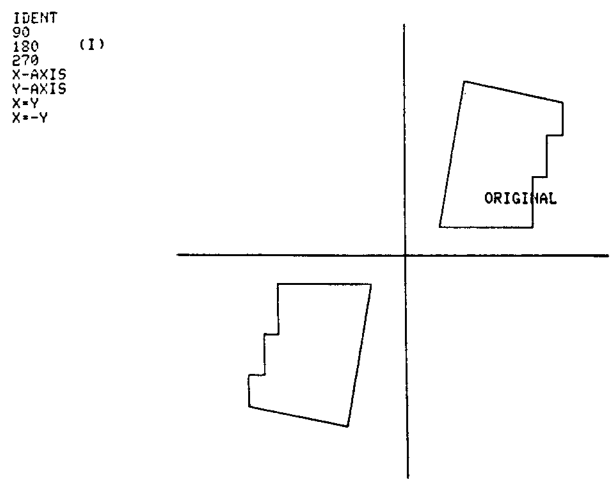

Figure 7

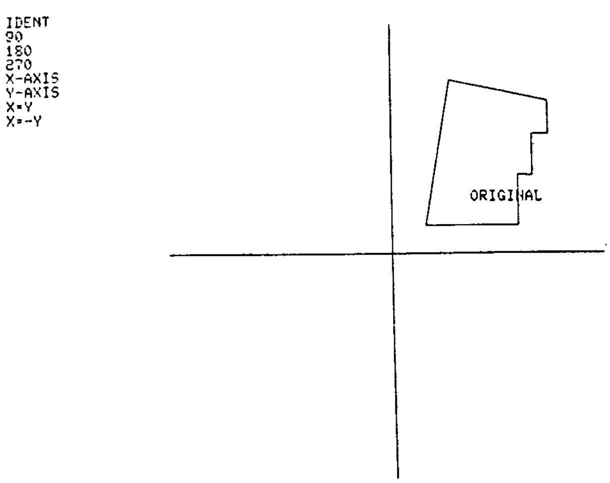

Figure 6

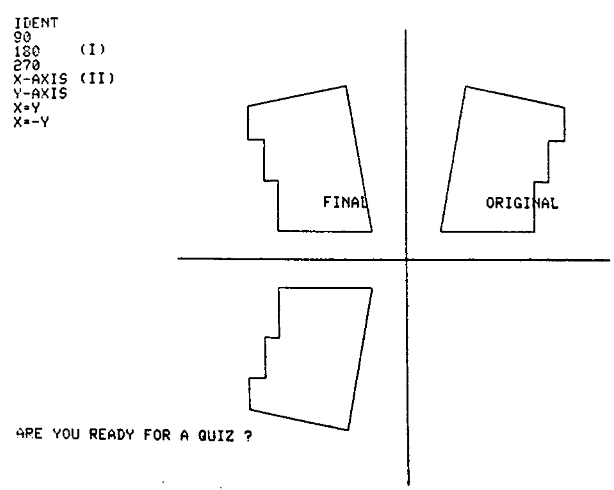

Figure 8

nal," along with a listing on the left hand side of the screen of the eight members of the set of transformations (see figure 6). The choice of transformation is made by positioning the crosshairs over the name of the desired transformation and striking any key on the keyboard. After the choice is made, the Roman numeral I appears beside the chosen transformation, and the transformed figure appears on the screen along with the original figure (see figure 7). The student then makes a second choice of transformations in the same way. A Roman numeral II appears beside the second choice, and the resulting figure is added to the screen with the label "final" (see figure 8). At this point, the student should fill in the proper position on the table.

The student now has three choices:

1) continue working on the same figure,
2) change the figure and continue working, or
3) indicate readiness for a quiz.

The student is expected to complete the table before making the third choice. However, at any point that the student understands the concepts, the table may be completed without the aid of the graphics terminal. Some students are quicker than others in observing relationships, so can complete this section in a shorter length of time.

The rules for passing this section are stricter than those for the first two sections. A perfect score on a quiz must be attained before permission is given to go on to the last section of the program. This is done to ensure that the student has the entire table properly completed. The student is instructed to note in the table any errors made while taking the quiz. If a perfect score is not attained, the student must redo at least one figure before retaking the quiz. The reasoning for this is based on the assumption that since the student made at least one error, he should recheck that particular relationship. There is no limit to the number of times the quiz may be retaken. The only escape is a perfect score. Such a score can easily be attained when the table is correctly completed.

The purpose of the fourth and last section of the program is to permit the student to observe what happens when more than two transformations are used in sequence, and to consolidate the student's ideas on the concepts of identity, inverse, closure, and associativity. The printout for part four (see figure 9) points out various relationships that should be observed in the completed and corrected table from part three. As before, the student may draw any figure on the graphics screen. This time, as many transformations as desired may be indicated. A record is not kept of first, second, etc.. Only the original and final figures are labeled. Hopefully, the student will observe that only eight different positions of the figure ever appear on the screen - no matter how many transformations have been performed (see figure 10). This reinforces the idea of closure by demonstrating that only elements of the original set appear when the operation 'followed by' is applied. Observing that one transformation 'followed by' another yields the original figure will reinforce the ideas of identity and inverse. Associativity can be tested for all possible combinations - given the necessary time and patience. Non-commutativity can be observed throughout sections three and four. All of these concepts are tested in the final quiz. Again, the student must attain a perfect score.

At the conclusion of the program, the student is given an accounting of the total number correct out of the total number attempted, and the percentage of correct responses. Some students complete the lesson using the minimum of forty questions; most do not. To date, no one has attained 100% the first time through the program.

Although the program has not yet been tested on a regular class, comments from students who have used the program indicate that the combination of a graphics screen and self-paced CAI is an effective method of presenting the topic of non-commutative groups as illustrated by the group of rotations and reflections in the plane. The following are typical of reactions to the program by users:

- good way to learn
- didn't realize that transformations formed a group
- learned twice as much twice as fast
- knew about rotations and reflections, but never knew they formed a group
- didn't know you could combine transformations
- hard to get a perfect score
- don't like to have to retake entire quiz after missing just one question
- fun to be able to draw your own figures

On the basis of the results received to date, we feel that the time and effort required to do a more rigorous evaluation is justified to test the hypothesis that the use of CAI is more effective than presenting the above topic in a traditional classroom setting. Such an evaluation is scheduled tentatively for the Fall of 1976.

Acknowledgments

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ASSOCIATIVITY

OBSERVE ON YOUR TABLE THAT CLOSURE HOLDS. ANY TRANSFORMATION FOLLOWED BY ANOTHER YIELDS ONE OF THE TRANSFORMATIONS IN THE SET.

FOR ASSOCIATIVITY WE WANT TO CONSIDER THREE TRANSFORMATIONS AT A TIME. CALL THEM I, II, AND III. USE AN ASTERISK(*) TO SYMBOLIZE 'FOLLOWED BY'. IS (I*(II*III)) THE SAME AS ((I*II)*III) ?

I = ROTATION OF 90 DEGREES
II = REFLECTION IN LINE X=Y
III = REFLECTION IN X-AXIS

I * (II * III) IS
(ROTATION OF 90)*(REFLECTION IN X=Y * REFLECTION IN X--AXIS)
(ROTATION OF 90)*(ROTATION OF 270)
IDENTITY

(I * II) * III IS
(ROTATION OF 90 * REFLECTION IN X=Y) * (REFLECTION IN X--AXIS)
(REFLECTION IN X-AXIS) * (REFLECTION IN X--AXIS)
IDENTITY

SINCE BOTH YIELD THE SAME RESULT (THE IDENTITY), WE CAN SAY THAT ASSOCIATIVITY HOLDS IN THIS CASE.

INVERSES

IN YOUR TABLE, YOU SHOULD HAVE EXACTLY ONE I IN EACH ROW AND COLUMN. OBSERVE THAT 90 IS THE INVERSE OF 270, AND VICE VERSA, BECAUSE 90 FOLLOWED BY 270 GIVES THE IDENTITY, AND 270 FOLLOWED BY 90 ALSO GIVES THE IDENTITY. EACH OF THE OTHER TRANSFORMATIONS IS ITS OWN INVERSE. THIS MEANS THAT PERFORMING THE TRANSFORMATION TWICE IN A ROW BRINGS YOU BACK TO THE ORIGINAL FIGURE.

COMMUTATIVITY

YOU SHOULD HAVE NOTICED BY NOW THAT OUR GROUP IS NOT COMMUTATIVE. WHILE IT IS TRUE THAT SOME TRANSFORMATIONS CAN BE REVERSED WITHOUT CHANGING THE RESULT, IT IS NOT POSSIBLE TO DO THIS FOR ALL PAIRS OF TRANSFORMATIONS. IN ORDER FOR A GROUP TO BE COMMUTATIVE, YOU MUST BE ABLE TO SAY THAT I*II = II*I FOR ALL TRANSFORMATIONS I AND II. ONE COUNTEREXAMPLE IS SUFFICIENT TO PROVE A GROUP NON-COMMUTATIVE. FROM YOUR TABLE, YOU SHOULD BE ABLE TO PICK OUT SEVERAL COUNTEREXAMPLES TO COMMUTATIVITY.

YOU MAY NOW TRY THREE OR FOUR TRANSFORMATIONS(ONE FOLLOWED BY ANOTHER). YOU WILL SEE THE ORIGINAL (LABELED)
EACH OF THE INTERMEDIATE FIGURES
THE FINAL FIGURE (LABELED)
HIT THE LETTER 'L' TO INDICATE THE LAST TRANSFORMATION IN EACH SET
YOU WILL BE GIVEN THE OPPORTUNITY TO CHANGE THE FIGURE BETWEEN SETS.

TRY TO CHECK THE RESULTS WITH YOUR TABLE TO BE SURE THAT YOUR TABLE IS CORRECT.

Figure 9

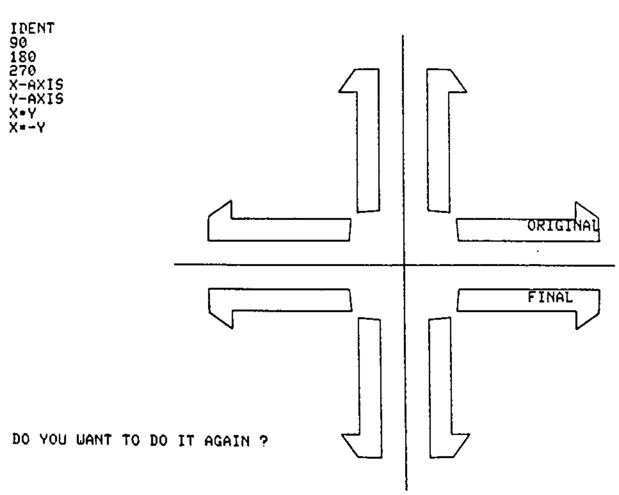

Figure 10

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