01 Jan 1998

Arc Approximation Property and Confluence of Induced Mappings

W. J. Charatonik
Missouri University of Science and Technology, wjcharat@mst.edu

Recommended Citation
W. J. Charatonik, "Arc Approximation Property and Confluence of Induced Mappings," *Rocky Mountain Journal of Mathematics*, Rocky Mountain Mathematics Consortium, Jan 1998.
The definitive version is available at https://doi.org/10.1216/rmjm/1181071825

ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

ARC APPROXIMATION PROPERTY AND CONFLUENCE OF INDUCED MAPPINGS

WŁODZIMIERZ J. CHARATONIK

ABSTRACT. We say that a continuum X has the arc approximation property if every subcontinuum K of X is the limit of a sequence of arcwise connected subcontinua of X all containing a fixed point of K. This property is applied to exhibit a class of continua Y such that confluence of a mapping $f : X \to Y$ implies confluence of the induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$. The converse implications are studied and similar interrelations are considered for some other classes of mappings, related to confluent ones.

1. Introduction. For a metric continuum X we denote by 2^X and $C(X)$ the hyperspaces of all nonempty compact and of all nonempty compact connected subsets of X, respectively. Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ to denote the corresponding induced mappings. Let $\mathfrak{M}$ stand for a class of mappings between continua. A general problem which is discussed in this paper is to find all possible interrelations between the following three statements:

$$f \in \mathfrak{M}; \tag{1.1}$$

$$2^f \in \mathfrak{M}; \tag{1.2}$$

$$C(f) \in \mathfrak{M}. \tag{1.3}$$

We do not intend to discuss the problem in full, for a wide spectrum of various classes $\mathfrak{M}$ of mappings. On the contrary, we concentrate our

Received by the editors on November 14, 1994, and in revised form on June 21, 1995.
1991 *Mathematics Subject Classification.* 54F15, 54B20, 54C10.
Key words and phrases. Approximation, arcwise connected, confluent, continuum, hyperspace, induced mapping, joining, pseudo-confluent, semi-confluent, weakly confluent, n-weakly confluent.

attention on the class of confluent mappings and on a few other classes that are relatively close to confluent ones. On the other hand, however, we consider several additional conditions that concern domain and/or range spaces as well as the mappings. under which some implications between statements (1.1), (1.2) and (1.3), while not true in general, are valid.

2. Preliminaries. All spaces considered in this paper are assumed to be metric. A *continuum* means a compact connected metric space. A property of a continuum is said to be *hereditary* provided that every subcontinuum of the continuum has the property.

Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X with the Hausdorff metric

$$H(A,B) = \max\{\sup\{(d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\};$$

equivalently, with the Vietoris topology, see, e.g., [**19** (0.1) and (0.12)]. Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X, and by $F_1(X)$ the *space of singletons*, i.e., $F_1(X) = \{\{p\} \in 2^X : p \in X\}$. Note that $F_1(X) \subset C(X)$, and that the following holds, see [**19**, Theorem (1.13)].

Proposition 2.1. *For each continuum X the hyperspaces 2^X and $C(X)$ are arcwise connected continua. In particular, the hyperspace $C(X)$ is a subcontinuum of the hyperspace 2^X.*

Proposition 2.2. *For each continuum X the space $F_1(X)$ of singletons is homeomorphic (even isometric) to X, and thus it is a subcontinuum of the hyperspaces 2^X and $C(X)$.*

By an *order arc* in 2^X we mean an arc Φ in 2^X such that if $A, B \in \Phi$, then either $A \subset B$ or $B \subset A$. The following facts are known, see [**19**, Theorem (1.8) and Lemma (1.11)].

Fact 2.3. Let $A, B \in 2^X$ with $A \neq B$. Then there exists an order arc in 2^X from A to B if and only if $A \subset B$ and each component of B intersects A.

Fact 2.4. If an order arc Φ in 2^X begins with $A \in C(X)$, then $\Phi \subset C(X)$.

The reader is referred to Nadler's book [**19**] for needed information on the structure of hyperspaces.

For a given point p of a space X and for a number $\varepsilon > 0$ we denote by $B(p, \varepsilon)$ the open ε-ball about the point p in the space X. Further, for a given subset A of X we denote by $N(A, \varepsilon)$ the ε-neighborhood of A in X, i.e., $N(A, \varepsilon) = \cup\{B(p, \varepsilon) : p \in A\}$.

Let a subset A and a sequence of subsets A_n of a metric space X be given. We use the notation $A = \operatorname{Lim} A_n$ in the usual sense as in [**12**, p. 339]. For compact spaces this notion of limit is equivalent to that using Hausdorff metric (see [**19**, Theorem (0.7)]). A continuum X is said to be *irreducible* provided that there are two points of X such that if a subcontinuum X' of X contains these points, then $X' = X$. Each of these points is called a *point of irreducibility* of X.

A *mapping* means a continuous function. A mapping $f : X \to Y$ between continua X and Y is said to be: *confluent* provided that for each subcontinuum L of Y each component of $f^{-1}(L)$ is mapped under f onto L; *semi-confluent* provided that for each subcontinuum L of Y and for every two components K_1 and K_2 of $f^{-1}(L)$ either $f(K_1) \subset f(K_2)$ or $f(K_2) \subset f(K_1)$ (or both); *joining* provided that for each subcontinuum L of Y and for every two components K_1 and K_2 of $f^{-1}(L)$ their images intersect, i.e., $f(K_1) \cap f(K_2) \neq \varnothing$; *weakly confluent* provided that for each subcontinuum L of Y some component of $f^{-1}(L)$ is mapped under f onto L; *pseudo-confluent* provided that for each irreducible subcontinuum L of Y some component of $f^{-1}(L)$ is mapped under f onto L.

Recently the following concept has been introduced in [**17**, p. 236]. Let a nonnegative integer n be given. A mapping $f : X \to Y$ between continua X and Y is said to be: 0-*weakly confluent* provided that it is a surjection; $(n+1)$-*weakly confluent* provided that for each subcontinuum L of Y there exists a component of $f^{-1}(L)$ such that the partial mapping $f|K : K \to L$ is n-weakly confluent; $\aleph_0$-*weakly confluent* (or *inductively weakly confluent*) provided that it is n-weakly confluent for every nonnegative integer n.

Thus it is evident that 1-weakly confluent mappings coincide with weakly confluent ones. Further, the following fact is a consequence of the definition, see [**17**, p. 236].

Fact 2.5. For every nonnegative integer n, an $(n+1)$-weakly confluent mapping is n-weakly confluent.

We have the following corollary.

Corollary 2.6. *For every nonnegative integer n an n-weakly confluent (and $\aleph_0$-weakly confluent) mapping is surjective.*

The following proposition is known.

Proposition 2.7. *Let a mapping between continua be given.*

(a) *If it is confluent, then it is semi-confluent and surjective.*

(b) *If it is semi-confluent, then it is joining.*

(c) *If it is semi-confluent and surjective, then it is weakly confluent.*

(d) *If it is weakly confluent, then it is pseudo-confluent.*

(e) *If it is pseudo-confluent, then it is surjective.*

Proof. Implications (a), (b), (d) and (e) are consequences of the definitions; (c) is shown in [**16**, Theorem (3.8)]. □

The next result is proved as Theorem 1 of [**17**, p. 236].

Theorem 2.8. *Each surjective semi-confluent mapping between continua is $\aleph_0$-weakly confluent.*

Recall that a mapping $r : X \to Y \subset X$ is called a *retraction* if the partial mapping $r|Y : Y \to Y$ is the identity.

Observation 2.9. Each retraction between continua is $\aleph_0$-weakly confluent.

Remark 2.10. The concept of a joining mapping has been introduced by T. Maćkowiak in [**15**, p. 288]. In that paper, as well as in [**16**, pp. 12–14], the reader can find some (but rather scanty) information on these mappings. Note that a mapping of an arc onto a simple closed curve that identifies the end points of the arc is joining while not pseudo-confluent. A weakly confluent but not joining mapping $f : [0,1] \to [0,1]$ can be defined as one that is linear on the intervals $[a_i, a_{i+1}]$, where $0 = a_0 < a_1 < a_2 < a_3 = 1$, and such that for $0 = b_0 < b_1 < b_2 < b_3 = 1$ we have $f(a_0) = b_1$; $f(a_1) = b_0$; $f(a_2) = b_3$; $f(a_3) = b_2$.

We will need the following general fact about confluent mappings.

Proposition 2.11. *If a mapping $f : X \to Y$ between continua X and Y is not confluent, then there are a continuum L in Y and a component C of $f^{-1}(L)$ such that $f(C)$ is a nondegenerate proper subset of L.*

Proof. Assume that f is not confluent and let L' and C' be two continua in Y and X, respectively, such that C' is a component of $f^{-1}(L')$ and $f(C') \neq L'$. If $f(C')$ is nondegenerate, we are done; so assume that $f(C') = \{p\}$ for some point p in L'. Let $q \in L' \setminus \{p\}$. Since the decomposition of $f^{-1}(L')$ into the components is 0-dimensional, there are two open subsets U_1 and U_2 of X such that

$$U_1 \cap U_2 = \varnothing, \quad f^{-1}(L') \subset U_1 \cup U_2, \quad C' \subset U_1 \quad \text{and} \quad f^{-1}(q) \subset U_2.$$

Let $\{V_n : n \in \mathbf{N}\}$ be a sequence of open subsets of Y such that

$$V_{n+1} \subset V_n \quad \text{for each} \quad n \in \mathbf{N} \quad \text{and} \quad \cap \{V_n : n \in \mathbf{N}\} = L'.$$

Then $\{f^{-1}(\operatorname{cl} V_n) : n \in \mathbf{N}\}$ is a decreasing sequence of compact subsets of X whose intersection equals $f^{-1}(L')$. Thus there is an index n_0 such that $f^{-1}(\operatorname{cl} V_{n_0}) \subset U_1 \cup U_2$. Denote by C'' a continuum properly containing C' and contained in $f^{-1}(V_{n_0})$, so $C'' \subset U_1$. Finally, put $L = L' \cup f(C'')$ and let C be the component of $f^{-1}(L)$ containing C''. Because C' is a component of $f^{-1}(L')$, the continuum $f(C'')$ is nondegenerate, and so is $f(C)$. Since $C \subset U_1$, we have $C \cap f^{-1}(q) = \varnothing$, i.e., $q \in L \setminus f(C)$. The proof is complete. □

Let a mapping $f : X \to Y$ between continua X and Y be given. A continuum $L \subset Y$ is said to be 1° a *continuum of confluence* for f provided that for each component K of $f^{-1}(L)$ the equality $f(K) = L$ holds; 2° a *continuum of semi-confluence* for f provided that for every two components K_1 and K_2 of $f^{-1}(L)$ either $f(K_1) \subset f(K_2)$ or $f(K_2) \subset f(K_1)$ (or both); 3° a *continuum of joining* for f provided that for every two components K_1 and K_2 of $f^{-1}(L)$ we have $f(K_1) \cap f(K_2) \neq \varnothing$; 4° a *continuum of weak confluence* for f provided that there exists a component K of $f^{-1}(L)$ such that the equality $f(K) = L$ holds.

Therefore, a mapping $f : X \to Y$ between continua X and Y is confluent (semi-confluent, joining, weakly confluent) if and only if each subcontinuum L of Y is a continuum of confluence (of semi-confluence, of joining, of weak confluence, respectively) for f.

Observation 2.12. Let a mapping $f : X \to Y$ between continua X and Y be given. A subcontinuum L of Y is a continuum of confluence for f if and only if for each point $x \in f^{-1}(L)$ there is a subcontinuum K of X such that $x \in K$ and $f(K) = L$.

Observation 2.13. Let a mapping $f : X \to Y$ between continua X and Y be given. A subcontinuum L of Y is a continuum of weak confluence for f if and only if there is a subcontinuum K of X such that $f(K) = L$.

Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \quad \text{for every} \quad A \in 2^X$$

and

$$C(f)(A) = f(A) \quad \text{for every} \quad A \in C(X).$$

The following two facts are known, see [**19**, Theorem (0.49.1)].

Fact 2.14. The induced mapping $2^f : 2^X \to 2^Y$ is surjective if and only if f is surjective.

Fact 2.15. The induced mapping $C(f) : C(X) \to C(Y)$ is surjective if and only if f is weakly confluent.

3. Arc approximation property. A space X is said to be *arcwise connected* if every two points of X can be joined by an arc in X.

Let a subcontinuum K of a continuum X and a point $p \in K$ be given. We say that K is *arcwise approximated at the point* p provided that there is a sequence of arcwise connected subcontinua K_n of X such that $p \in K_n$ for each $n \in \mathbf{N}$ and $K = \operatorname{Lim} K_n$. A subcontinuum K of a continuum X is said to be *arcwise approximated* provided that it is arcwise approximated at each point of K. In particular, this condition is satisfied in the case when $K \subset K_n$ for each $n \in \mathbf{N}$. Then we say that K is *strongly arcwise approximated*. In other words, a continuum $K \subset X$ is strongly arcwise approximated provided that there is a sequence of arcwise connected subcontinua K_n of X such that $K = \operatorname{Lim} K_n \subset \cap\{K_n : n \in \mathbf{N}\}$. A continuum X is said to have the (strong) arc approximation property provided every subcontinuum of X is (strongly) arcwise approximated.

As a direct consequence of the definition, we get the following characterization.

Proposition 3.1. *Let a continuum X, its subcontinuum $K \subset X$ and a point $p \in K$ be given. Then K is arcwise approximated at p if and only if for each $x \in K$ and each $\varepsilon > 0$ there is an arc $J \subset N(K, \varepsilon)$ with one end point at p and the other in $B(x, \varepsilon)$.*

Proof. If K is arcwise approximated at p and $\varepsilon > 0$, let K' be an arcwise connected continuum containing p such that $H(K, K') < \varepsilon$. Take a point $x' \in B(x, \varepsilon) \cap K'$. Then the arc J joining p and x' in K' satisfies the required conditions.

On the other hand, assume that for each point $x \in K$ and for each $\varepsilon > 0$ there is an arc $J(x) \subset N(K, \varepsilon)$ with one end point at p and the other in $B(x, \varepsilon)$. Consider the cover $\{B(x, \varepsilon) : x \in K\}$ of K and choose a finite subcover $\{B(x_i, \varepsilon) : i \in \{1, 2, \dots, n\}\}$. Then $K' = \cup\{J(x_i) : i \in \{1, 2, \dots, n\}\}$ is an arcwise connected continuum such that $H(K, K') < \varepsilon$. The proof is complete. □

Putting $K = X$ in the definition of the strong arc approximation property, we get the following fact.

Fact 3.2. If a continuum has the strong arc approximation property, then it is arcwise connected.

Proposition 3.3. *Let a continuum X, its subcontinuum $K \subset X$ and a point $p \in K$ be given, and let $r : X \to Y \subset X$ be a retraction with $p \in Y$. If K is arcwise approximated at p, then $r(K)$ is arcwise approximated at p, too.*

Proof. Let a sequence of arcwise connected continua K_n converging to the continuum K be given with $p \in K_n$ for each $n \in \mathbf{N}$. Then $r(K_n)$ are arcwise connected continua containing p and converging to $r(K)$. □

Corollary 3.4. *Every retract of a continuum having the* (*strong*) *arc approximation property has the* (*strong*) *arc approximation property.*

The above corollary can be generalized as follows.

Theorem 3.5. *Let a continuum X have the* (*strong*) *arc approximation property, and let a mapping $f : X \to Y$ be weakly confluent. Then Y has the* (*strong*) *arc approximation property.*

Proof. Take a continuum $L \subset Y$ and a point $q \in L$. Denote by K a subcontinuum of X with $f(K) = L$ and by p a point of K with $f(p) = q$. Let arcwise connected continua containing p (containing K) converge to K. Then $f(K_n)$ are arcwise connected continua containing q (containing L) and converging to L. □

Proposition 3.6. *Let a continuum X be given, and let K be a subcontinuum of X. If every point of K has an arbitrarily small arcwise connected closed neighborhood, then K is strongly arcwise approximated.*

Proof. Given a positive number ε, we have to show that there is an arcwise connected continuum L of X such that $K \subset L$ and $H(K, L) < \varepsilon$. By assumption, for each point q of K there exists a closed arcwise connected neighborhood $V(q)$ of q such that $V(q)$ is contained in the ε-ball about q. Thus $\{\operatorname{int} V(q) : q \in K\}$ is an open covering of K and, by compactness of K, there is a finite set of points $\{q_1, \dots, q_m\}$ of K, such that $K \subset \cup\{\operatorname{int} V(q_i) : i \in \{1, \dots, m\}\}$. Put

$$L = \cup\{V(q_i) : i \in \{1, \dots, m\}\}.$$

Note that L is an arcwise connected continuum which contains K and which is contained in the ε-ball about K in X. Therefore, $H(K, L) < \varepsilon$, and the proof is complete. □

Since each point in a locally connected continuum has arbitrarily small arcwise connected closed neighborhoods, see, e.g., [**13**, p. 254], we get the following known result, see [**7**, Lemma 2.4] as a corollary to Proposition 3.6.

Corollary 3.7. *Each locally connected continuum has the strong arc approximation property.*

The following observations are direct consequences of definitions.

Fact 3.8. If every nonempty proper subcontinuum of a continuum is arcwise connected, then the continuum has the arc approximation property, and each of its proper subcontinua is strongly arcwise approximated.

In particular, any solenoid, see, e.g., [**19**, p. 202], as well as the simplest indecomposable continuum, see, e.g., [**13**, p. 204], compare [**19**, p. 201], have the arc approximation property.

As a special case of Fact 3.8, we get the following fact.

Fact 3.9. Every hereditarily arcwise connected continuum has the strong arc approximation property.

In particular, any *dendroid*, i.e., an arcwise connected and hereditarily unicoherent continuum, see [**19**, p. 16], has the strong arc approximation property. Note that heredity of arcwise connectedness of the continuum is an indispensable assumption in Fact 3.9.

Proposition 3.10. *If a continuum X has the arc approximation property, then every arc component of X is dense.*

Proof. Suppose there is an arc component A of a point p of X that is not a dense subset of X. Then there exists a continuum K such that $p \in A \subset \operatorname{cl} A \subsetneq K \subset X$. Therefore, for each sequence of arcwise connected subcontinua K_n of X such that $p \in K_n$ for each $n \in \mathbf{N}$, we have $K_n \subset A$, and thus no such sequence converges to K, contrary to the assumption. □

Remark 3.11. The $\sin 1/x$-circle, i.e., $\operatorname{cl}\{(x, \sin(1/x)) : 0 < x \leq 1\}$ with points $(0,-1)$ and $(1, \sin 1)$ identified, is an example of an arcwise connected continuum which does not have the arc approximation property. Namely some subcontinua containing points at which the curve is not locally connected are not arcwise approximated.

Theorem 3.12. *Let a continuum X be given such that there are a subcontinuum C_0 and a family of subcontinua $\{C_\alpha : \alpha \in A\}$ having the following properties*:

$$X = C_0 \cup \bigcup \{C_\alpha : \alpha \in A\}; \tag{3.13}$$

$$C_0 \bigcap C_\alpha \neq \varnothing \quad \textit{for every} \quad \alpha \in A; \tag{3.14}$$

(3.15) *for every $\alpha \in A$ each point of the intersection $C_0 \cap C_\alpha$ has arbitrarily small arcwise connected neighborhoods in both C_0 and C_α*;

(3.16) *if a subcontinuum K of X is such that $K \cap C_\alpha \neq \varnothing \neq K \cap C_\beta$ for some α, $\beta \in A$ and $\alpha \neq \beta$, then $K \cap C_0 \neq \varnothing$;*

(3.17) *C_0 and C_α for every $\alpha \in A$ have the (strong) arc approximation property.*

Then the continuum X has the (strong) arc approximation property.

Proof. We have to show that, for every continuum $K \subset X$ and a point $p \in K$, there is a sequence of arcwise connected continua K_n containing p (containing K) and converging to K. Consider a family $\mathcal{L}$ of subcontinua of K containing the point p and such that for each member L of $\mathcal{L}$ there exists a sequence of arcwise connected continua L_n with $p \in L_n$ (with $L \subset L_n$) and $L = \operatorname{Lim} L_n$. The proof will be completed if we show that $K \in \mathcal{L}$.

Observe that the family $\mathcal{L}$ has the following properties:

(3.18) $\mathcal{L}$ is nonempty because the singleton $\{p\}$ is in $\mathcal{L}$;

(3.19) $\mathcal{L}$ is a closed subset of $C(K)$ by the standard diagonal argument;

(3.20) if $L_1, L_2 \in \mathcal{L}$, then $L_1 \cup L_2 \in \mathcal{L}$ because the union of two arcwise connected continua containing p (containing L_1 and containing L_2, respectively) is an arcwise connected continuum containing p (containing $L_1 \cup L_2$).

Note that conditions (3.18)–(3.20) imply that there is the greatest (with respect to the inclusion) element M in $\mathcal{L}$, i.e., such an element $M \in \mathcal{L}$ that if $L \in \mathcal{L}$, then $L \subset M$. We will show that $M = K$. Suppose that M is a proper subset of K. Consider two cases.

Case 1. $(K \backslash M) \cap C_0 \neq \varnothing$. Let x be a point of $M \cap \operatorname{cl}(K \backslash M) \cap C_0$. Note that (3.17) implies that M is not contained in C_0. Denote by K_0 the component of $K \cap C_0$ that contains x. Thus there exists an index $\alpha \in A$ such that $K_0 \cap M \cap C_\alpha \neq \varnothing$. Denote by b a point of $K_0 \cap M \cap C_\alpha$ and consider a sequence of arcwise connected continua M_n containing p (containing M) and converging to M. Since C_0 has the (strong) arc approximation property according to (3.17), there is a sequence of arcwise connected continua $B_n \subset C_0$ containing b (containing K_0) and converging to K_0. By (3.15) there is a sequence of arcs $D_n \subset C_\alpha$ containing the point b and tending to $\{b\}$ such that

$D_n \cap M_n \neq \varnothing$. Then $M_n \cup D_n \cup B_n$ is a sequence of arcwise connected continua each containing p (containing $M \cup K_0$) and converging to $M \cup K_0$. Thus, $M \cup K_0 \in \mathcal{L}$ and (by Case 1) properly contains M, contrary to maximality of M.

Case 2. $(K \backslash M) \cap (C_\alpha \backslash C_0) \neq \varnothing$ for some $\alpha \in A$. Let x be a point in $(K \backslash M) \cap (C_\alpha \backslash C_0)$. If $M \subset C_\alpha \backslash C_0$ then we are done. So assume $M \backslash C_\alpha \neq \varnothing$. Denote by K_0 the component of $K \cap C_\alpha$ that contains x; and denote by b a point of $K_0 \cap M \cap C_0$. Let M_n be a sequence of arcwise connected continua containing p (containing M) and converging to M. Since C_α has the (strong) arc approximation property according to (3.17), there is a sequence of arcwise connected continua $B_n \subset C_\alpha$ containing b (containing K_0) and converging to K_0. As previously, by (3.15) there is a sequence of arcs $D_n \subset C_\alpha$ containing the point b and tending to $\{b\}$ such that $D_n \cap M_n \neq \varnothing$. Then $M_n \cup D_n \cup B_n$ is a sequence of arcwise connected continua each containing p (containing $M \cup K_0$) and converging to $M \cup K_0$, contrary to maximality of the continuum M. The proof is finished. □

Remarks 3.21. We shall show that the assumptions (3.15) and (3.16) in Theorem 3.12 are necessary.

1) To see that (3.15) is essential, take as X the wedge (also called the one-point union) of a solenoid C_0 and an arc C_1:

$$X = C_0 \cup C_1 \quad \text{and} \quad C_0 \cap C_1 = \{p\}$$

and note that C_0 and C_1 have the arc approximation property while X does not by Proposition 3.10; condition (3.15) evidently does not hold.

2) Assumption (3.16) is indispensable by the following example. Define X as the cone over the pseudo-arc P. Denote by v the vertex of the cone X; let C_0 be the singleton $\{v\}$, and for each $\alpha \in P$ let C_α mean the straight line segment in X joining v and α. Then (3.13) holds, X does not have the arc approximation property, and taking $K = P$ we see that (3.16) is not satisfied.

Now we intend to discuss the arc approximation property for hyperspaces. Recall the following result, see [**19**, Theorem (1.92)].

Theorem 3.22. *The following three statements are equivalent*:

(a) *the continuum X is locally connected*:

(b) *the hyperspace 2^X is locally connected*;

(c) *the hyperspace $C(X)$ is locally connected*.

A general problem can be posed which is related in a way to Theorem 3.22.

Problem 3.23. Find necessary and/or sufficient conditions under which some implications are true between the three statements below:

(a) the continuum X has the arc approximation property;

(b) the hyperspace 2^X has the arc approximation property;

(c) the hyperspace $C(X)$ has the arc approximation property.

To see that the arc approximation property is weaker than local connectedness for hyperspaces for continua, we need an example of a not locally connected continuum X whose hyperspaces 2^X and/or $C(X)$ have the arc approximation property. It can be observed that the harmonic fan, i.e., the cone over $\{0\} \cup \{1/n : n \in \mathbf{N}\}$, is such a continuum. However, a more general result can be shown.

Theorem 3.24. *Let a continuum X be given such that there are a subcontinuum C_0 and a family of subcontinua $\{C_\alpha : \alpha \in A\}$ having the following properties*:

(3.13) $$X = C_0 \cup \bigcup\{C_\alpha : \alpha \in A\};$$

(3.14) $$C_0 \cap C_\alpha \neq \varnothing \quad \text{for every} \quad \alpha \in A;$$

(3.25) *for every $\alpha \in A$ each point of the intersection $C_0 \cap C_\alpha$ has arbitrarily small arcwise connected neighborhoods in C_α*;

(3.16) *if a subcontinuum K of X is such that $K \cap C_\alpha \neq \varnothing \neq K \cap C_\beta$ for some $\alpha, \beta \in A$ and $\alpha \neq \beta$, then $K \cap C_0 \neq \varnothing$*;

(3.26) $$\{K \in C(X) : K \cap C_0 \neq \varnothing\} \quad \textit{is locally connected};$$

(3.27) *$C(C_\alpha)$ has the (strong) arc approximation property for every $\alpha \in A$.*

Then the hyperspace $C(X)$ has the (strong) arc approximation property.

Proof. Observe that (3.16) implies

$$C(X) = \{K \in C(X) : K \cap C_0 \neq \varnothing\} \cup \bigcup \{C(C_\alpha) : \alpha \in A\}. \tag{3.28}$$

Put

$$\mathcal{H} = \{K \in C(X) : K \cap C_0 \neq \varnothing\},$$

and note that $\mathcal{H}$ is a continuum according to [**19**, p. 200]. We apply Theorem 3.12 with $C(X)$ in place of X, with $\mathcal{H}$ in place of C_0, and with $C(C_\alpha)$ in place of C_α. Thus, (3.28) stands for (3.13) in Theorem 3.12. We have to verify that all the other assumptions, i.e., (3.14)–(3.17), of that theorem are satisfied, which now run as follows.

$$\mathcal{H} \bigcap C(C_\alpha) \neq \varnothing \quad \text{for every} \quad \alpha \in A; \tag{3.29}$$

(3.30) for every $\alpha \in A$ each point of the intersection $\mathcal{H} \cap C(C_\alpha)$ has arbitrarily small arcwise connected neighborhoods in both $\mathcal{H}$ and $C(C_\alpha)$;

(3.31) if a subcontinuum $\mathcal{K}$ of $C(X)$ is such that

$$\mathcal{K} \cap C(C_\alpha) \neq \varnothing \neq \mathcal{K} \cap C(C_\beta)$$

for some $\alpha, \beta \in A$ and $\alpha \neq \beta$, then $\mathcal{K} \cap \mathcal{H} \neq \varnothing$;

(3.32) $\mathcal{H}$ and $C(C_\alpha)$ for every $\alpha \in A$ have the (strong) arc approximation property.

Indeed, (3.14) implies (3.29) because, if $x \in C_0 \cap C_\alpha$, then $\{x\} \in \mathcal{H} \cap C(C_\alpha)$. To show (3.31), assume that $\mathcal{K}$ is a subcontinuum of $C(X)$ that intersects both $C(C_\alpha)$ and $C(C_\beta)$ for some $\alpha, \beta \in A$ and $\alpha \neq \beta$. Denote by K the union $\cup\mathcal{K}$. Then $K \cap C_\alpha \neq \varnothing \neq K \cap C_\beta$, so $K \cap C_0 \neq \varnothing$ by (3.16). Whence there is a continuum L in $\mathcal{K}$ that intersects C_0, i.e., $L \in \mathcal{H}$, so $L \in \mathcal{H} \cap \mathcal{K}$, which establishes (3.31). Further, since the continuum $\mathcal{H}$ is locally connected by assumption

(3.26), it has the strong arc approximation property according to Corollary 3.7. This fact, together with (3.27) gives (3.32). So only (3.30) needs an argument. Again, since $\mathcal{H}$ is locally connected by (3.26), it has arbitrarily small arcwise connected neighborhoods of each point. To show the other half of (3.30), take a continuum K contained in $C_0 \cap C_\alpha$, and let $L \in C(C_\alpha)$ be close to K. We have to construct a small arc in $C(C_\alpha)$ that joins K and L. By (3.25) there is a small arc D in C_α with $D \cap K \neq \varnothing \neq D \cap L$. Then, according to [**19**, pp. 59, 64], there are order arcs from K to $K \cup D \cup L$ and from L to $K \cup D \cup L$. The union f these two order arcs contains a small arc joining K and L in $C(C_\alpha)$. The proof is complete. □

Remark 3.33. If the continuum C_0 is locally connected, then condition (3.26) is satisfied, see [**19**, p. 200]. This fact will be used in Corollaries 3.38–3.41, while in Example 3.36 below we shall see that local connectedness of C_0 is not necessary to show the (strong) arc approximation property for $C(X)$.

Remark 3.34. If the continuum X is hereditarily unicoherent, then condition (3.16) in Theorems 3.12 and 3.24 may be replaced by a weaker one

$$C_\alpha \cap C_\beta \subset C_0 \quad \text{for every} \quad \alpha, \beta \in A \quad \text{and} \quad \alpha \neq \beta. \tag{3.35}$$

Indeed, if a point $p \in X$ is in the intersection $C_\alpha \cap C_\beta$ for some $\alpha, \beta \in A$ with $\alpha \neq \beta$, then putting $K = \{p\}$ in condition (3.16) we infer that $p \in C_0$, and thus (3.35) holds. On the other hand, if (3.16) is not satisfied, then taking a continuum K that intersects C_α and C_β for some $\alpha, \beta \in A$ and $\alpha \neq \beta$ and is disjoint with C_0 we see that the continuum $C_0 \cup C_\alpha \cup C_\beta \cup K$ is not unicoherent because (3.35) implies that the intersection $(C_0 \cup C_\alpha \cup C_\beta) \cap K = (C_\alpha \cap K) \cup (C_\beta \cap K)$ is not connected.

Example 3.36. For each continuum P and for each zero-dimensional compact set A there exists a continuum X, its subcontinuum C_0 and a family of subcontinua $\{C_\alpha : \alpha \in A\}$ such that all assumptions of Theorem 3.24 are satisfied and yet C_0 is homeomorphic to P.

Proof. Let an arbitrary continuum P be considered as a subset of the Hilbert cube Q, and let a zero-dimensional compact set A be given. In the product $Q \times A$ define an equivalence relation $\sim$ putting

$$(x,t) \sim (y,s) \iff x = y \in P \quad \text{or} \quad x = y \quad \text{and} \quad t = s.$$

Let $X = Q \times A/\sim$, and let $\pi : Q \times A \to X$ stand for the quotient mapping. Further, for every $\alpha \in A$ define $C_\alpha = \pi(Q \times \{\alpha\})$ and $C_0 = \pi(P \times \{\alpha\})$, and note that C_0 does not depend on α, as well as that it is homeomorphic to P simply by its definition. The reader can observe that assumptions (3.13), (3.14), (3.25), (3.16) and (3.27) of Theorem 3.24 are satisfied. We show also that (3.26) holds true. As previously put

$$\mathcal{H} = \{K \in C(X) : K \cap C_0 \neq \varnothing\}$$

and note that $\mathcal{H}$ is a continuum according to [**19**, p. 200]. Let continua K and L of $\mathcal{H}$ be close to each other. We will construct a small arc in $\mathcal{H}$ joining K and L. Take a point $x_0 \in K \cap C_0$ and let a point $x_1 \in L$ be close to x_0. Then, for some $\alpha \in A$, we have $x_1 \in C_\alpha$. Since $x_0 \in C_\alpha$ too, there is a small arc D in C_α joining x_0 and x_1. Then, according to [**19**, pp. 59, 64], there are an order arc from K to $K \cup D \cup L$ and an order arc from L to $K \cup D \cup L$ in $\mathcal{H}$. The union of these two order arcs contains a small arc joining K and L in $\mathcal{H}$. So $\mathcal{H}$ is locally arcwise connected, and thus (3.26) is shown. The proof is complete. □

Corollary 3.37. *Let (X_1, p_1) and (X_2, p_2) be pointed continua which are locally connected at points p_1 and p_2, respectively. Assume that the hyperspaces $C(X_1)$ and $C(X_2)$ have the (strong) arc approximation property. Then the hyperspace of the one-point union, $C(X_1 \vee X_2)$, has the (strong) arc approximation property.*

Before we formulate further corollaries to Theorem 3.24, we recall some concepts concerning dendroids. By an *end point* of a dendroid X we mean such a point p of X that each arc containing p ends at p. A locally connected dendroid is called a *dendrite*. By a *finite dendrite* we mean a dendrite having finitely many end points. A dendroid X having exactly one *ramification point*, i.e., a point being the center of a simple triod contained in X, is called a *fan*, and the point is called the *top* of the fan. A dendroid X is said to be *smooth at a point* $p \in X$

provided that if $\{a_n : n \in \mathbf{N}\}$ is any convergent sequence of points of X, converging to a point $a_0 \in X$, then the sequence of the (unique) arcs pa_n converges to the arc pa_0. A dendroid X is said to be *smooth* provided that it is smooth at some point $p \in X$, see, e.g., [**19**, p. 117]. Thus the harmonic fan is an example of a smooth fan. The structure of the hyperspace $C(X)$ of a smooth fan X is described in a detailed way in Theorem 3.1 of [**4**, p. 282]. In particular, it is known that if X is a smooth fan with its top v, then the continuum

$$\mathcal{H} = \{K \in C(X) : v \in K\}$$

is homeomorphic to the Hilbert cube, see [**4**, Theorem 3.1]; compare [**3**, Theorem 8]. However, even if X is an arbitrary fan, not necessarily a smooth one, the structure of its hyperspace $C(X)$ can be described in a similar way. This will be presented in the next corollary to Theorem 3.24.

Corollary 3.38. *For every fan X the hyperspace $C(X)$ has the strong arc approximation property.*

Proof. It is enough to consider the fan X as the union of arcs from the top v to end points α of X. Putting $C_0 = \{v\}$ and $C_\alpha = v\alpha$ for each end point α of X, we see that all the assumptions of Theorem 3.24 are satisfied. In particular, (3.26) holds by Remark 3.33. □

Corollary 3.39. *If a dendroid X has all its ramification points in a dendrite $C_0 \subset X$, in particular if X has finitely many ramification points, then the hyperspace $C(X)$ has the strong arc approximation property.*

Corollary 3.40. *Let a dendroid X be given such that there are a dendrite C_0 and a family of dendrites $\{C_\alpha : \alpha \in A\}$ with*

(3.13) $$X = C_0 \cup \bigcup \{C_\alpha : \alpha \in A\};$$

(3.14) $$C_0 \cap C_\alpha \neq \varnothing \quad \textit{for every} \quad \alpha \in A;$$

(3.35) $$C_\alpha \cap C_\beta \subset C_0 \quad \textit{for every} \quad \alpha, \beta \in A \quad \textit{and} \quad \alpha \neq \beta.$$

Then the hyperspace $C(X)$ *has the strong arc approximation property.*

Corollary 3.41. *Let a dendroid* X *be given such that there are a dendrite* C_0 *and a family of dendroids* $\{C_\alpha : \alpha \in A\}$ *having the following properties*:

(3.13) $$X = C_0 \cup \bigcup \{C_\alpha : \alpha \in A\};$$

(3.42) $$C_0 \cap C_\alpha \quad \textit{is a singleton for every} \quad \alpha \in A;$$

(3.43) *for every* $\alpha \in A$ *the dendroid* C_α *is locally connected at the only point of the intersection* $C_0 \cap C_\alpha$;

(3.35) $$C_\alpha \cap C_\beta \subset C_0 \quad \textit{for every} \quad \alpha, \beta \in A \quad \textit{and} \quad \alpha \neq \beta;$$

(3.27) $C(C_\alpha)$ *has the* (*strong*) *arc approximation property for every* $\alpha \in A$.

Then the hyperspace $C(X)$ *has the* (*strong*) *arc approximation property.*

Proof. Note that (3.16) of Theorem 3.24 follows from hereditary unicoherence of the dendroid X, and (3.26) holds by Remark 3.33. All other assumptions are also satisfied. □

The following theorem gives a method of constructing continua with the arc approximation property.

Theorem 3.44. *Let a continuum* X *be given such that, for each point* $p \in X$, *there are a sequence of continua* $X_n \subset X$ *and a sequence of retractions* $r_n : X \to X_n$ *such that*

(3.45) $$p \in X_n \quad \textit{for every} \quad n \in \mathbf{N};$$

(3.46) *the sequence of retractions* r_n *converges to the identity, i.e.,* $x = \lim r_n(x)$ *for each point* $x \in X$;

(3.47) *X_n has the arc approximation property for every $n \in \mathbf{N}$.*

Then the continuum X has the arc approximation property.

Proof. Let K be a subcontinuum of X, and let $p \in K$. Put $K_n = r_n(K)$. According to (3.47) for every $n \in \mathbf{N}$ there are in X_n arcwise connected continua $K_{n,m}$ (where $m \in \mathbf{N}$) containing p and such that $\mathrm{Lim}_{m\to\infty} K_{n,m} = K_n$. For sufficiently large n and m, the continuum $K_{n,m}$ is close to K, so the proof is finished. □

Now we intend to show another theorem which says that if a continuum X can be approximated in a special way by a sequence of its subcontinua X_n, the hyperspaces $C(X_n)$ of which have the arc approximation property, then the hyperspace $C(X)$ has the arc approximation property. Namely, we have the following result.

Theorem 3.48. *Let a continuum X be given such that, for each point $p \in X$, there are a sequence of continua $X_n \subset X$ and a sequence of retractions $r_n : X \to X_n$ such that*

$$(3.45) \qquad p \in X_n \quad \textit{for every} \quad n \in \mathbf{N};$$

(3.46) *the sequence of retractions r_n converges to the identity, i.e., $x = \lim r_n(x)$ for each point $x \in X$;*

(3.49) *$C(X_n)$ has the arc approximation property for every $n \in \mathbf{N}$. Then the hyperspace $C(X)$ has the arc approximation property.*

Proof. Let $\mathcal{K}$ be a subcontinuum of $C(X)$, and let $p \in K \in \mathcal{K}$. We have to construct arcwise connected continua containing K and approximating $\mathcal{K}$. Put $K_n = r_n(K)$ and $\mathcal{K}_n = C(r_n)(\mathcal{K})$. According to (3.49) for every $n \in \mathbf{N}$ there are in $C(X_n)$ arcwise connected continua $\mathcal{L}_{n,m}$ (where $m \in \mathbf{N}$) containing K_n and such that $\mathrm{Lim}_{m\to\infty}\mathcal{L}_{n,m} = \mathcal{K}_n$. Thus, by (3.46), we have

$$(3.50) \qquad \mathrm{Lim}\, \mathcal{K}_n = \mathcal{K}.$$

Since $K_n \cap K \neq \varnothing$, for every $n \in \mathbf{N}$ (the intersection contains p), there is an arc $\mathcal{D}_n$ in $C(X_n)$ from K_n to K (it can be taken in the union of an order arc from K_n to $K_n \cup K$ and an order arc from K to $K_n \cup K$). Moreover, since K_n's tend to K by (3.46), we et

$$\operatorname{Lim} \mathcal{D}_n = \{K\}. \tag{3.51}$$

Thus, for every $n, m \in \mathbf{N}$, the union $\mathcal{L}_{n,m} \cup \mathcal{D}_n$ is an arcwise connected continuum containing K. For sufficiently large n and m, it is close to $\mathcal{K}$ by (3.50) and (3.51). The proof is then complete. □

Remark 3.52. We cannot conclude that $C(X)$ has the strong arc approximation property if we assume the strong arc approximation property for each $C(X_n)$. To see this, it is enough to consider the simplest indecomposable continuum X, see, e.g., [**13**, p. 204], compare [**19**, p. 201], and for each point $p \in X$ a sequence of arcs X_n containing p and tending to X. The same example shows that in Theorem 3.44 we cannot conclude that X has the strong arc approximation property if we assume the strong arc approximation property for each X_n.

Remark 3.53. Note that the continua X_n in Theorems 3.44 and 3.48 depend on the choice of the point p. It is not enough to have X_n and r_n for some point $p \in X$, or even for points p forming a dense subset of X, as can be seen from the following example.

Let X stand for the $\sin(1/x)$-curve, p an arbitrary point at which X is locally connected, and let X_n be a sequence of arcs in X containing p and tending to X. Then all the assumptions of Theorems 3.44 or 3.48 are satisfied with this choice of p, while X and $C(X)$ do not have the arc approximation property, namely, if q is a point at which X is not locally connected, then X and $F_1(X)$ are not arcwise approximated at q and at $\{q\}$, respectively.

Corollary 3.54. *Let a continuum X be given such that, for each point $p \in X$, there are a sequence of continua $X_n \subset X$ containing p and a sequence of retractions $f_n : X_{n+1} \to X_n$ such that X is homeomorphic to the inverse limit* $\operatorname{Lim\,inv}\{X_n, f_n\}$. *If $C(X_n)$ has the arc approximation property for every $n \in \mathbf{N}$, then the hyperspace $C(X)$ has the arc approximation property.*

Proof. It is enough to apply Theorem 3.48 with r_n as the projection of $\operatorname{Lim\,inv}\{X_n, f_n\}$ onto X_n. □

Remark 3.55. Again we would like to stress that the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has to be chosen for each point $p \in X$ separately, and it is not enough to have one such sequence. Really, in the previous example we can choose arcs X_n in such a way that $X_n \subset X_{n+1}$ and that X is homeomorphic to $\operatorname{Lim\,inv}\ \{X_n, f_n\}$.

To apply Theorem 3.48 to dendroids, we need the following result.

Theorem 3.56. *Let a dendroid X be smooth at a point $v \in X$, and let a point $p \in X$ be given. Then there exists a sequence of finite dendrites $X_n \subset X_{n+1} \subset X$ with $v, p \in X_n$ for every $n \in \mathbf{N}$, and a sequence of retractions $r_n : X \to X_n$ such that*

(3.46) *the sequence of retractions r_n converges to the identity, i.e., $x = \lim r_n(x)$ for each point $x \in X$.*

Proof. A universal smooth dendroid Y is described in [**18**, p. 538] as the inverse limit of an inverse sequence of finite dendrites Y_n with open bonding mappings $h_n : Y_{n+1} \to Y_n$. Since the bonding mappings may be thought of as retractions, we can consider each Y_n as a subcontinuum of Y and each projection $\pi_n : Y \to Y_n$ as a retraction. It can be observed from the construction of Y that, for every two end points e_1 and e_2 of Y, there is a homeomorphism of Y onto itself that maps e_1 onto e_2. Let g be an embedding of X into Y such that Y is smooth at $g(v)$. Denote by e an end point of Y such that $g(p) \in g(v)e$, and let h be a homeomorphism of Y onto itself that maps e onto the end point of Y_0. Put

$$X_n = g^{-1}(h^{-1}(\pi_n(h(g(X))))) \quad \text{and} \quad r_n = g^{-1} \circ h^{-1} \circ \pi_n \circ h \circ g.$$

Then all the required conditions are satisfied, and the proof is complete. □

Remark 3.57. It is not known if an arbitrary dendroid X is homeomorphic to the inverse limit of an inverse sequence of (finite) dendrites

$X_n \subset X$ with retractions as bonding mappings (compare, e.g., a remark after Theorem 2 of [**5**, p. 261], so it is not known whether Theorem 3.48 can be applied to all dendroids.

Corollary 3.58. *If a dendroid X is smooth, then the hyperspace $C(X)$ has the arc approximation property.*

Questions 3.59. a) For what dendroids X does the hyperspace $C(X)$ have the (strong) arc approximation property? b) Does $C(X)$ have the strong arc approximation property for every hereditarily arcwise connected continuum X? c) Does 2^X have the (strong) arc approximation property for every smooth fan X or for every smooth dendroid X?

Questions 3.60. Let an arcwise connected continuum X have the strong arc approximation property. Do the hyperspaces: a) $C(X)$ b) 2^X also have the strong arc approximation property?

Several other questions are related to the arc approximation property for hyperspaces 2^X and $C(X)$.

Question 3.61. For what continua X do the hyperspaces 2^X and $C(X)$ have the arc approximation property?

As the reader has certainly observed, all locally connected continua are such (by Theorem 3.22 and Corollary 3.7). In connection with Problem 3.23 we have some particular questions.

Questions 3.62. Are the three statements: (a) the continuum X has the arc approximation property, (b) the hyperspace 2^X has the arc approximation property, (c) the hyperspace $C(X)$ has the arc approximation property, equivalent? If not, what implications between (a), (b) and (c) are true?

Question 3.63. Let the hyperspace $C(X)$ of a continuum X have the arc approximation property. Does it follow that every arc component of X is a dense subset of X?

Question 3.64. Assume continua X and Y have the (strong) arc approximation property. (a) Does it follow that the product $X \times Y$ has the (strong) arc approximation property? (b) What if $Y = [0,1]$?

A continuum X is said to have the *property of Kelley* provided that, for each point $x \in X$, for each sequence of points $\{x_n\}$ converging to x and for each subcontinuum K of X containing x there exists a sequence $\{K_n\}$ of subcontinua of X containing x_n and converging to the continuum K.

To prove the next result we need the notion of a Whitney map. Let X denote a continuum. By a *Whitney map* for 2^X, or for $C(X)$, we mean any mapping $\omega : 2^X \to [0,\infty)$, or $\omega : C(X) \to [0,\infty)$, respectively, satisfying

$$\text{if} \quad A \subset B \quad \text{and} \quad A \neq B, \quad \text{then} \quad \omega(A) < \omega(B),$$

and

$$\omega(\{x\}) = 0 \quad \text{for each point} \quad x \in X.$$

The reader can find basic facts on these maps and a proof of their existence in [**19**, pp. 24–29 and 399–511].

Recall that the symbol Q stands for the Hilbert cube.

Theorem 3.65. *If a continuum X has the property of Kelley and if the product $X \times Q$ has the arc approximation property, then the hyperspace $C(X)$ has the arc approximation property.*

Proof. If X has the property of Kelley, then the space $\Lambda(X)$ of all maximal order arcs in $C(X)$ can be embedded in $X \times Q$ as a retraction of it, see [**9**, the last part of the proof of Corollary 3.3, p. 1148]. Let $r : X \times Q \to \Lambda(X) \times [0,1]$ be a retraction, and let $\omega : C(X) \to [0,1]$ be a Whitney map. Define a mapping $m : \Lambda(X) \times [0,1] \to C(X)$ by the condition: for every $\alpha \in \Lambda(X)$ and $t \in [0,1]$ the value $m(\mathcal{A}, t)$ is the only continuum $A \in \mathcal{A}$ with $\omega(A) = t$.

As is observed in [**9**, p. 1148], the mapping m is a monotone surjection. In fact, take a continuum $A \in C(X)$ and note that

$$m^{-1}(A) = \{(\mathcal{A}, \omega(A)) \in \Lambda(X) \times [0,1] : A \in \mathcal{A}\}.$$

Each order arc $\mathcal{A}$ satisfying $A \in \mathcal{A}$ is the union of two order arcs: one from $\{x\}$ to A for some point $x \in A$, and the other from A to X. Thus $m^{-1}(A)$ is homeomorphic to $\Lambda(A) \times \{\mathcal{B} \in \Lambda(X/A) : A \in \mathcal{B}\}$, so it is connected.

We have shown that the hyperspace $C(X)$ is the image of $X \times Q$ under the composition of a retraction and of a monotone mapping, thus under a weakly confluent mapping. Since the arc approximation property is preserved by weakly confluent mappings, see Theorem 3.5, the proof is complete. □

Usually it is not easy to verify whether $X \times Q$ has or does not have the arc approximation property. The next statement shows that a positive answer to Question 3.64 (b) implies that $X \times Q$ has the arc approximation property provided X does.

Statement 3.66. *If it is true that*

(3.67) *for each continuum X with the arc approximation property the product $X \times [0,1]$ has the arc approximation property, then it is also true that*

(3.68) *if a continuum X has the arc approximation property, then the product $X \times Q$ has the arc approximation property.*

Proof. Indeed, take an arbitrary point p of $X \times Q$. By homogeneity of Q, see [**10**], we can assume that p is of the form $(x, 0, 0, 0, \dots)$ for some $x \in X$. Substituting $X \times [0,1]^n$ for X_n and $X \times Q$ for X in Theorem 3.44, the conclusion follows. □

4. Confluent mappings. We begin our study of interrelations between conditions (1.1), (1.2) and (1.3) for the class $\mathfrak{M}$ of confluent mappings, recalling a known example which is due to Hiroshi Hosokawa and Kazuhiro Kawamura, see [**8**, Example 5.1].

Example 4.1. There exist continua X and Y and a confluent mapping $f : X \to Y$ such that the two induced mappings 2^f and $C(f)$ both are not pseudo-confluent.

Proof. Let (r, ϑ) denote a point of the Euclidean plane having r and ϑ as its polar coordinates. Take the unit circle

$$S = \{(1, \vartheta) : \vartheta \in [0, 2\pi]\}$$

and define two spiral lines H^1 and H^2, both homeomorphic to a closed half line, as follows:

$$H^1 = \left\{(r, \vartheta) : \vartheta = \frac{\pi}{2} \sin \frac{1}{r-1} \text{ and } r \in (1, 2]\right\},$$
$$H^2 = \left\{(r, \vartheta) : \vartheta = \frac{\pi}{2}\left(2 + \sin \frac{1}{r-1}\right) \text{ and } r \in \left[\frac{1}{2}, 1\right)\right\}.$$

Thus H^1 approaches the right semicircle $\{(1, \vartheta) : \vartheta \in [-\pi/2, \pi/2]\}$ from outside, while H^2 approaches the left one $\{(1, \vartheta) : \vartheta \in [\pi/2, 3\pi/2]\}$ from inside, and each of the two spiral lines has its turning points lying on the half lines $\vartheta = -\pi/2$ and $\vartheta = \pi/2$. Put

$$X = S \cup H^1 \cup H^2,$$

and note that X is a continuum.

Consider an equivalence relation on X such that the only nondegenerate equivalence classes are two-point sets composed of antipodal points of S, that is, equivalence classes of the relation are singletons in $H^1 \cup H^2$ and the sets of the form $\{(1, \vartheta), (1, \pi + \vartheta)\}$ for $\vartheta \in [0, \pi)$. Denote by Y the quotient space, and let $f : X \to Y = f(X)$ stand for the quotient mapping. One can define f using polar coordinates in the following way. For each point $(r, \vartheta) \in X$ we define a point $f((r, \vartheta)) = (r, 2\vartheta)$ in the plane, and we put $Y = f(X)$. Then the partial mapping $f : S : S \to S$ has two-point point inverses, while $f|H^1 : H^1 \to f(H^1)$ and $f|H^2 : H^2 \to f(H^2)$ are homeomorphisms. Therefore, the sets $f(H^1)$ and $f(H^2)$ are spiral lines, each of which approaches the whole circle $S = f(S)$, namely, $f(H^1)$ from outside and $f(H^2)$ from inside, and has its turning points all lying on the half line $\vartheta = \pi$.

Now we intend to show that the induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ both are not pseudo-confluent. We verify this for $C(f)$ first.

To this aim consider the following two arcs $\mathcal{A}^1$ and $\mathcal{A}^2$ in the hyperspace $C(S) \subset C(X)$. For $t \in \mathbf{R}$, put $A_t = \{(1, \vartheta) : \vartheta \in [t, t + \pi/4]\}$. Then the sets

$$\mathcal{A}^1 = \{A_t : t \in [-\pi/2, \pi/4]\}$$

and

$$\mathcal{A}^2 = \{A_t : t \in [\pi/2, 5\pi/4]\}$$

are the needed arcs. Note that $C(f)(\mathcal{A}^1) = C(f)(\mathcal{A}^2)$.

Now let us perform a similar construction in the spiral lines H^1 and H^2. Namely, consider subsets $\mathcal{B}^1$ and B^2 of $C(X)$, each of which is homeomorphic to the closed half line and such that each element of $\mathcal{B}^1$ (of $\mathcal{B}^2$) is contained in H^1 (in H^2, respectively), and that

$$\operatorname{cl}\mathcal{B}^1 \backslash \mathcal{B}^1 = \mathcal{A}^1 \quad \text{and} \quad \operatorname{cl}\mathcal{B}^2 \backslash \mathcal{B}^2 = \mathcal{A}^2.$$

The sets $\mathcal{C}^1 = \mathcal{A}^1 \cup \mathcal{B}^1$ and $\mathcal{C}^2 = \mathcal{A}^2 \cup \mathcal{B}^2$ are disjoint subcontinua of the hyperspace $C(X)$, each of which is homeomorphic to the familiar $\sin(1/x)$-continuum. Put $\mathcal{L} = C(f)(\mathcal{C}^1 \cup \mathcal{C}^2)$ and observe that $C(f)$ glues together the limit continua $\mathcal{A}^1$ and $\mathcal{A}^2$ of $\mathcal{C}^1$ and $\mathcal{C}^2$ only, whence it follows that $\mathcal{L}$ is an irreducible subcontinuum of $C(Y)$. Further, we have $C(f)^{-1}(\mathcal{L}) = \mathcal{C}^1 \cup \mathcal{C}^2$ and neither of $\mathcal{C}^1$ and $\mathcal{C}^2$ is mapped onto $\mathcal{L}$ under $C(f)$. Thus, $C(f)$ is not pseudo-confluent.

The reader can observe that the argument for 2^f is similar. Since the partial mappings $f|H^1$ and $f|H^2$ are one-to-one and $f|S$ is two-to-one, we infer that $(2^f)^{-1}(\mathcal{L})$ has three components: $\mathcal{C}^1$ and $\mathcal{C}^2$ as above, and the third component whose image is contained in the arc $2^f(\mathcal{A}^1) = 2^f(\mathcal{A}^2)$. Thus 2^f is not pseudo-confluent, too. The proof is finished. □

Remark 4.2. It will be shown later, in Example 4.24, that the induced mappings 2^f and $C(f)$ of the above Example 4.1 both are not joining.

Thus it is natural to ask under what conditions concerning the continua X and/or Y the implication from confluence of f to confluence of the induced mappings holds true. A partial answer to this question is presented below. The main idea of the proof is taken from one of Theorem 2.5 in [**7**, p. 3], and it exploits the following lemma which is shown in [**8**, Theorem 4.3] (for 2^f) and in [**7**, Corollary 2.2] (for $C(f)$).

Lemma 4.3. *Let a surjective mapping $f : X \to Y$ between continua X and Y be confluent. Then each arcwise connected subcontinuum of the hyperspace 2^Y (of the hyperspace $C(Y)$) is a continuum of confluence for 2^f (for $C(f)$, respectively).*

Theorem 4.4. *Let a surjective mapping $f : X \to Y$ between continua X and Y be confluent. Then the following two implications hold. If the hyperspace*

(i) 2^Y

(ii) $C(Y)$

has the arc approximation property, then the induced mapping

(i) $2^f : 2^X \to 2^Y$

(ii) $C(f) : C(X) \to C(Y)$

is confluent.

Proof. The argument for both versions (i) and (ii) is essentially the same. We shall argue for (i) using the characterization of confluence from Observation 2.12. Let $\mathcal{L}$ be a subcontinuum of 2^Y, and let $K \in (2^f)^{-1}(\mathcal{L})$. Since 2^Y has the arc approximation property by assumption, there are arcwise connected continua $\mathcal{L}_n$ containing $f(K)$ and such that $\mathcal{L} = \operatorname{Lim} \mathcal{L}_n$. Let $\mathcal{K}_n$ be the component of $(2^f)^{-1}(\mathcal{L}_n)$ containing K. Now Lemma 4.3 implies that $2^f(\mathcal{K}_n) = \mathcal{L}_n$. Taking a subsequence, if necessary, we can assume that the sequence $\{K_n\}$ is convergent. Let $\mathcal{K} = \operatorname{Lim} \ \mathcal{K}_n$. Then $\mathcal{K}$ is a continuum contained in 2^X containing K and such that $2^f(\mathcal{K}) = \mathcal{L}$. The proof is complete. □

As a consequence of Theorem 4.4 and Corollary 3.7, we get a well-known result, see [**8**, Theorem 4.4] and [**7**, Theorem 2.5].

Corollary 4.5. *Let a surjective mapping $f : X \to Y$ between continua X and Y be confluent. If Y is locally connected, then*

(i) $2^f : 2^X \to 2^Y$ *is confluent;*

(ii) $C(f) : C(X) \to C(Y)$ *is confluent.*

Let us accept the following definition. A subcontinuum K of a continuum X is said to be *weakly arcwise approximated* provided that there is a sequence of arcwise connected subcontinua K_n of X such that $K = \operatorname{Lim} K_n$. A continuum X is said to have the *weak arc approximation property* provided every subcontinuum of X is weakly arcwise approximated. We will show that Theorem 4.4 cannot be strengthened by replacing the arc approximation property by the weak arc approximation property and getting the same conclusion.

Example 4.6. There are continua X_0 and Y_0 both having the weak arc approximation property, and a surjective confluent mapping $f : X_0 \to Y_0$ such that neither 2^f nor $C(f)$ are pseudo-confluent.

Proof. Let continua X and Y have the same meaning as in Example 4.1. Let H be a ray, i.e., a one-to-one continuous image of $[0, \infty)$. There exists a compactification X_0 of H such that the remainder $\operatorname{cl} H \setminus H$ is just X, and moreover that each subcontinuum of X is approximated by a sequence of arcs contained in H, see [**6**, Theorems 3.3 and 3.5]. Consider, as in Example 4.1, an equivalence relation on X the only nondegenerate equivalence classes are two-point sets composed of antipodal points of S, that is, equivalence classes of the relation are singletons in $H^1 \cup H^2 \cup H$ and the sets of the form $\{(1, \vartheta), (1, \pi + \vartheta)\}$ for $\vartheta \in [0, \pi)$. Denote by Y_0 the quotient space, and let $f : X_0 \to Y_0 = f(X_0)$ stand for the quotient mapping. Arguing exactly as in Example 4.1 one can verify that neither 2^f nor $C(f)$ are pseudo-confluent, while both X_0 and Y_0 have the weak arc approximation property. □

Let us recall that the property of Kelley is closely related to confluent mappings, see [**20**, Theorem 2.2 and 4.2] and that the property of Kelley is weaker than local connectedness [**19**, p. 538]. So it could be interesting to know if the property of Kelley can be substituted in place of local connectedness in Corollary 4.5. The negative answer can be seen from the next example.

Example 4.7. There are continua X and Y both having the property of Kelley, and a confluent surjection $f : X \to Y$ such that neither 2^f not $C(f)$ are pseudo-confluent.

Proof. In polar coordinates (r, ϑ) in the plane, put

$$H^1 = \{(r, \vartheta) : r = 1 + \vartheta^{-1} \text{ and } \vartheta \in [1, +\infty)\},$$
$$H^2 = \{(r, \vartheta) : r = 1 - \vartheta^{-1} \text{ and } \vartheta \in (-\infty, -1]\},$$

and $S = \{(1, \vartheta) : \vartheta \in [0, 2\pi]\}$, and define $X = S \cup H^1 \cup H^2$. Observe that H^1 and H^2 are rays approximating the circle S smoothly from outside, H^1 and H^2 are mutually symmetric with respect to the straight line $\vartheta = \pm\pi$ under the symmetry $(r, \vartheta) \to (r, -\vartheta)$, and that

$$H^1 \cap H^2 = \{(1 + (2k\pi)^{-1}, 0) : k \in \{1, 2, 3, \dots\}\}$$
$$\cup \{(1 + ((2k + 1)\pi)^{-1}, \pi) : k \in \{0, 1, 2, \dots\}\}.$$

One can verify that X has the property of Kelley.

Consider again, as in Examples 4.1 and 4.6, an equivalence relation on X the only nondegenerate equivalence classes are two-point sets composed of antipodal points of S, that is, equivalence classes of the relation are singletons in $H^1 \cup H^2$ and the sets of the form $\{(1, \vartheta), (1, \pi + \vartheta)\}$ for $\vartheta \in [0, \pi)$. Denote by Y the quotient space, and let $f : X \to Y = f(X)$ stand for the quotient mapping. To verify it is confluent, observe that the preimage of a proper subcontinuum of $f(S)$ has two components both contained in S, each of which is mapped onto the considered continuum. Preimages of other subcontinua of Y are connected.

Now we intend to show that the induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ both are not pseudo-confluent. We verify this for $C(f)$ first.

To this aim consider the following two arcs $\mathcal{A}^1$ and $\mathcal{A}^2$ in the hyperspace $C(S) \subset C(X)$. For $t \in \mathbf{R}$, put $A_t = \{(1, \vartheta) : \vartheta \in [t, t + \pi/4]\}$. Then the sets

$$\mathcal{A}^1 = \{A_t : t \in [0, 3\pi/4]\} \quad \text{and} \quad \mathcal{A}^2 = \{A_t : t \in [\pi, 7\pi/4]\}$$

are the needed arcs. Note that $C(f)(\mathcal{A}^1) = C(f)(\mathcal{A}^2)$.

Denote by D^1 and D^2 the upper and the lower halves of the continuum X, respectively, i.e., (in the Cartesian rectangular coordinates (x, y) in the plane)

$$D^1 = X \cap \{(x, y) \in \mathbf{R}^2 : y \geq 0\}$$

and

$$D^2 = X \cap \{(x, y) \ni \mathbf{R}^2 : y \leq 0\}.$$

Consider subsets $\mathcal{B}^1$ and $\mathcal{B}^2$ of $C(X)$, each of which is homeomorphic to the closed half line, and such that each element of $\mathcal{B}^1$ (of $\mathcal{B}^2$) is nondegenerate and contained in D^1 (in D^2, respectively), and that

$$\operatorname{cl} \mathcal{B}^1 \backslash \mathcal{B}^1 = \mathcal{A}^1 \quad \text{and} \quad \operatorname{cl} \mathcal{B}^2 \backslash \mathcal{B}^2 = \mathcal{A}^2.$$

The sets $\mathcal{C}^1 = \mathcal{A}^1 \cup \mathcal{B}^1$ and $\mathcal{C}^2 = \mathcal{A}^2 \cup \mathcal{B}^2$ are disjoint subcontinua of the hyperspace $C(X)$, each of which is homeomorphic to the familiar $\sin(1/x)$-continuum. Put $\mathcal{L} = C(f)(\mathcal{C}^1 \cup \mathcal{C}^2)$ and observe that $C(f)$ glues together the limit continua $\mathcal{A}^1$ and $\mathcal{A}^2$ of $\mathcal{C}^1$ and $\mathcal{C}^2$ only, whence it follows that $\mathcal{L}$ is an irreducible subcontinuum of $C(Y)$. Further, we have $C(f)^{-1}(\mathcal{L}) = \mathcal{C}^1 \cup \mathcal{C}^2$ and none of $\mathcal{C}^1$ and $\mathcal{C}^2$ is mapped onto $\mathcal{L}$ under $\mathcal{C}(f)$. Thus, $C(f)$ is not pseudo-confluent.

Exactly as in the proof of Example 4.1, the reader can observe that the argument for 2^f is similar. Since the partial mappings $f|D^1$ and $f|D^2$ are one-to-one and $f|S$ is two-to-one, we infer that $(2^f)^{-1}(\mathcal{L})$ has three components: $\mathcal{C}^1$ and $\mathcal{C}^2$ as above, and the third component whose image is contained in the arc $2^f(\mathcal{A}^1) = 2^f(\mathcal{A}^2)$. Thus 2^f is not pseudo-confluent, too. The proof is finished. □

Now we will show that there is a confluent mapping f between some continua such that $C(f)$ is, while 2^f is not confluent. We start with a proposition concerning a property of confluent mappings.

Proposition 4.8. *Let a continuum Y, its subcontinuum $L \subset Y$ and a point $y \in L$ be given. If a mapping $f : X \to Y$ between continua X*

and Y is confluent, then the number of components of $f^{-1}(L)$ is less than or equal to the number of components of $f^{-1}(y)$.

Proof. Assign to each component of $f^{-1}(y)$ a component of $f^{-1}(L)$ containing it. Since f is confluent, the assigning is well-defined and onto. $\square$

The following fact is well-known [**19**, p. 30], but the special description of the hyperspace of the circle given in its proof will be used in the sequel.

Fact 4.9. The hyperspace $C(S)$ of the unit circle

$$S = \{(1, \vartheta) : \vartheta \in [0, 2\pi]\}$$

is homeomorphic to the unit disk

$$D = \{(r, \vartheta) : r \in [0, 1] \text{ and } \vartheta \in [0, 2\pi]\}.$$

If A is a proper subcontinuum of S, then let $m(A)$ be the midpoint of the arc A and let $l(A)$ be the length of A. Assign to A the point $(1 - l(A)/2\pi, m(A)) \in D$, and assign to S the point 0.

Two lemmas will be needed to show the result. Let $\mathbf{C}$ stand for the complex plane.

Lemma 4.10. *Let $D = \{z \in \mathbf{C} : |z| \leq 1\}$ be the unit disk, and let $s : D \to D$ be the central symmetry defined by $s(z) = -z$. If a continuum $K \subset D$ satisfies the condition $K \cap s(K) = \emptyset$, then K is contractible in $D \backslash \{0\}$.*

Proof. Denote by W the component of $D \backslash K$ containing the center 0 of D, and note that $0 \in W \cap s(W)$. If K is not contractible in $D \backslash \{0\}$ and $K \cap s(K) = \emptyset$, then either $s(K) \subset W$ or $K \subset s(W)$. Observe that the two inclusions are equivalent by taking the mapping s for both members of the inclusion. If $s(K) \subset W$, then also $s(W) \subset W$, whence $W \subset s(W)$, and so $s(W) = W$. Since $\operatorname{bd} W \subset K$ and since

$s(\operatorname{bd} W) = \operatorname{bd} s(W)$, the mapping s being a homeomorphism, we infer that $\varnothing \neq \operatorname{bd} W = \operatorname{bd} s(W) \subset K \cap s(K)$, a contradiction. □

Put in polar coordinates (r, ϑ)

$$H = \{(r, \vartheta) : r = 1 + \vartheta^{-1} \text{ and } \vartheta \in [1, \infty)\}$$

and

$$S = \{(1, \vartheta) : \vartheta \in [0, 2\pi]\}.$$

Lemma 4.11. *If a continuum $C \subset S \times [0,1] \subset (S \cup H) \times [0,1]$ is contractible in $S \times [0,1]$, then there is an open set $U \subset (S \cup H) \times [0,1]$ such that the component of U containing C is contained in $S \times [0,1]$.*

Proof. If C is contractible in $S \times [0,1]$, then there is in $S \times [0,1]$ an open set G containing C such that G is homeomorphic to an open disk. Then C is contained in an open set $U \subset (S \cup H) \times [0,1]$ homeomorphic to $G \times \{0, 1, 1/2, 1/3, \dots\}$ satisfying the conclusion of the lemma.

Example 4.12. There are continua X and Y and a confluent mapping $f : X \to Y$ such that $C(f)$ is confluent while 2^f is not.

Proof. Let $X = S \cup H$, where S and H are defined above. Consider again an equivalence relation on X (the only nondegenerate equivalence classes are two-point sets composed of antipodal points of S), that is, equivalence classes of the relation are singletons in H and the sets of the form $\{(1, \vartheta), (1, \pi + \vartheta)\}$ for $\vartheta \in [0, \pi)$. Denote by Y the quotient space, and let $f : X \to Y = f(X)$ stand for the quotient mapping. To verify it is confluent observe that the preimage of a proper subcontinuum of $f(S)$ has two components both contained in S, each of which is mapped onto the considered continuum. Preimages of other subcontinua of Y are connected.

To verify that 2^f is not confluent, consider the continuum $F_1(Y) \subset 2^Y$. Then $(2^f)^{-1}(F_1(Y))$ has two components: $F_1(X)$ and $\{\{x, -x\} : x \in S\}$. The latter is mapped onto $F_1(f(S))$ which is a proper subset of $F_1(Y)$. Thus 2^f is not confluent.

Now we will prove that $C(f)$ is confluent. Let $\mathcal{L} \subset C(Y)$ be any continuum. Consider three cases.

(1) $\mathcal{L} \subset C(f(S))$. Note that the partial mapping $C(f)|C(S)$ is confluent by Corollary 4.5 (ii), so we are done.

(2) $\mathcal{L} \subset C(f(H))$. Since $f|H$ is one-to-one, we see that $C(f)^{-1}(\mathcal{L})$ is connected.

To finish the proof, we have to consider the last case.

(3) $\mathcal{L} \cap C(f(S)) \neq \varnothing \neq \mathcal{L} \cap C(f(H))$. In this case we will show that again $C(f)^{-1}(\mathcal{L})$ is connected. Suppose on the contrary that there are nonempty, closed and disjoint sets $\mathcal{A}$ and $\mathcal{B}$ such that $C(f)^{-1}(\mathcal{L}) = \mathcal{A} \cup \mathcal{B}$. Then their images $C(f)(\mathcal{A})$ and $Cf(\mathcal{B})$ are nonempty and closed sets, the union of which is $\mathcal{L}$, so they cannot be disjoint. Thus there is a continuum L belonging to $C(f)(\mathcal{A}) \cap C(f)(\mathcal{B})$. Let $K_1 \in \mathcal{A}$ and $K_2 \in \mathcal{B}$ be such that

$$L = f(K_1) = f(K_2). \tag{4.13}$$

We claim that

$$L \quad \text{is a proper subcontinuum of} \quad f(S). \tag{4.14}$$

Indeed, if not, then either $L \subset f(H)$ or $f(S) \subset L$. If $L \subset f(H)$ or L contains $f(S)$ as a proper subset, then $C(f)^{-1}(L)$ is a singleton in $C(X)$ by Proposition 4.8, contrary to the assumption (4.13); if $L = f(S)$, then $C(f)^{-1}(L)$ is homeomorphic to $\{(r, \vartheta) : r \in [0, 1/2] \text{ and } \vartheta \in [0, 2\pi]\}$ using the homeomorphism described in the proof of Fact 4.9, so it is connected, contrary to (4.13). So (4.14) is shown.

Denote by $\mathcal{C}$ the component of $C(f(S)) \cap \mathcal{L}$ containing L. Thus $\mathcal{C}$ is a proper subcontinuum of $\mathcal{L}$. We will show that

$$C(f)^{-1}(\mathcal{C}) \quad \text{has exactly two components.} \tag{4.15}$$

Since the partial mapping $f|S : S \to f(S)$ is confluent by its definition, the induced mapping $C(f)|C(S) : C(S) \to C(f(S))$ is also confluent according to Corollary 4.5 (ii). Thus, by Proposition 4.8, the inverse image $C(f)^{-1}(\mathcal{C})$ has at most two components. It has at least two components by (4.13) and (4.14). So (4.15) is shown.

Denote by $\mathcal{K}$ and $\mathcal{K}'$ the two components of $C(f)^{-1}(\mathcal{C})$, and note that each of them satisfies the assumptions of Lemma 4.10, so they are contractible in $C(S)\setminus\{S\}$ according to Fact 4.9. Therefore, the component of $C(S)\setminus(\mathcal{K}\cup\mathcal{K}')$ containing the point S intersects $F_1(S)$. Thus there is an arc $\mathcal{F}$ contained in $C(S)$ with one end point in $F_1(S)$ and the other at S and such that $\mathcal{F}\cap(\mathcal{K}\cup\mathcal{K}')=\varnothing$. Hence, $C(f(S))\setminus C(f)(\mathcal{F})$ is an open set in $C(f(S))$ containing $\mathcal{C}$. Since $f(S)\in C(f)(\mathcal{F})$, we infer that $\mathcal{C}$ is contractible in $C(f(S)\setminus\{f(S)\}$.

Because $f(S)$ is not in $\mathcal{C}$, there is a small open ball $\mathcal{G}$ about $f(S)$ in $C(f(S))$, and therefore we can define an embedding e of $C(f(S))\setminus\mathcal{G}$ into $f(S)\times[0,1]$ by $e(P)=(m(P),1-l(P)/2\pi)$, where $m(P)$ is the midpoint of an arc P in the circle $f(S)$ and $l(P)$ is its length (compare the proof of Fact 4.9). Continua close to elements of $C(f(S))\setminus\mathcal{G}$ are arcs contained in $f(H)$ or in $f(S)$, so the embedding e can be extended to an embedding e^* of some neighborhood of $C(f(S))\setminus\mathcal{G}$ into $Y\times[0,1]$. One can verify that all the assumptions of Lemma 4.11 are satisfied with $e^*(\mathcal{C})$ in place of C, $f(S)$ in place of S and $f(H)$ in place of H. So there is an open in $C(Y)$ neighborhood $\mathcal{U}$ of $\mathcal{C}$ such that $e^*(\mathcal{U})$ satisfies the conclusion of the lemma. Considering $(e^*)^{-1}$ we see that the component $\mathcal{V}$ of $\mathcal{U}$ containing $\mathcal{C}$ is contained in $C(f(S))$.

Denote by $\mathcal{D}$ the component of $\mathcal{L}\cap\mathcal{U}$ containing $\mathcal{C}$ and observe that $\mathcal{D}\subset\mathcal{V}\subset C(f(S))$. Note that $(\mathrm{cl}\,\mathcal{D})\cap(\mathrm{bd}\,\mathcal{U})\neq\varnothing$ by the boundary bumping theorem, see [**19**, p. 626], so $\mathcal{C}$ is a proper subset of $\mathcal{D}$, which contradicts the definition of $\mathcal{C}$. This finishes the proof. □

A continuum is said to be *indecomposable* provided that it is not the union of two proper subcontinua. It is known, see [**13**, p. 207], that a continuum is indecomposable if and only if each proper subcontinuum has empty interior. Let us recall characterizations of hereditarily indecomposable continua in terms of confluent mappings, see, e.g., [**16**, p. 53], and of structure of hyperspaces, see, e.g., [**19**, p. 111].

Theorem 4.16. *The following conditions are equivalent for a continuum Y:*

(i) *Y is hereditarily indecomposable*;

(ii) *each mapping $f: X\to Y$ of a continuum X onto Y is confluent*;

(iii) *each confluent mapping $f : X \to Y$ of a continuum X onto Y is hereditarily confluent*;

(iv) *the hyperspace $C(Y)$ is uniquely arcwise connected.*

Further, let us recall that each hereditarily indecomposable continuum has the property of Kelley [**20**, Theorem 3.1].

Example 4.17. There are two hereditarily indecomposable continua X and Y and a surjective (confluent) mapping $f : X \to Y$ such that 2^f is not confluent.

Proof. Denote by S the pseudo-circle and recall that there is a special embedding of S into the plane such that S is preserved under the central rotation of the plane by the angle $\pi/3$, see [**11**, Sections 1 and 2]. Consequently, there is a homeomorphism $h : S \to S$ of S onto itself such that the composition $h \circ h$ is the identity on S. Replace one point of the pseudo-arc P by the pseudo-circle S, see [**1**, p. 35], and let X be the obtained space. Note that since S is nowhere dense in X, the continuum X is hereditarily indecomposable. Define an equivalence relation on X, the only nondegenerate equivalence classes of which are sets of the form $\{x, h(x)\}$ for all $x \in S$. Denote by Y the quotient space, and let $f : X \to Y$ be the quotient mapping. We will verify that f is confluent. Indeed, if a continuum is contained in $f(S)$, then its preimage has two components, each of which is mapped onto the continuum; otherwise, the preimage is connected. The continuum Y is hereditarily indecomposable as a confluent image of a hereditarily indecomposable continuum X.

We will show that 2^f is not confluent. Note that $(2^f)^{-1}(F_1(Y))$ has two components: $F_1(X)$ and $\{\{x, h(x)\} : x \in S\}$. The latter one is not mapped onto $F_1(Y)$. The proof is complete. □

Questions 4.18. Let $f : X \to Y$ be the mapping of Example 4.17. a) Is $C(f)$ confluent? b) Is 2^f weakly confluent?

Question 4.19. More generally, let $f : X \to Y$ be any (confluent) mapping between hereditarily indecomposable continua. a) Is $C(f)$

confluent? b) Is 2^f weakly confluent?

Now let us discuss implications in the opposite direction: from confluence of the induced mappings 2^f and/or $C(f)$ to one of f. It can be shown that not only both of these implications are true but, moreover, confluence of the mapping f is a consequence of a much weaker assumption that either of the two induced mappings is joining. Namely, we have the following result.

Theorem 4.20. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping either*

(i) $2^f : 2^X \to 2^Y$ *or*

(ii) $C(f) : C(X) \to C(Y)$

is surjective and joining, then f is confluent.

Proof. Argument for both versions (i) and (ii) is essentially the same. We shall show (ii). Suppose on the contrary that f is not confluent. Thus, by Proposition 2.11, there is a continuum $L \subset Y$ and a component K_0 of $f^{-1}(L)$ such that $f(K_0)$ is a nondegenerate proper subcontinuum of L.

Let $p \in L \setminus f(K_0)$. Then, according to Facts 2.3 and 2.4, there are in $C(Y)$ an order arc $\mathcal{L}_1$ from $\{p\}$ to L and an order arc $\mathcal{L}_2$ from $f(K_0)$ to L. It follows from the definition of $\mathcal{L}_2$ that

$$\mathcal{L}_2 \cap F_1(L) = \varnothing. \tag{4.21}$$

Since the space $F_1(L)$ of singletons of L is a continuum, see Proposition 2.2, we see that the union $\mathcal{L} = F_1(L) \cup \mathcal{L}_1 \cup \mathcal{L}_2$ is a subcontinuum of $C(L)$. Since $f(K_0) \subset L$, we have $C(f)(F_1(K_0)) \subset F_1(L) \subset \mathcal{L}$. Further, since $F_1(K_0)$ is a continuum (again by Proposition 2.2), there is a component $\mathcal{K}_1$ of $C(f)^{-1}(\mathcal{L})$ which contains $F_1(K_0)$. We claim that

$$C(f)(\mathcal{K}_1) = F_1(f(K_0)). \tag{4.22}$$

Indeed, the union $\cup\mathcal{K}_1$ is a continuum in X containing K_0 and contained in $f^{-1}(L)$. Since K_0 is a component of $f^{-1}(L)$, we have

$\cup\mathcal{K}_1 = K_0$, whence it follows that $\mathcal{K}_1 \subset C(K_0)$. Therefore, $C(f)(\mathcal{K}_1)$ is a continuum contained in $\mathcal{L}$, containing $F_1(f(K_0))$ by the definition of $\mathcal{K}_1$, and such that $C(f)(\mathcal{K}_1) \subset C(f)(C(K_0)) \subset C(f(K_0))$. Note that the only continuum having all these properties of $C(f)(\mathcal{K}_1)$ is the continuum $F_1(f(K_0))$ itself. Thus, (4.22) is shown.

Similarly, there is a component $\mathcal{K}_2$ of $C(f)^{-1}(\mathcal{L})$ which contains the singleton $\{K_0\}$. And again we claim that

$$C(f)(\mathcal{K}_2) = \{f(K_0)\}. \tag{4.23}$$

Indeed, the union $\cup\mathcal{K}_2$ is a continuum in X containing K_0 and contained in $f^{-1}(L)$. Since K_0 is a component of $f^{-1}(L)$, we have $\cup\mathcal{K}_2 = K_0$, whence it follows that $\mathcal{K}_2 \subset C(K_0)$. Therefore, $C(f)(\mathcal{K}_2)$ is a continuum contained in $\mathcal{L}$, containing $\{(f(K_0)\}$ by the definition of $\mathcal{K}_2$, and such that $C(f)(\mathcal{K}_2) \subset C(f)(C(K_0)) \subset C(f(K_0))$. Note that the only continuum having all these properties of $C(f)(\mathcal{K}_2)$ is the singleton $\{(f(K_0)\}$. Thus (4.23) is shown.

Now (4.22) and (4.23) imply that $C(f)(\mathcal{K}_1) \subset F_1(L)$ and $C(f)(\mathcal{K}_2) \in \mathcal{L}_2$, whence it follows by (4.21) that $C(f)(\mathcal{K}_1) \cap C(f)(\mathcal{K}_2) = \varnothing$, contrary to the assumption that $C(f)$ is joining. The proof is finished. □

The converse implication to that of Theorem 4.20 is not true, that is, confluence of f does not imply that any of the two induced mappings 2^f or $C(f)$ is joining. Namely, the following statement holds true.

Example 4.24. Let $f : X \to Y$ be the confluent mapping of Example 4.1. Then the two induced mappings 2^f and $C(f)$ both are not joining.

Proof. We keep all the notation of Example 4.1. Moreover, for $t \in \mathbf{R}$ put $A_t^* = \{(1, \vartheta) : \vartheta \in [t, t + \pi/2]\}$, and

$$\mathcal{A}^3 = \{A_t^* : t \in [-\pi/2, 0]\} \quad \text{and} \quad \mathcal{A}^4 = \{A_t^* : t \in [\pi/2, \pi]\}.$$

Similarly, let $\mathcal{B}^3$ and $\mathcal{B}^4$ be subsets of $C(X)$ each of which is homeomorphic to the closed half line, and such that each element of $\mathcal{B}^3$ (of $\mathcal{B}^4$) is contained in H^1 (in H^2, respectively), and that

$$\operatorname{cl} \mathcal{B}^3 \setminus \mathcal{B}^3 = \mathcal{A}^3 \quad \text{and} \quad \operatorname{cl} \mathcal{B}^4 \setminus \mathcal{B}^4 = \mathcal{A}^4.$$

Further, we require that $\mathcal{B}^1 \cap \mathcal{B}^3 = \varnothing$ and that $\mathcal{B}^2 \cap \mathcal{B}^4$ is a singleton which corresponds to the common end point of the considered half lines. Then each of the sets $\mathcal{C}^3 = \mathcal{A}^3 \cup \mathcal{B}^3$ and $\mathcal{C}^4 = \mathcal{A}^4 \cup \mathcal{B}^4$ is also homeomorphic to the $\sin(1/x)$-continuum, and

$$\mathcal{C}^i \cap \mathcal{C}^j \neq \varnothing \Longleftrightarrow i = j \quad \text{or} \quad \{i, j\} = \{2, 4\}.$$

Put $\mathcal{M} = C(f)(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3 \cup \mathcal{C}^4)$ and observe that $C(f)$ glues together the limit continua $\mathcal{A}^1$ and $\mathcal{A}^2$ of $\mathcal{C}^1$ and $\mathcal{C}^2$, and the limit continua $\mathcal{A}^3$ and $\mathcal{A}^4$ of $\mathcal{C}^3$ and $\mathcal{C}^4$, respectively. Further, we have

$$C(f)^{-1}(\mathcal{M}) = \mathcal{C}^1 \cup (\mathcal{C}^2 \cup \mathcal{C}^4) \cup \mathcal{C}^3,$$

and we see that $\mathcal{C}^1$, $\mathcal{C}^2 \cup \mathcal{C}^4$ and $\mathcal{C}^3$ are components of $C(f)^{-1}(\mathcal{M})$. The images of the components $\mathcal{C}^1$ and $\mathcal{C}^3$ under $C(f)$ are disjoint, so $C(f)$ is not a joining mapping.

Since $\mathcal{C}^1$ and $\mathcal{C}^3$ are components of $(2^f)^{-1}(\mathcal{M})$ also, we see that 2^f is not joining either. The proof is complete. □

Question 4.25. Let a mapping $f : X \to Y$ be given such that the induced mapping $2^f : 2^X \to 2^Y$ is confluent. Does it imply that the induced mapping $C(f) : C(X) \to C(Y)$ is confluent?

5. Semi-confluent and joining mappings. Since each confluent mapping is semi-confluent, and each semi-confluent surjective mapping is weakly confluent, see Proposition 2.7 (a) and (b), Example 4.1 shows that semi-confluence of a mapping $f : X \to Y$ between continua X and Y does not imply semi-confluence of any of the two induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$. A similar statement holds true if the property "to be joining" is considered in place of "to be semi-confluent," see Example 4.24. So it is natural to ask whether the implication holds true if the range hyperspace is assumed to have the arc approximation property, or even to be locally connected, similarly as it was shown for confluent mappings, see Theorem 4.4 and Corollary 4.5. However, it is not the case: there is no analogy for the considered implications between confluent and semi-confluent mappings. Indeed, it is enough to consider a semi-confluent (thus joining) but not confluent mapping f defined on a continuum to conclude, according to Theorem 4.20 and Proposition 2.7 (b), that neither 2^f nor $C(f)$ is joining, in particular, semi-confluent. For example, such is the mapping

$$(5.1) \qquad f : [-1, 2] \longrightarrow [0, 2] \quad \text{defined by} \quad f(t) = |t|$$

[**16**, Example 3.12]. So, even when both the domain and the range space for f is as simple as an arc, the implication does not hold.

Concerning the implications in the opposite direction, i.e., from semi-confluence of the induced mapping 2^f or $C(f)$ to semi-confluence of f, the situation has already been clarified in the above-mentioned Theorem 4.20, where a stronger conclusion of confluence of f has been obtained.

Questions 5.2. Let a mapping $f : X \to Y$ be given. Consider the following assertions:

(i) the induced mapping $2^f : 2^X \to 2^Y$ is semi-confluent,

(ii) the induced mapping $C(f) : C(X) \to C(Y)$ is semi-confluent.

(a) Does (i) imply (ii)?

(b) Does (ii) imply (i)?

Partial answers to the above questions can be derived from Theorems 4.4 and 4.20 and Proposition 2.7 (b). Namely, as direct consequences of these theorems, we infer the following two results.

Proposition 5.3. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $2^f : 2^X \to 2^Y$ is joining, and if the hyperspace $C(Y)$ has the arc approximation property, then the induced mapping $C(f) : C(X) \to C(Y)$ is confluent.*

Proposition 5.4. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $C(f) : C(X) \to C(Y)$ is surjective and joining, and if the hyperspace 2^Y has the arc approximation property, then the induced mapping $2^f : 2^X \to 2^Y$ is confluent.*

6. Weakly confluent and related mappings. Let us observe that, since each confluent mapping is weakly confluent, Example 4.1

shows that weak confluence of a mapping $f : X \to Y$ between continua X and Y does not imply weak confluence of any of the two induced mappings 2^f and $C(f)$. Even if $C(Y)$ is locally connected, we can have an example of a weakly confluent mapping $f : X \to Y$ between continua X and Y such that the induced mapping $C(f)$ is not weakly confluent. However, an argument used to show that the example is correct (which can obviously be proved directly) is a very particular case of a much more general result that is interesting by itself. Thus we prove this result first.

Theorem 6.1 *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $C(f) : C(X) \to C(Y)$ is n-weakly confluent, then the mapping f is $(n+1)$-weakly confluent.*

Proof. We proceed by induction. For $n = 0$ the implication is known, see Fact 2.15 above. Postulate that it holds for all integers $k < n$ where $n \geq 1$. We have to verify that it is true for the integer n. So, assume that the induced mapping $C(f)$ is n-weakly confluent. We will show $(n+1)$-weak confluence of f.

Take an arbitrary subcontinuum L of Y and consider the hyperspace $C(L) \subset C(Y)$. By virtue of n-weak confluence of $C(f)$, there exists a component $\mathcal{K}$ of $C(f)^{-1}(C(L))$ such that $C(f)|\mathcal{K} : \mathcal{K} \to C(L)$ is $(n-1)$-weakly confluent. Put $K = \cup\mathcal{K}$. We claim that

$$K \in \mathcal{K}. \tag{6.2}$$

Indeed, since $C(f)|\mathcal{K}$ is $(n-1)$-weakly confluent, it is surjective, see Fact 2.6, whence there is a continuum $P \in \mathcal{K}$ such that $C(f)(P) = L$. Note that $P \subset K$. Thus, there is an order arc $\mathcal{I}$ from P to K in $C(X)$. Then the image of $\mathcal{I}$ under $C(f)$ is $\{L\}$, so $C(F)(\mathcal{I} \cup \mathcal{K}) = C(L)$, and since $\mathcal{K}$ is a component of $C(f)^{-1}(C(L))$, we have $\mathcal{I} \subset \mathcal{K}$, and (6.2) follows.

Further, we claim that

$$\mathcal{K} = C(K). \tag{6.3}$$

To show the claim take an arbitrary subcontinuum A of K, and let $\mathcal{I}$ be an order arc from A to K. Then we have $C(f)(\mathcal{J} \cup \mathcal{K}) = C(L)$, and since $\mathcal{K}$ is a component of $C(f)^{-1}(C(L))$, we have $\mathcal{J} \subset \mathcal{K}$, whence $A \in \mathcal{K}$. This shows that $C(K) \subset \mathcal{K}$. The other inclusion is obvious, and so (6.3) holds true.

Therefore, we have proved that the partial mapping $C(f)|C(K) : C(K) \to C(L)$ is $(n-1)$-weakly confluent. By the inductive assumption this implies that $f|K : K \to L$ is n-weakly confluent. So, according to the definition, the mapping f is $(n+1)$-weakly confluent. The proof is finished. □

Let a mapping $f : X \to Y$ between continua X and Y be given. Define inductively:

$$C^0(X) = X \quad \text{and} \quad C^0(f) = f,$$

and, for every nonnegative integer n,

$$C^{n+1}(X) = C(C^n(X)) \quad \text{and} \quad C^{n+1}(f) = C(C^n(f)).$$

Thus, for every nonnegative integer n we have $C^n(f) : C^n(X) \to C^n(Y)$.

As a straightforward consequence of Theorem 6.1 we get the following corollary.

Corollary 6.4. *Let k, m and n be integers such that*

$$-1 \le k \le m \le n-1.$$

Consider the following conditions:

(a) $C^{n+1}(f)$ *is a surjection*;

(b) $C^{n-k}(f)$ *is* $(k+1)$*-weakly confluent*;

(c) $C^{n-m}(f)$ *is* $(m+1)$*-weakly confluent*;

(d) f *is* $(n+1)$*-weakly confluent.*

Then the following implications are true: $(a) \to (b) \to (c) \to (d)$.

Remark 6.5. The converse implication to that of Theorem 6.1 does not hold. Even in the case when f is confluent, thus $\aleph_0$-weakly confluent according to Theorem 2.8, the induced mapping $C(f)$ does not have to be 1-weakly confluent, as can be seen by Example 4.1 and Proposition 2.7 (d). Moreover, also if the continua X and Y are locally connected, the converse implication does not hold, as can be seen from an example below.

Example 6.6. There is an $\aleph_0$-weakly confluent and joining mapping f from an arc onto a circle such that neither 2^f nor $C(f)$ are pseudo-confluent (thus they are not weakly confluent).

Proof. As previously, let (r, ϑ) denote a point of the Euclidean plane having r and ϑ as its polar coordinates. Take the unit circle

$$S = \{(1, \vartheta) : \vartheta \in [0, 2\pi]\}$$

and define $f : [0,1] \to S$ by $f(t) = (1, 4\pi t)$ for each $t \in [0,1]$. Then f is $\aleph_0$-weakly confluent (because for every proper subcontinuum L of S there is a subinterval of $[0,1]$ which is mapped onto L homeomorphically) and joining.

The induced mapping $C(f) : C([0,1]) \to C(S)$ is not pseudo-confluent. In fact, let

$$A(\varphi) = \{(1, \vartheta) : \vartheta \in [\varphi, (13\varphi + \pi)/12]\} \subset S,$$

and put

$$\mathcal{L} = \{A(\varphi) : \varphi \in [-\pi, 5\pi]\} \subset C(S).$$

Then $\mathcal{L}$ is an arc in $C(S)$. Note that $f^{-1}(A(-\pi)) = \{1/4, 3/4\}$, so there are two components of $(C(f))^{-1}(\mathcal{L})$ whose images contain the point $A(-\pi)$. The reader can verify that none of them is mapped onto $\mathcal{L}$ under $C(f)$. The argument for 2^f is similar, but

$$(2^f)^{-1}(A(-\pi)) = \{\{1/4\}, \{3/4\}, \{1/4, 3/4\}\},$$

so there are three components of $(2^f)^{-1}(\mathcal{L})$ whose images contain the point $A(-\pi)$; and again none of them is mapped onto $\mathcal{L}$ under 2^f. This finishes the proof. □

If the mapping $f : X \to Y$ is n-weakly confluent only (without being $(n+1)$-weakly confluent), then $C(f)$ does not have to be n-weakly confluent, even if all considered spaces are locally connected. Namely, we have the following example.

Example 6.7. For every positive integer n there are trees X and Y and a surjective mapping $f : X \to Y$ such that f is n-weakly confluent and $C(f)$ is not n-weakly confluent.

Proof. It is known, see [**17**, p. 236], that for every positive integer n there are trees X and Y and a surjective mapping $f : X \to Y$ such that f is n-weakly confluent but not $(n+1)$-weakly confluent. Thus $C(f)$ is not n-weakly confluent by Theorem 6.1. □

As a particular case of Example 6.7, namely, for $n = 1$, we have the following one, which is described in [**17**, p. 236].

Example 6.8. There exists a weakly confluent mapping $f : X \to Y$ of the one-point union X of two triods onto a triod Y such that the induced mappings 2^f and $C(f)$ are not weakly confluent.

Proof. Let X be the union of two triods obtained by identifying some two of their end points, and let Y be a triod. A mapping $f : X \to Y$ is defined in [**17**, p. 236] which is weakly confluent while not 2-weakly confluent. Thus, $C(f)$ is not weakly confluent by Theorem 6.1. To see that 2^f is not weakly confluent, take the following subcontinuum $\mathcal{K}$ of $C(Y) \subset 2^Y$ (we keep notation as in Figure 1 of [**17**, p. 236]):

$$\mathcal{K} = \{\{y\} : y \in DV \cup VE\} \bigcup \{P \in C(Y) : P \subset EV \cup VB \text{ and } e \in P\}.$$

Then there exists exactly one point $d \in X$ with $f(d) = D$, and it can be seen that the component of $(2^f)^{-1}(\mathcal{K})$ that contains $\{d\}$ does not contain the arc $EV \cup VB$. □

Question 6.9. What are relations between n-weak confluence of a mapping $f : X \to Y$ and m-weak confluence of the induced mapping $2^f : 2^X \to 2^Y$ for nonnegative integers n and m?

A partial answer to this question is given by the following result.

Notwithstanding a stronger result will be shown later, see Theorem 7.2, we present its proof now because it is much simpler than the proof of Theorem 7.2.

Proposition 6.10. *If the induced mapping $2^f : 2^X \to 2^Y$ is weakly confluent, then the mapping $f : X \to Y$ is weakly confluent, too.*

Proof. Suppose, on the contrary, that f is not weakly confluent. Thus, there exists a subcontinuum L of Y such that no component of its inverse image $f^{-1}(L)$ is mapped onto L under f. Consequently, there are two disjoint closed subsets K_1 and K_2 of $f^{-1}(L)$ such that $K_1 \cup K_2 = f^{-1}(L)$ and $f(K_1)\backslash f(K_2) \neq \varnothing$. Let $q \in f(K_1)\backslash f(K_2)$. There is in 2^Y an order arc $\mathcal{L}$ from $\{q\}$ to L (see Fact 2.3). Since 2^f is weakly confluent, there is a continuum $\mathcal{K}$ in 2^X such that $2^f(\mathcal{K}) = \mathcal{L}$. Denote by A and B the elements of $\mathcal{K}$ such that $2^f(A) = \{q\}$ and $2^f(B) = L$. Since $\mathcal{K} \subset C(X)$, see Fact 2.4, the sets A and B are subcontinua of X with $f(A) = \{q\}$ and $f(B) = L$. Thus, we may assume that $A \subset K_1$ and $B \cap K_2 \neq \varnothing$. Define

$$\mathcal{K}_1 = \mathcal{K} \cap 2^{K_1} \quad \text{and} \quad \mathcal{K}_2 = \mathcal{K} \bigcap \{P \in 2^X : P \cap K_2 \neq \varnothing\}.$$

Thus, $\mathcal{K}_1$ and $\mathcal{K}_2$ are disjoint closed subsets of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, contrary to connectedness of $\mathcal{K}$. The proof is complete. □

Observation 6.11. Let continua X and $Y \subset X$ be given. If $r : X \to Y$ is a retraction, then both $2^r : 2^X \to 2^Y$ and $C(r) : C(X) \to C(Y)$ are retractions, so they are $\aleph_0$-weakly confluent.

Questions 6.12. Let a mapping $f : X \to Y$ be given. Consider the following assertions:

(i) the induced mapping $2^f : 2^X \to 2^Y$ is weakly confluent,

(ii) the induced mapping $C(f) : C(X) \to C(Y)$ is weakly confluent.

(a) Does (i) imply (ii)?

(b) Does (ii) imply (i)?

7. Pseudo-confluent mappings. We start with two results that concern pseudo-confluence of the induced mappings. Observe that if

$C(f)$ is pseudo-confluent, then it is surjective by Proposition 2.7 (e), whence f is weakly confluent by Fact 2.15. So we have the following result.

Theorem 7.1. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $C(f) : C(X) \to C(Y)$ is pseudo-confluent, then the mapping f is weakly confluent.*

A similar result for 2^f, which we intend to prove now, forms a stronger version of Proposition 6.10. Its proof, which is more complicated than that of 6.10, uses Whitney maps for 2^Y.

Theorem 7.2. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $2^f : 2^X \to 2^Y$ is pseudo-confluent, then the mapping f is weakly confluent.*

Proof. Assume on the contrary that 2^f is pseudo-confluent while f is not weakly confluent, i.e., there is a continuum $L \subset Y$ such that no component of $f^{-1}(L)$ is mapped onto L. Denote by d the metric in Y and by H the Hausdorff metric in 2^Y induced by d, and put

$$t_0 = (1/2) \cdot \inf\{H(f(K), L) : K \text{ is a component of } f^{-1}(L)\}.$$

We claim that $t_0 > 0$. In fact, if t_0 was zero, then there would be a sequence of components of $f^{-1}(L)$ whose images under f were converging to L and then the limit of a subsequence would be a continuum mapped onto L under f. Thus, the claim is shown.

Denote by $\omega : 2^Y \to [0, \infty)$ a Whitney map. Continuity of ω implies that there is a number $t^* > 0$ such that

$$(7.3) \qquad \text{for each} \quad B \in 2^Y \quad \text{if} \quad \omega(B) < t^*, \quad \text{then} \quad \operatorname{diam} B < t_0.$$

Let $\{a_1, a_2, \dots\}$ be a dense subset of L, and let $\{t_n\}$ be an increasing sequence of numbers tending to t^*. For each $n \in \mathbf{N}$ define subcontinua L_n and L'_n of L such that

$$a_n \in L_n, \qquad L_n \subset L'_n, \quad \omega(L_n) = t_n, \qquad \omega(L'_n) = t_{n+1}.$$

Denote by $\mathcal{D}_n$ an order arc in $C(L) \subset C(Y)$ from L_n to L'_n. Since $\omega^{-1}(t_{n+1}) \cap C(L)$ is a Whitney level in $C(L)$, it is a continuum (see [**19**, Theorem (14.2)], and therefore there is a continuum

$$\mathcal{L}_n \subset \omega^{-1}(t_{n+1}) \cap C(L)$$

irreducible between L'_n and L_{n+1}. Then

$$\mathcal{L} = \operatorname{cl} \bigcup \{\mathcal{L}_n \cup \mathcal{D}_n : n \in \mathbf{N}\}$$

is an irreducible subcontinuum of $C(L)$. Since 2^f is pseudo-confluent, there is a continuum $\mathcal{K} \subset 2^X$ which is mapped onto $\mathcal{L}$ under 2^f. Denote by K a component of $f^{-1}(L)$ such that $K \cap (\cup \mathcal{K}) \neq \varnothing$. Then all members of $\mathcal{K}$ are in the same component of $(2^f)^{-\lambda}(L)$, and thus by Lemma 23 of [**2**, p. 214], each of them intersects K, i.e.,

$$\text{(7.4)} \qquad \text{for each} \quad A \in 2^X \quad \text{if} \quad A \in \mathcal{K}, \quad \text{then} \quad A \cap K \neq \varnothing.$$

Because $H(f(K), L) \geq 2t_0$, density of the set $\{a_1, a_2, \dots\}$ implies that there is a number $m \in \mathbf{N}$ such that

$$\text{(7.5)} \qquad d(f(p), a_m) > t_0 \quad \text{for each point} \quad p \in K.$$

Denote by P a member of $\mathcal{K}$ such that $f(P) = L_m$. Thus, by (7.4), there is a point $x \in K \cap P$. Then, since both $f(x)$ and a_m are in $f(P)$, we have $\omega(f(P)) \geq \omega(\{f(x), a_m\})$, and because $d(f(x), a_m) > t_0$ by (7.5), we get $\omega(\{f(x), a_m\}) > t^*$ according to implication (7.3) in the definition of t^*. Consequently, $\omega(f(P)) > t^*$, contrary to the fact that $f(P) \in \mathcal{L}$. The proof is complete. □

As previously for confluent, semi-confluent and weakly confluent mappings, and also for pseudo-confluent ones, Example 4.1 solves the question concerning the implication from f to 2^f and $C(f)$ in the negative. And, again, a question can be asked concerning conditions under which pseudo-confluence of f implies pseudo-confluence of 2^f and/or $C(f)$. Let us recall that there exists a weakly confluent mapping f from an arc onto a circle such that neither 2^f nor $C(f)$ is pseudo-confluent, see Example 6.6. Example 6.6 shows that the implication from pseudo-confluence of f to pseudo-confluence of an induced mapping is not true even if all continua under consideration are locally connected. However, the continuum Y of Example 6.6 is a cyclic graph, so it would be interesting to know whether the implication holds true if these spaces are acyclic graphs. An example of a pseudo-confluent but not weakly confluent mapping of an arc onto a simple triod can be described as follows, see [**14**, Example 3.6]. Let $X = [0, 1]$ and Y be the union of three straight line segments a_0a_k for $k \in \{1, 2, 3\}$ having the point a_0 in common only. Define $f : X \to Y$ by the following conditions: $f(0) = a_1$ and f maps the intervals $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$ homeomorphically onto the arcs $a_1a_0 \cup a_0a_2$, $a_2a_0 \cup a_0a_3$ and $a_3a_0 \cup a_0a_1$, respectively. The reader can verify that the mapping f just defined is pseudo-confluent while not weakly confluent. Then $C(f)$ as well as 2^f are not pseudo-confluent by Theorems 7.1 and 7.2, respectively.

Acknowledgment. The notion of the arc approximation property and the idea of the investigation of this notion is due to Janusz J. Charatonik, to whom I am very grateful. I would like to express my sincere thanks to the referee who read the paper very carefully and suggested changes which contributed to clarifying some proofs.

REFERENCES

1. J.M. Aarts and P. van Emde Boas, *Continua as remainders in compact extensions*, Nieuw Arch. Wisk. (4) **15** (1967), 34–37.

2. W.J. Charatonik, *R^i-continua and hyperspaces*, Topology Appl. **23** (1986), 207–216.

3. C. Eberhart, *Intervals of continua which are Hilbert cubes*, Proc. Amer. Math. Soc. **68** (1978), 220–224.

4. C. Eberhart and S.B. Nadler, Jr., *Hyperspaces of cones and fans*, Proc. Amer. Math. Soc. **77** (1979), 279–288.

5. J.B. Fugate, *Small retractions of smooth dendroids onto trees*, Fund. Math. **71** (1971), 255–262.

6. J. Grispolakis and E.D. Tymchatyn, *Continua which admit only certain classes of onto mappings*, Topology Proc. **3** (1978), 347–362.

7. H. Hosokawa, *Induced mappings between hyperspaces*, Bull. Tokyo Gakugei Univ. **41** (1989), 1–6.

8. ———, *Induced mappings between hyperspaces* II, Bull. Tokyo Gakugei Univ. **44** (1992), 1–7.

9. H. Kato, *A note on continuous mappings and the property of J.L. Kelley*, Proc. Amer. Math. Soc. **112** (1991), 1143–1148.

10. O.H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

11. J. Kennedy and J.T. Rogers, Jr., *Orbits of the pseudo-circle*, Trans. Amer. Math. Soc. **296** (1986), 327–340.

12. K. Kuratowski, *Topology*, Vol. I, Academic Press and PWN, 1966.

13. ———, *Topology*, Vol. II, Academic Press and PWN, 1968.

14. A. Lelek and E.D. Tymchatyn, *Pseudo-confluent mappings and a classification of continua*, Canad. J. Math. **27** (1975), 1336–1348.

15. T. Maćkowiak, *Mappings of a constant degree*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. **23** (1975), 285–291.

16. ———, *Continuous mappings on continua*, Dissert. Math. **158** (1979), 1–95.

17. M.M. Marsh and E.D. Tymchatyn, *Inductively weakly confluent mappings*, Houston J. Math. **18** (1992), 235–250.

18. L. Mohler and J. Nikiel, *A universal smooth dendroid answering a question of J. Krasinkiewicz*, Houston J. Math. **14** (1988), 535–541.

19. S.B. Nadler, Jr., *Hyperspaces of sets*, Marcel Dekker, New York, 1978.

20. R.W. Wardle, *On a property of J.L. Kelley*, Houston J. Math. **3** (1977), 291–299.

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: wjcharat@hera.math.uni.wroc.pl

Current Address: Departamento de Matemáticas, Facultad de Ciencias, U.N.A.M., Ciudad Universitaria, 04510, México, D.F., México
E-mail address: wjcharat@lya.fciencias.unam.mx