Regular Singular Differential Equations Whose Conjugate Equation Has Polynomial Solutions

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REGULAR SINGULAR DIFFERENTIAL EQUATIONS WHOSE CONJUGATE EQUATION HAS POLYNOMIAL SOLUTIONS*

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Abstract. Consider the n-dimensional singular differential system defined by the operator $L: (Ly)(z) = z^p y'(z) + A(z)y(z)$, where z is a complex variable and p is a positive integer. The solvability of the nonhomogeneous system $Ly = g$ depends on the solutions of the homogeneous conjugate system, $L^*f = 0$, where L^* is the operator conjugate to L. We show that $L^*f = 0$ has polynomial solutions if the constant matrix in the series expansion of $A(z)$ has at least one nonpositive integer eigenvalue. Also, we show that if $L^*f = 0$ has a polynomial solution, then a finite number of the coefficients of $A(z)$ must satisfy certain properties. These results are then used to obtain a solvability condition for the nonhomogeneous Bessel equation of integer order.

1. Introduction. Let $A_{q,n}$ be the space of n-vector functions whose components are analytic in the open unit disc and q times continuously differentiable on the closed unit disc. A norm can be defined so that $A_{q,n}$ is a Banach space. Let p be a positive integer, let $A(z)$ be an $n \times n$ matrix with columns in $A_{0,n}$, and define the operator $L: A_{1,n} \to A_{0,n}$ by

$$Ly(z) = z^p y'(z) + A(z)y(z). \tag{1.1}$$

The following theorem, due to Grimm and Hall [2], states necessary and sufficient conditions for the nonhomogeneous system $Ly = g$, $g \in A_{0,n}$, to have a solution in $A_{1,n}$.

THEOREM A. *The system $Ly = g$ has a solution in $A_{1,n}$ if and only if*

$$\lim_{r \to 1^-} B(g, f; r) = 0 \tag{1.2}$$

for all f belonging to the conjugate space $A^*_{0,n}$ such that

$$\lim_{r \to 1^-} B(Ly, f; r) = 0 \tag{1.3}$$

for all y in $A_{1,n}$.

If $u(z) = \sum_{k=0}^{\infty} u_k z^k$ and $v(z) = \sum_{k=0}^{\infty} v_k z^k$ are n-vector functions analytic in the unit disc, $B(u, v; z)$ denotes the Hadamard product of u and v, i.e., $B(u, v; z) = \sum_{k=0}^{\infty} (u_k \cdot v_k) z^k$, where $u_k \cdot v_k = u_k^1 v_k^1 + u_k^2 v_k^2 + \cdots + u_k^n v_k^n$. A detailed treatment of the relationship between the Hadamard product and the space $A^*_{0,1}$ has been given by Taylor [5], and his results were extended to the vector case by Grimm and Hall [2].

Equation (1.3) characterizes $K(L^*)$, the kernel of the conjugate operator L^*, and (1.2) characterizes the annihilator of $K(L^*)$. In this paper we shall study systems for which (1.3) has polynomial solutions, and we shall also study the relationship between the regular singular property at $z = 0$ and the existence of polynomials in $K(L^*)$ for such systems.

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2. Preliminaries. We first rewrite (1.3) in a more useful form. Let $f(z)=\sum_{k=0}^{\infty} f_k z^k$, $A(z)=\sum_{k=0}^{\infty} A_k z^k$, and T denote transpose. Then (1.3) is equivalent to the following infinite system of equations:

$$\tag{2.1}\begin{aligned}&\sum_{k=0}^{\infty} f_k^T A_k = 0\\ &\sum_{k=0}^{\infty} f_{k+1}^T A_k = -f_p^T\\ &\qquad\vdots\\ &\sum_{k=0}^{\infty} f_{k+l}^T A_k = -l f_{p+l-1}^T\\ &\qquad\vdots\end{aligned}$$

Also, we shall assume that the first nonzero coefficient in the power series expansion of $A(z)$ is in Jordan normal form. This can be done without loss of generality, and will facilitate several proofs.

3. Results for nilpotent A_0. In this section we assume that the matrix A_0 is nilpotent. In this case, if the rank of A_0 is r,

$$A_0 = \operatorname{diag}\{J_1, \cdots, J_{n-r}\},$$

where each J_i is an elementary Jordan matrix of dimension ρ_i, $\sum_{i=1}^{n-r} \rho_i = n$. We can arrange the matrices J_i so that $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_{n-r}$. Hence A_0 is nilpotent of index ρ_1, and each J_i is nilpotent of index ρ_i. Now define the $n\times n$ matrices $\tilde{J}_i$, $i=1, \cdots, n-r$, as the matrices formed by replacing each elementary Jordan matrix in A_0 except J_i by the zero matrix of corresponding dimension. Clearly, each $\tilde{J}_i$ is nilpotent of index ρ_i. In case $\rho_i = 1$, define $\tilde{J}^{\rho_i-1}$ to be the $n\times n$ matrix with a one in the $((\sum_{j=1}^{i-1} \rho_j)+1, \sum_{j=1}^{i} \rho_j)$th position, and zeros elsewhere.

THEOREM 3.1. *If A_0 is as given above, and $p=1$, then there exist $n-r$ linearly independent vectors f_{0_i} such that the constant functions $f_i(z)=f_{0_i}$ belong to $K(L^*)$, $i=1, \cdots, n-r$. Also, no other polynomials belong to $K(L^*)$ in this case.*

Proof. Since the rank of A_0 is r, there exist exactly $n-r$ linearly independent vectors f_{0_i} such that $f_{0_i}^T A_0 = 0$. These vectors can be written as

$$f_{0_i} = (0, \cdots, 0, 1, 0, \cdots, 0)^T,$$

where the 1 is in the $(\sum_{j=1}^{i} \rho_j)$th position. Therefore the functions $f_i(z)=f_{0_i}$ satisfy (2.1), $i=1, \cdots, n-r$.

Now suppose that $f(z)=\sum_{k=0}^{N} f_k z^k$ belongs to $K(L^*)$ with $N \geqq 1$ and $f_N \neq 0$. Then from (2.1) f_N must satisfy $f_N^T A_0 = -N f_N^T$. This is impossible since A_0 is nilpotent. If we let $N=0$ then we obtain the same vectors as before, and so the proof is complete.

THEOREM 3.2. *Let A_0 be as given above and let $p \geqq 2$. If there exists a nonnegative integer N such that, for $i=1, \cdots, n-r$,*

$$\tilde{J}_i^{\rho_i-1} A_k = 0, \qquad k=1, \cdots, N, \quad k \neq p-1,$$

and

$$\tilde{J}_i^{\rho_i-1}[A_{p-1}+(N-p+1)I]=0 \quad (\text{if } N \geqq p-1),$$

then there exist $n-r$ linearly independent vectors f_{N_i} such that $f_i(z)=f_{N_i}z^N$, $i=1,\cdots,n-r$, belongs to $K(L^*)$.

Proof. As in the proof of Theorem 3.1 we can define the vectors

$$f_{N_i}=(0,\cdots,0,1,0,\cdots,0)^T, \qquad i=1,\cdots,n-r,$$

such that $f_{N_i}^T A_0=0$. If $N=0$, then we are done. If $N>0$, the condition $\tilde{J}_i^{\rho_i-1}A_k=0$ is equivalent to requiring that the $(\sum_{j=1}^i \rho_j)$th row of A_k is all zeros, $i=1,\cdots,n-r$, $k=1,\cdots,N$, $k\neq p-1$. Hence $f_{N_i}^T A_k=0$ for $i=1,\cdots,n-r$ and $k=1,\cdots,N$, $k\neq p-1$. Similarly, the condition $\tilde{J}_i^{\rho_i-1}[A_{p-1}+(N-p+1)I]=0$ implies that $f_{N_i}^T A_{p-1}=-(N-p+1)f_{N_i}^T$ for $i=1,\cdots,n-r$. Hence, the functions $f_i(z)=f_{N_i}z^N$ belong to $K(L^*)$, and the proof is complete.

The two preceding theorems do not, however, completely describe $K(L^*)$ unless $\dim K(L^*)=n-r$. Since we know from [2] that

$$\dim K(L^*)=n(p-1)+\dim K(L), \tag{3.1}$$

then when $p=1$, $\dim K(L^*)=n-r$ if and only if $\dim K(L)=n-r$, and when $p=2$, $\dim K(L^*)=n-r$ if and only if $r=0$ and $\dim K(L)=0$.

4. Results for systems with a regular singular point. In this section we drop the assumption that A_0 is nilpotent. However, if $p\geqq 2$ and (1.1) has a regular singular point at $z=0$, Harris [3] has shown that A_0 must be nilpotent, and so the results of § 3 will apply to such systems.

THEOREM 4.1. *Let $f(z)=\sum_{k=0}^N f_k z^k$ belong to $K(L^*)$ with $N>p-1$ and every component of f_N nonzero. Then $A(z)$ must satisfy*:

$$\text{(i)} \qquad A_k=0, \qquad k=0,\cdots,p-2,$$

$$\text{(ii)} \qquad A_{p-1}=-(N-p+1)I,$$

and

$$\text{(iii)} \qquad \sum_{k=p}^N f_k^T A_k=(N-p+1)f_{p-1}^T$$

$$\cdot \quad \cdot \quad \cdot$$

$$\sum_{k=p}^N f_{k+N-p-1}^T A_k=2f_{N-2}^T$$

$$f_N^T A_p=f_{N-1}^T. \tag{4.1}$$

Proof. We first remark that when $p=1$ condition (i) is vacuously satisfied. Since f is a solution of (2.1) we must have, for $p\geqq 2$, $f_N^T A_0=0$. Because A_0 is in Jordan normal form, each eigenvalue of A_0 must occur at least once as the only nonzero element of some column. But this means that every eigenvalue of A_0 is zero, since no component of f_N is zero and $f_N^T A_0=0$. The same argument can now be used again to show that every element of the superdiagonal of A_0 is zero and that, consequently, $A_0=0$. The same argument yields $A_1=A_2=\cdots=A_{p-2}=0$ and (i) is proved.

Counting from the bottom up, the first equation from (2.1) for this f with a nonzero right-hand side is

$$f^T_{N-p+1}A_0+\cdots+f^T_NA_{p-1}=-(N-p+1)f^T_N. \tag{4.2}$$

Condition (i) implies that (4.2) is equivalent to $f^T_N[A_{p-1}+(N-p+1)I]=0$, and the same argument as before yields $[A_{p-1}+(N-p+1)]=0$, or $A_{p-1}=-(N-p+1)I$. This proves (ii).

The remaining equations from (2.1) for this f are equivalent to (4.1). Since f belongs to $K(L^*)$, these equations must be satisfied by $A_p, \cdots, A_N$ and so (iii) holds.

The next theorem provides conditions which guarantee that $K(L^*)$ contains only polynomials. Further, these polynomials will be constructed.

THEOREM 4.2. *In* (1.1) *let* $p=1$ *and let* $A_0=-NI$, N *a positive integer. Then*

(i) $K(L^*)=\{f_i(z)\}$, $i=1,\cdots,n$, *where the* f_i *are linearly independent polynomials of degree* N,

(ii) $\dim K(L)=n$, *and* $z=0$ *is an apparent singularity* (*see* [1]) *for* $Ly=0$.

Proof. Let $\{f_{N_i}\}$, $i=1,\cdots,n$, be an arbitrary set of n linearly independent constant vectors. We can now uniquely define the vectors $\{f_{k_i}\}$, $k=N-1,\cdots,0$, and $i=1,\cdots,n$ by the system (4.1). The functions $f_i(z)=\sum_{k=0}^N f_{k_i}z^k$, $i=1,\cdots,n$, all satisfy (2.1), and so belong to $K(L^*)$. Since the f_i are linearly independent, $\dim K(L^*)\geqq n$. From (3.1) we have $\dim K(L^*)=\dim K(L)$. But $\dim K(L)\leqq n$, so $\dim K(L^*)=\dim K(L)=n$. This proves (i). To complete the proof we note that $\dim K(L)=n$ implies that every fundamental matrix for $Ly=0$ is analytic at $z=0$. Hence $z=0$ is an apparent singularity as defined in [1].

The condition that A_0 be a multiple of the identity matrix in Theorem 4.2 is quite restrictive. In the next theorem we find that a weaker hypothesis still guarantees the existence of polynomials in $K(L^*)$. However, these polynomials no longer span $K(L^*)$.

THEOREM 4.3. *Let* A_0 *have a nonpositive integer eigenvalue, let* $-N$ *be the largest such eigenvalue, and let* m *be the number of linearly independent eigenvectors of* A_0^T *corresponding to* $-N$. *Then if* $z=0$ *is a regular singular point for* $Ly=0$, *there exist* m *linearly independent polynomials of degree* N *which belong to* $K(L^*)$.

Proof. If $p\geqq 2$, then, since $z=0$ is a regular singular point, A_0 is nilpotent, and so we have $N=0$. Hence, if the rank of A_0 is r, then $m=n-r$ and we can apply Theorem 3.2.

Assume $p=1$. Then $Ly=0$ always has a regular singular point at $z=0$. Let $\{f_{N_i}\}$, $i=1,\cdots,m$, be m linearly independent eigenvectors of A_0^T corresponding to the eigenvalue $-N$, and therefore satisfying $f^T_{N_i}A_0=-Nf^T_{N_i}$. Now successively define the vectors $\{f_{k_i}\}$, $k=N-1,\cdots,0$ and $i=1,\cdots,m$, by

$$f^T_{k_i}=\left(-\sum_{j=1}^{N-k} f^T_{(j+k)_i}A_j\right)(A_0+kI)^{-1}. \tag{4.3}$$

These vectors are uniquely defined for each i in terms of f_{N_i} and $A(z)$ since $-N$ is the largest nonpositive integer eigenvalue of A_0. Hence the m polynomials $f_i(z)=\sum_{k=0}^{N} f_{k_i}z^k$, $i=1,\cdots,m$, belong to $K(L^*)$ and the proof is complete.

We shall now give two theorems which provide necessary and sufficient conditions for $K(L^*)$ to contain a nontrivial polynomial in the regular singular and irregular singular cases, respectively.

THEOREM 4.4. *Let $z=0$ be a regular singular point for $Ly=0$. Then $K(L^*)$ contains a nontrivial polynomial if and only if A_0 has a nonpositive integer eigenvalue.*

Proof. Assume $K(L^*)$ contains a nontrivial polynomial of degree N. If $p=1$, the coefficient of z^N, call it f_N, must satisfy $f_N^TA_0=-Nf_N^T$. Hence $-N$ is an eigenvalue of A_0. If $p\geqq 2$, A_0 is nilpotent by the result of Harris mentioned before, and so zero is an eigenvalue of A_0.

The converse is a direct application of Theorem 4.3.

THEOREM 4.5. *Let $z=0$ be an irregular singular point for $Ly=0$. Then $K(L^*)$ contains a nontrivial polynomial if and only if A_0 is singular.*

Proof. Since $z=0$ is an irregular singular point, we must have $p\geqq 2$.

Assume $f(z)=\sum_{k=0}^{N} f_kz^k$ belongs to $K(L^*)$, with $N\geqq 0$ and $f_N\neq 0$. Then (2.1) implies that $f_N^TA_0=0$ and hence A_0 is singular.

Assume A_0 is singular. Then there exists a nonzero vector, f_0, such that $f_0^TA_0=0$. Let $f(z)=f_0$. The quantity f satisfies (2.1) and hence belongs to $K(L^*)$.

5. Examples. To illustrate some of the preceding results, we consider the following linear second order equation, with a, b, and g in $A_{0,1}$:

$$z^2y''+za(z)y'+b(z)y=g(z). \tag{5.1}$$

This equation clearly has a regular singular point at $z=0$ and, as a system, has the form

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} y^1 \\ y^2 \end{bmatrix}'+\begin{bmatrix} 0 & -1 \\ b(z) & a(z)-1 \end{bmatrix}\begin{bmatrix} y^1 \\ y^2 \end{bmatrix}=\begin{bmatrix} 0 \\ g(z) \end{bmatrix}, \tag{5.2}$$

where $y^1=y$ and $y^2=zy'$. Hence the matrix A_0 is given by

$$A_0=\begin{bmatrix} 0 & -1 \\ b(0) & a(0)-1 \end{bmatrix}. \tag{5.3}$$

A_0, as given in (5.3), is not in Jordan normal form. However, since the eigenvalues of a constant matrix are invariant under similarity transformations, (5.3) will be used to calculate the eigenvalues of A_0. If J is the Jordan normal form of A_0, then there exists a nonsingular matrix P such that $J=P^{-1}A_0P$. In the remainder of this section, L will be the operator corresponding to the system obtained from (5.2) after A_0 has been converted to Jordan normal form.

If $b(0)=0$ and $a(0)=1$, then A_0 will be nilpotent of index two, and we can apply Theorem 3.1. In this case, any constant two-dimensional vector whose first component is zero belongs to $K(L^*)$, and so, by Theorem A, if $g(0)\neq 0$, then (5.1) is not solvable in $A_{1,2}$.

If $a(z)=1$ and $b(z)=z^2-\nu^2$, then (5.1) becomes the nonhomogeneous Bessel equation of order ν. In this case the eigenvalues of A_0, from (5.3), are $-\nu$

and ν and so, if ν is a positive integer, $K(L^*)$ contains a polynomial of degree ν. Moreover, this polynomial spans $K(L^*)$ since, from (3.1), $\dim K(L^*) = \dim K(L)$ and $\dim K(L) = 1$ for Bessel's equation. We shall now construct this polynomial using Theorem 4.3 with $\nu = N$.

In (5.2) let $Y = (y^1, y^2)^T$, let $U = PY$, and multiply both sides of (5.2) by P^{-1}, where

$$P = \begin{bmatrix} 1 & 1 \\ \nu & -\nu \end{bmatrix}.$$

System (5.2) then becomes

$$zU' + \left\{ \begin{bmatrix} -\nu & 0 \\ 0 & \nu \end{bmatrix} + \frac{1}{2\nu} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} z^2 \right\} U = G, \tag{5.4}$$

where $G = (1/(2\nu))(g, -g)^T$. The operator equation $LU = G$ will now refer to (5.4), and we shall rename A_0 so that $A_0 = \operatorname{diag}(-\nu, \nu)$.

The one linearly independent eigenvector of A_0^T corresponding to $-\nu$ can be written as $f_\nu = (1, 0)^T$. Then (4.3) yields

$$f_k^T = \frac{1}{2\nu} f_{k+2}^T \begin{bmatrix} \dfrac{1}{\nu - k} & \dfrac{-1}{\nu + k} \\ \dfrac{-1}{\nu - k} & \dfrac{1}{\nu + k} \end{bmatrix}$$

or, equivalently,

$$f_{\nu-2j}^T = \frac{1}{\nu!} \left(\frac{1}{2}\right)^{2j} \left(\frac{(\nu - j)!}{j!}, \frac{-(\nu - j - 1)!}{(j-1)!} \right), \qquad j = 1, \cdots, \left[\frac{\nu}{2}\right]. \tag{5.5}$$

Hence, $K(L^*)$ is spanned by the polynomial

$$f(z) = \begin{pmatrix} z^\nu \\ 0 \end{pmatrix} + \sum_{j=1}^{[\nu/2]} f_{\nu-2j} z^{\nu-2j},$$

where $f_{\nu-2j}$ is given in (5.5).

If we now apply Theorem A to the system (5.4) we see that (5.4) has a solution in $A_{1,2}$ if and only if ν is a positive integer, and

$$g_\nu + \sum_{j=1}^{[\nu/2]} \left(\frac{(\nu - j)! + j(\nu - j - 1)!}{\nu! j! 2^{2j}} \right) g_{\nu-2j} = 0. \tag{5.6}$$

Clearly, (5.4) has a solution in $A_{1,2}$ if and only if (5.1), with $a(z) = 1$ and $b(z) = z^2 - \nu^2$, has a solution in $A_{1,1}$.

Remark. Condition (5.6) corresponds to a condition given by Ibragimov and Kušnirčuk [4], who obtained their result by using the equivalence of the Bessel and Euler operators. They also gave a solvability condition for certain nonhomogeneous Euler equations which we can obtain by using Theorem 4.3 and Theorem A as we did with Bessel's equation.

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