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Design of multivariable deadbeat controllers using reduced order models

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DESIGN OF MULTIVARIABLE DEADBEAT CONTROLLERS
USING REDUCED ORDER MODELS

BY

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ABSTRACT

This study is concerned with the design of minimum Frobenius norm deadbeat controllers using reduced order models. Deadbeat controllers force the non-zero initial states of a linear multivariable discrete time system to the origin of the state space in minimum time. A controller of minimum Frobenius norm achieves such response with least effort.

The deadbeat controllers are designed on the basis of eigenspectrum assignment. In general, this approach permits a freedom in the selection of the closed-loop Jordan matrix which is used in the design of parametric controllers.

In this presentation, it is demonstrated that the task of deadbeat controller design assignment can be achieved via reduced order models. The computational requirement involved with the selection of the free parameters that yield a minimum Frobenius norm deadbeat controller are greatly simplified.

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I. INTRODUCTION

In the past two decades the topic of deadbeat control has attracted the attention of various researchers. The deadbeat control concept is unique to discrete time systems. A controller which drives all the states of a multivariable linear discrete time control system to the origin of the state space in the minimum number of time steps and with zero steady state error is called a deadbeat controller [1].

For a single-input single-output (SISO) system the design of deadbeat controllers is well established and available in standard textbooks [1,2]. The salient approaches for design of deadbeat controllers for multivariable systems are based on controllability concepts and eigenspectrum assignment. The methods available are the Kalman deadbeat controller [3,4], the Ludyk and Leden controller [5,6], the Kucera controller [7], and the Tou, Farrison and Fu controller [8], among others. The eigenvalue assignment approaches are presented by Ackermann and Prepilita [9], Pachter and Ichikawa [10], and Fahmy and O'Reilly [11,12,13,14]. Even though Fahmy and O'Reilly base their work principally on closed-loop eigenstructure, they have provided a unified approach for the design of deadbeat controllers.

In multivariable deadbeat design, the parametric controller is of significant importance. This controller is designed on the basis of closed loop eigenstructure assignment. In this approach a number of free parameters are to be selected by minimizing a Frobenius norm. As such, the control effort is reduced to a minimum. The selection of free parameters requires a considerable computational effort. In this thesis a procedure is developed to design a parametric controller using reduced order models. A considerable computational simplicity is accomplished in the proposed method.

The problem in determining a constant feedback gain matrix that places the eigenvalues of a multivariable system to new locations has

been researched considerably [12]. Algorithms for the design of such controllers are based on an n^{th} order discrete time linear system with the assumption that all open loop eigenvalues need to be relocated. However, in the field of large scale control systems, techniques have been developed to relocate only d out of the n values to the new locations. Using these techniques the remaining (n-d) eigenvalues remain at their original locations. In this thesis a procedure has been developed to relocate all eigenvalues by use of a recursive method.

A brief outline of the thesis is as follows. The important results of deadbeat controller design methods and reduced order models are presented in chapter II. Chapter III contains a detailed description of multivariable deadbeat controller design techniques based on eigenspectrum assignment. The design procedure of parametric controllers by minimizing the Frobenius norm is included in this chapter.

The parametric controller design procedure using reduced order models is presented in chapter IV. This procedure is illustrated with an example. The results of the original system are compared with the reduced order model approach. It is interesting to note that the deadbeat controller designed by reduced order models gave almost the same closed loop system response. Chapter V contains the conclusions and suggestions for future work.

II. REVIEW OF LITERATURE

A number of control systems are designed with the purpose of having an output response that reaches the desired steady-state value in a minimum time and stays at that value thereafter. In such systems, the steady-state value is attained with zero overshoot. This type of response, generally referred to as deadbeat response, is unique to discrete-time systems. The important results of SISO and multivariable deadbeat controller design techniques are reviewed in this chapter [1,2,3]. An effort has been made to design multivariable deadbeat controllers using reduced order models. A brief review of the dominant eigenvalue reduction technique is also presented [8].

A. DESIGN OF DEADBEAT CONTROLLERS FOR SISO SYSTEMS

The design of digital control systems with deadbeat response for SISO systems is reviewed in this section. The digital control system configuration is shown in figure 1.

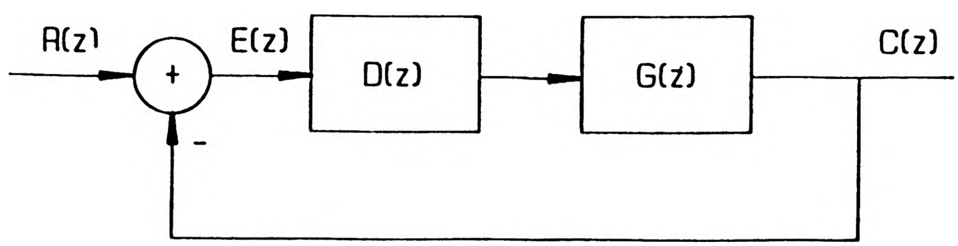

Figure 1. A Typical Digital Control System.

For a unit step input R(z), the desired deadbeat response C(z) sampled at every time interval T, is given in Fig. 2.

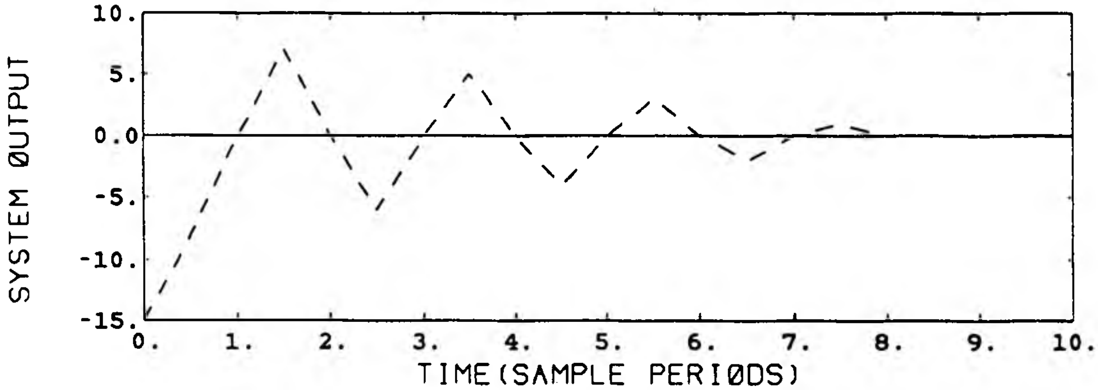

Figure 2. Example of Deadbeat Time Response.

The desired output transform of the closed-loop system is given by

$$C(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots = \frac{1}{(z-1)} \tag{2.1}$$

and the transfer function of the closed-loop system is written in terms of the digital controller D(z) and the z-transform of the plant G(z) as

$$M(z) = \frac{C(z)}{R(z)} = \frac{D(z)G(z)}{[1+D(z)G(z)]} \tag{2.2}$$

Then, the deadbeat controller is designed with the following criteria :

1. The system must have zero steady state error at the sampling instants for the specified reference input signal.

2. Response time should be minimum.

3. Settling time should be finite.

An expression for the digital controller D(z), can be evaluated from (2.2) in the following manner

$$D(z) = \frac{M(z)}{[1-M(z)]G(z)} \tag{2.3}$$

The closed-loop transfer function M(z) is dependent on the input to the system and the desired output response. A general form for the inputs of the type t^{N-1} can be written as

$$R(Z) = \frac{A(z)}{(1-z^{-1})^N} \tag{2.4}$$

where A(z) is a polynomial in z^{-1} with no zeros at $z = 1$ and N is a positive integer. For example, for a unit-step function input, $A(z) = 1$ and $N = 1$; for a unit-ramp function input $A(z) = Tz^{-1}$ and $N = 2$.

The steady state error of a closed loop-system is defined by :

$$e_{ss} = \lim_{T\to\infty} e(kT) = \lim_{z\to 1} (1-z^{-1})E(z) \tag{2.5}$$

For a system configuration shown in Fig. 2.1, the steady-state error transform E(z) is given by

$$E(z) = [1-M(z)]R(z) \tag{2.6}$$

Substitution of (2.4) and (2.6) in (2.5) yields

$$e_{ss} = \lim_{z\to 1} (1-z^{-1})^{1-N} [1-M(z)] A(z) \tag{2.7}$$

Since the polynomial A(z) does not contain any zeros at $z = 1$, the necessary condition for the steady-state error to be zero is that [1-M(z)] must contain the factor $(1-z^{-1})^N$. Thus, [1-M(z)] takes the

form

$$[1-M(z)] = (1-z^{-1})^N F(z) \tag{2.8}$$

where, F(z) is a polynomial in z^{-1}. Solving for M(z) in the preceding equation one obtains

$$M(z) = \frac{z^N - (z-1)^N F(z)}{z^N} \tag{2.9}$$

The fact that F(z) is a polynomial in z^{-1} indicates that the closed-loop system has poles only at $z = 0$.

Substitution of (2.9) in (2.3) yields the deadbeat controller D(z) as

$$D(z) = \frac{z^N - (z-1)^N F(z)}{(z-1)^N F(z)} \frac{1}{G(z)} \tag{2.10}$$

For the design and implementation of the deadbeat controller D(z), the following comments are to be taken into consideration.

1. Inexact Pole-Zero Cancellation : It is evident from (2.3) and (2.9) that the deadbeat response is realized by designing a controller with pole-zero cancellations. Attempts at cancelling poles and zeros of G(z) which lie outside the unit circle should be avoided, for each dipole arising from an inexact cancellation, a root may be generated which falls outside the unit circle.

2. Non-Realizable Controller : The controller of (2.3) should be a causal controller. A non causal controller, responding to anticipatory information, can not be implemented.

3. Input Based Controller : A compensator designed on the presented technique is for a particular type of input. If the input is changed, the closed-loop system does not yield the desired response.

B. MULTIVARIABLE DEADBEAT CONTROLLERS

There have been several approaches to the design of a state feedback controller which drives a linear multivariable discrete-time system from any arbitrary initial condition to the origin of the state space. A controller designed in such way that forces each state to the origin, in the minimum possible number of time-steps and zero steady-state error is termed a deadbeat controller.

Most of the early research in the theory of the deadbeat controller was focused on the time-optimal control problems. There are two salient approaches taken by researchers. One is based on system controllability and the other on eigenvalue assignment. The methods available are the Kalman deadbeat controller, the Ludyk and Leden controller, the Kucera controller, and the Tou, Farison and Fu controller, among others [3]. Approaches based on eigenvalue assignment are presented by Ackermann and Prepilita, Pachter and Ichikawa [3], and Fahmy and O'Reilly [3,4,5,6]. Even though Fahmy and O'Reilly base their work principally on closed-loop eigenstructure, they have provided a unified approach for the design of deadbeat controllers.

In this section, a brief review of time optimal-deadbeat controller design based on system controllability is presented. The derivation is in the manner of the original contributions of Kalman and Tou [2]. A detailed discussion of deadbeat controller design methods based on eigenstructure assignment is presented in Chapter III.

Consider a finite dimensional, shift invariant, linear discrete-time system described by the difference equation

$$x(i+1) = Ax(i) + Bu(i) \tag{2.11}$$

$$x(0) = x_0 \tag{2.12}$$

where $x(i) \in R^n$ is the state vector, $u(i) \in R^r$ is the control vector, and the input matrix B is assumed to be of full rank. The initial conditions are taken as arbitrary. A necessary and sufficient condition for (2.11) to be completely p state reachable at some time p_1, where $(p \leq p_1 \leq n)$, is that

$$\text{rank}[B \ AB \ A^2B \ \ldots \ A^{p-1}B] = n, \quad \text{for some } p \leq n \tag{2.13}$$

The criterion (2.13) is a sufficient but not necessary condition for complete p-step controllability. If A is invertible, a necessary and sufficient condition for the system of (2.11) to be completely p-state controllable is that

$$\text{rank}[A^{-1}B \ A^{-2}B \ \ldots. \ A^{-p}B] = n, \text{ for some } p \leq n \tag{2.14}$$

Depending on how it is used, p is called the controllability [7] or reachability index. In the present approach, A is assumed to be nonsingular and p is the controllability index.

Consider the linear discrete system (2.11) to be controllable with controllability index p. It is possible to show that

$$\begin{aligned}
x(1) &= Ax(0) + Bu(0) \\
x(2) &= Ax(1) + Bu(1) = A^2x(0) + [B \ AB][u(1) \ u(0)]^T \\
x(3) &= Ax(2) + Bu(2) = A^3x(0) + [B \ AB \ A^2B][u(2) \ u(1) \ u(0)]^T \\
&\ \ \vdots \\
x(p_1) &= A^{p_1}x(0) + [B, AB, A^2B, \ldots A^{p_1-1}B][u(p_1-1), \ldots u(0)]^T
\end{aligned} \tag{2.15}$$

Then

$$f_{p_1}(A,B) = \{x(0) = -[A^{-1}Bu(0) + \ldots + A^{-p_1}Bu(p_1-1)]\} \tag{2.16}$$

and for some input vector $[u(0),u(1),\ldots,u(p_1)]$, it can be observed from (2.18) that

$$f_{p_1} = \text{range}[A^{-1}B,\ldots,A^{-p_1}B] \tag{2.17}$$

The function f_{p_1} has the following properties :

$$f_0 = \{0\} \tag{2.18}$$

$$f_{p_1+1} = A^{-1}(f_{p_1} + \text{range}(B)) \tag{2.19}$$

$$f_0 \subset f_1 \subset f_2 \subset \ldots . \; R^n \tag{2.20}$$

Moreover, from (2.14) and (2.17) it is implied that $f_{p1} = f_p$ for all $p_1 \geq p$. This relation was first noted by Kalman [3] for single input systems. It means that the initial state may be transferred to the origin in p time-steps and remains thereafter for $p_1 \leq p$.

By choosing an ordered selection of $\{D_{p1}\}$ for (2.17) such that

$$\text{range}[A^{-1}B,\ldots,A^{-p_1}B] = \text{range}[A^{-1}BD_1,\ldots,A^{-p}BD_p] \tag{2.21}$$

then, the initial condition x(0) is written as

$$x(0) = -[A^{-1}BD_1,\ldots,A^{-p}BD_p] \; [v(0),\ldots,v(p-1)]^T \tag{2.22}$$

where

$$u(0) = D_1v(0), \; u(1) = D_2v(1),\ldots, \; u(p-1) = D_pv(p-1) \tag{2.23}$$

In a more compact form, (2.22) is

$$x(0) = -S[v(0),\ldots,v(p-1)]^T \tag{2.24}$$

from (2.24)

$$[v(0),\ldots,v(p-1)]^T = -S^{-1}x(0) = [M_1,\ldots,M_p]^T x(0) \tag{2.25}$$

where

$$S = [A^{-1}BD_1,\ldots A^{-p}BD_p] \tag{2.26}$$

from (2.25)

$$-[M_1,\ldots,M_p]^T S = I_n \tag{2.27}$$

The matrix M_1 is given by

$$M_1[A^{-1}BD_1,\ldots,A^{-p}BD_p] = [I_r,0,\ldots,0] \tag{2.28}$$

By combining (2.23), (2.25), and (2.28) the control input u(0) is of the form

$$u(0) = D_1v(0) = D_1M_1x(0) = Kx(0) \tag{2.29}$$

where K is a rxn matrix that satisfies

$$K[A^{-1}BD_1,\ldots,A^{-p}BD_p] = [-D_1,0,\ldots,0] \tag{2.30}$$

It has been shown [3] that the system of (2.11) can be driven from the initial state x(0) to the origin in p steps. Similarly, the state x(1) can be driven to the origin in (p-1) time-steps.

From (2.22), (2.24) and noting $v(p) = 0$, one obtains

$$x(1) = -S[v(1),\dots,v(p)]^T \tag{2.31}$$

$$[v(1),\dots,v(p)]^T = -S^{-1}x(1) = [M_1,\dots,M_p]^T x(1) \tag{2.32}$$

and

$$u(1) = Kx(1) \tag{2.33}$$

In general it can be written as

$$u(i) = Kx(i), \quad \text{for } i = 1,2,\dots,(p-1) \tag{2.34}$$

In conclusion, the discrete-time system of (2.11) with controllability index p, and having an ordered selection of $D_1,D_2,\dots,D_p$ has a feedback controller of the form (2.34). Such a feedback controller drives all the non-zero states to the origin in the minimum possible time.

If the linear state-feedback controller of (2.34) is applied to the system (2.11), the resultant closed-loop system is :

$$x(i+1) = (A + BK)x(i) \tag{2.35}$$

The time response at the p^{th} time step is given by

$$x(p) = (A + BK)^p x(0) = 0 \tag{2.36}$$

Equation (2.36) implies that the closed-loop system matrix (A + BK) is a nilpotent matrix of order p [5,6,7].

The characteristic equation of the closed-loop system is given by [4]

$$CP(z) = Det(zI_n-A-BK) = Det(zI_n-A)Det[I_n-(zI_n-A)^{-1}BK] = 0 \tag{2.37}$$

$$CP(z) = z^p = 0 \tag{2.38}$$

Although, the technique presented here for the design of multivariable deadbeat controllers may not seem difficult, the procedure does not give insight of the freedom available in choosing the controller.

C. DOMINANT EIGENVALUE REDUCED ORDER MODEL

It is often desirable to choose a reduced-order model of a dynamic system which reflects a significant portion of the behavior of the original system. This concept is called aggregation [8]. In control systems, aggregation is of great interest because it offers the possibility of having simplified models.

The aggregated model description is considered satisfactory if for a given class of inputs, the aggregated outputs are good approximations to the original outputs of the large model. Both, the original and aggregated models use the same input-output information, although their dynamic descriptions are different.

Many methods for deriving reduced-order models for large systems are available in the literature [14]. In general these methods are based either on the state-space description or on a frequency domain description. In this investigation, the interest lies in the discrete-time state space methods.

Consider a linear multivariable discrete-time system described by the state-space equation :

$$x(i+1) = Ax(i) + Bu(i) \tag{2.39}$$

where $x(i) \in R^n$ is the state vector and $u(i) \in R^r$ is the control input at the i^{th} time-sample. The matrices A, and B are constant with dimensions nxn and nxr respectively. It is desired to replace the large

model description of (2.39) by a satisfactory aggregated model. Using the modal matrix M, a linear transformation is introduced

$$x(i) = Mw(i) \tag{2.40}$$

where the modal matrix M [8] of A is found from the eigenvectors and generalized eigenvectors of A. For asymptotically stable discrete systems, the eigenvalues lie within the unit circle in the complex plane. Therefore, the significant eigenvalues with large magnitudes, are located near the circumference and correspond to the first few columns on the left (eigenvectors) of the modal matrix. Also, the insignificant eigenvalues, with small magnitudes, are grouped close to the origin of the complex plane.

Writing (2.40) in partitioned form :

$$\begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix} = \begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix} \begin{bmatrix} w_1(i) \\ w_2(i) \end{bmatrix} \tag{2.41}$$

From (2.40)

$$w(i) = M^{-1}x(i) \tag{2.42}$$

then

$$w(i+1) = M^{-1}Ax(i) + M^{-1}Bu(i) \tag{2.43}$$

$$w(i+1) = M^{-1}AMw(i) + M^{-1}Bu(i) \tag{2.44}$$

The transformed system can be rewritten as

$$w(i+1) = Jw(i) + Kw(i) \tag{2.45}$$

where

$$J = M^{-1}AM \tag{2.46}$$

$$\Gamma = M^{-1}B \tag{2.47}$$

Equation (2.46) is the Jordan form [5,8,10] of A. The diagonal elements of J contain the eigenvalues the matrix A, from left to right, in descending order of dominance.

Partitioning (2.45)

$$\begin{bmatrix} w_1(i+1) \\ w_2(i+1) \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} w_1(i) \\ w_2(i) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} u(i) \tag{2.48}$$

Due to the arrangement of the modal matrix M, the canonical state subvector $w_1(i)$ is associated with the dominant modes (contained in J_1) which contribute to the system behavior over a longer period. On the other hand, non-dominant eigenvalues contained in the Jordan form J_2 die out after a short period. A technique to reduce the order of the model, by retaining the dominant modes of the system, shall be derived.

Let P be a (dxn) matrix of the form

$$P = [I_d \mid 0] \tag{2.49}$$

and let the reduced order system state vector be a subset of w(i). Then

$$v(i) = [I_d \mid 0]w(i) = Pw(i) \tag{2.50}$$

From (2.45) and (2.50)

$$v(i+1) = PJw(i) + P\Gamma u(i) \tag{2.51}$$

or

$$v(i+1) = PJP^{+}v(i) + P\Gamma u(i) \tag{2.52}$$

where P^{+} is the generalized-inverse [7,8] of P given by :

$$P^{+} = P^{T}(PP^{T})^{-1} \tag{2.53}$$

The reduced-order model is given by

$$v(i+1) = J_1 v(i) + \Gamma_1 u(i) \tag{2.54}$$

where

$$J_1 = PJP^{+} \tag{2.55}$$

$$\Gamma_1 = P\Gamma \tag{2.56}$$

Now, to transform the model from (2.54) into a general form:

$$z(i) = M_0 v(i) \tag{2.57}$$

where

$$M_0 = PMP^{+} \tag{2.58}$$

From (2.54) and (2.58)

$$z(i+1) = M_0 J_1 v(i) + M_0 \Gamma_1 u(i) \tag{2.59}$$

which can be written as

$$z(i+1) = M_0 J_1 M_0^{-1} z(i) + M_0 \Gamma_1 u(i) \tag{2.60}$$

The reduced order model is represented as

$$z(i+1) = Fz(i) + Gu(i) \tag{2.61}$$

where

$$F = M_0 J_1 M^{-1}{}_0 \tag{2.62}$$

$$G = M_0\Gamma_1 \tag{2.63}$$

A relationship between the original system of (2.39) and the reduced order model of (2.61) is established by the aggregation matrix. The aggregation matrix L, is an (dxn) constant matrix of rank d. Then

$$z(i) = Lx(i) \tag{2.64}$$

From (2.40), (2.50), and (2.57) :

$$z(i) = M_0v(i) = M_0Pw(i) = M_0PM^{-1}x(i) \tag{2.65}$$

then

$$L = M_0PM^{-1} \tag{2.66}$$

The eigenspectrum of F is contained in that of A. In particular, the matrix F retains some dominant eigenvalues of A. This can be observed by letting $\{z_1,...,z_n\}$ be the set of eigenvalues of A with associated eigenvectors $\{v_1,...,v_n\}$. From (2.64), the equivalence between the models of (2.39) and (2.61) is achieved provided that

$$FL = LA \tag{2.67}$$

$$G = LB \tag{2.68}$$

and

$$z(0) = Lx(0) \tag{2.69}$$

then it follows that

$$\begin{aligned} LAv_i &= FLv_i \\ &= z_iLv_i \end{aligned} \tag{2.70}$$

indicating that if $Lv_i \neq 0$, then the vector Lv_i is an eigenvector of F with the sane eigenvalue z_i. Another property of the aggregated system matrix F is that any polynomial matrix in A, P(A), has P(F) as its aggregation. In other words, LP(A) = P(F)L [14].

In the subsequent chapter, the reduced order models are used for eigenvalue assignment.

III. DESIGN OF MULTIVARIABLE DEADBEAT CONTROLLERS

A. INTRODUCTION

A controller which drives all the states of a multivariable linear discrete-time control system to the origin of the state space in the minimum number of time-steps and with zero steady state error is termed a deadbeat controller [5]. The salient approaches to the theory of the such controllers are based either on system controllability or eigenvalue assignment. In the previous chapter an approach based on system controllability was presented. In this chapter, the design of deadbeat controllers based on eigenvalue assignment is introduced [4].

Early contributions to the design of deadbeat controllers from an eigenvalue assignment approach were given by Ackermann, Prepilita, Patcher, Ichikawa [3], Fahmy and O'Reilly [4]. The algorithms they presented may be viewed as special techniques for eigenvalue assignment. In this chapter, a deadbeat controller design approach is presented in the manner of the contributions given by Fahmy and O'Reilly [5,6].

Properties common to all multivariable deadbeat controllers are studied. The non-uniqueness of the structure of the deadbeat controllers is utilized to develop a parametric controller. A controller of minimum Frobenius norm is obtained by proper selection of the parameters. In this manner, the control effort is minimized.

B. PROBLEM FORMULATION AND ASSUMPTIONS

Consider a linear, time invariant system represented by

$$x(i+1) = Ax(i) + Bu(i) \tag{3.1}$$

where $x(i) \in R^n$ is a state vector, $u(i) \in R^r$ is a control vector and A and B are real matrices of compatible dimensions. The object of deadbeat

control is to find a time invariant state feedback controller [13].

$$u(i) = Kx(i) \tag{3.2}$$

such that the system is driven from any arbitrary initial condition x_0 to the origin of the state space in minimum time. This is achieved in p_1 time steps ($p < p_1 < n$), where p is the controllability index [15].

The closed loop-system obtained is of the form

$$x(i+1) = A_c x(i), \tag{3.3}$$

where $A_c = A + BK$

The state response at the p_1^{th} sample is given by

$$x(p_1) = (A_c)^{p_1} x(0) = 0 \tag{3.4}$$

The characteristic equation [15] of the system is

$$CP(z) = z^n \tag{3.5}$$

and the minimum polynomial [18]

$$\mu(z) = z^{p_1} \tag{3.6}$$

In the design of a deadbeat controller, three basic assumptions are being made:

(i) The system matrix A is invertible.

(ii) The input matrix B is of full rank, which means no redundant inputs are present.

(iii) The pair (A,B) is completely controllable.

C. EIGENSTRUCTURE ASSIGNMENT ON MULTIVARIABLE SYSTEMS

A very important problem in multivariable control is that of assigning a prescribed set of eigenvalues to the closed-loop system by using linear state feedback [1,13,15]. In this section, a method is developed to design a controller that achieves this task. The problem is solved for three cases :

(a) for the assignment of distinct eigenvalues

(b) for the general case, and

(c) for the specialized assignment of an all-zero eigenvalue spectrum.

The controller that assigns all the modes of the closed loop system to zero is in fact a deadbeat controller.

1. Assignment of Distinct Eigenvalues to the Closed Loop System

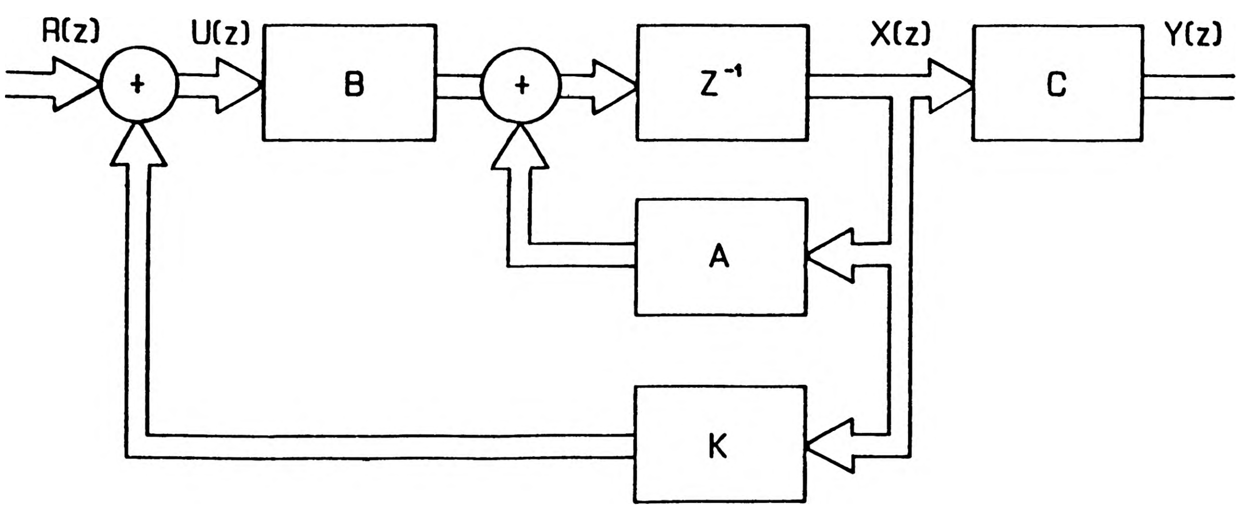

Figure 3. Realization modified by state variable feedback.

Consider the discrete-time constant linear system:

$$x(i+1) = Ax(i) + Bu(i) \qquad (3.7)$$

$$y(i) = Cx(i) \qquad (3.8)$$

where $x(i) \in R^n$ is the state vector, $u(i) \in R^r$ is the control vector and A and B are real constant matrices of appropriate sizes. The pair (A,B) is assumed to be controllable with the further restrictions that B has full rank, and A is non-singular.

The objective is to find a feedback control law :

$$u(i) = Kx(i) + r(i) \qquad (3.9)$$

such that the closed loop system is assigned an arbitrary set of eigenvalues, together with a permissible set of eigenvectors. The resultant feedback system is :

$$x(i) = A_c x(i) + Bu(i) \qquad (3.10)$$

where $$A_c = A + BK \qquad (3.11)$$

Figure 3. is a representation of the system described by equations (3.7) and (3.8) in the z-domain. By taking the z-transform of equations (3.7), (3.8) and (3.9) we have

$$zX(z) = AX(z) + BU(z) \qquad (3.12)$$

$$U(z) = KX(z) + R(z) \qquad (3.13)$$

$$Y(z) = CX(z) \qquad (3.14)$$

The transfer function is

$$\frac{Y(z)}{R(z)} = \frac{CAdj(zI_n-A-BK)^{-1}B}{det[zIn-A-BK]} \tag{3.15}$$

Let the closed-loop characteristic polynomial [4,7] of (3.15) be represented by :

$$CP(z) = Det[zI_n-A-BK] = Det[zI_n-A]Det[I_r-KS(z)] \tag{3.16}$$

where

$$S(z) = (zI_n-A)^{-1}B \tag{3.17}$$

If one considers a desired closed loop system in which the eigenvalues $\{Z_i\}$ are distinct, then the characteristic polynomial takes the form :

$$CP(z) = (z-z_1)(z-z_2)\cdots(z-z_n) = 0 \tag{3.18}$$

The minimum polynomial of A_c is equal to the characteristic polynomial.

$$\begin{gathered} Det[I_r - K(z_iI_n - A)^{-1}B] = 0 \\ \text{for } i = 1,2,..,n \end{gathered} \tag{3.19}$$

The determinant of (3.19) can be removed if there exists some vectors f_i of r-dimensions, which span the null space of $[I_r - K(z_iI_n - A)^{-1}B]$. That is,

$$[I_r - K(z_iI_n - A)^{-1}B]f_i = 0 \tag{3.20}$$

$$f_i = K(z_iI_n - A)^{-1}Bf_i \tag{3.21}$$

Using (3.17) and (3.21), the expression for f_i can be written

$$f_i = KS(z_i)f_i, \quad \text{for } i = 1,2,\ldots,n \tag{3.22}$$

in matrix notation an expression for (3.22) can be written as

$$K = FV^{-1} \tag{3.23}$$

where

$$V = [S(z_1)f_1,\ S(z_2)f_2, \ldots,\ S(z_n)f_n] \tag{3.24}$$

and

$$F = [f_1, f_2, \ldots . f_n] \tag{3.25}$$

The matrices V and F play an important role in the eigenstructure assignment of the closed-loop system.

<u>a. The free Parameter Vectors f_i</u> : Equation (3.23) is a parametric expression for K in terms of the free parameter vectors f_i. These vectors are called free parameter vectors because they are arbitrarily chosen under the sole condition that V must be invertible. That is, the n vectors $[S(z_i)f_i]$ must be linearly independent. To meet the requirement that K must be real, one needs to choose $f_i \in R^r$, for a real eigenvalue z_i, where as for a complex conjugate pair of eigenvalues $(z_i, z_j = z_i^*)$ the free parameter vector should be chosen as either $f_j = f_i \in R^r$ or $f_j = f_i^* \in C^r$ [12].

The freedom in choosing the parametric vectors f_i reflects the freedom offered by state feedback in assigning the closed loop eigenvalues.

<u>b. The Modal Matrix V of the closed loop system</u> : The solution to the real feedback gain matrix is not unique because of the freedom

available in choosing the n linearly independent vectors $[S(z_i)f_i]$. From (3.17) and (3.22) it is obvious that $[S(z_i)f_i]$ belongs to the range space of $(z_iI_n\text{-}A)^{-1}B$ [13]. Intuitively, one can see that the vectors $[S(z_i)f_i]$ are the assigned closed-loop eigenvectors v_i (cc. Appendix A), defined by the eigenvalue-eigenvector equation [15,17,18]

$$A_c v_i = z_i v_i, \quad \text{for } i = 1,2,3,\ldots,n \tag{3.26}$$

The fact that the eigenvectors $v_i = [S(z_i)f_i]$ are the eigenvectors of the closed-loop system implies that the (nxn) matrix

$$V = [v_1,v_2,\ldots v_n] \tag{3.27}$$

called the modal matrix of the closed-loop system, is not unique. As a result, one has the freedom in choosing the eigenvectors of the closed-loop system. A unique Jordan form is obtained from the modal matrix V :

$$J = V^{-1}A_cV = \mathrm{diag}(z_1,z_2,\ldots,z_n) \tag{3.28}$$

<u>2. The Case of Repeated Eigenvalues (the General Case)</u> : A more general and important problem in multivariable feedback control is the assignment of repeated eigenvalues to a closed-loop system. When distinct eigenvalues are assigned to the closed-loop system, a unique Jordan form is found from the modal matrix. However, if the eigenvalues being assigned to the closed loop system are repeated, one may not find n independent eigenvectors to form the modal matrix V. In such case the so called generalized eigenvectors are used [17]. The structure of the Jordan form in this case is dependent upon the choice of eigenvectors and generalized eigenvectors.

Consider the closed-loop characteristic equation [12]:

$$CP(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_s)^{m_s} = 0, \; s < n \tag{3.29}$$

and

$$(z-z_i)^{mi} = (z-z_i)^{p_{i1}}(z-z_i)^{p_{i2}}..(z-z_i)^{p_{ij}}..(z-zi)^{p_{iq_i}} \quad (3.30)$$

where m_i is the algebraic multiplicity of z_i, $i = 1,2,...,s$, and $m_1 + m_2 +...+ m_s = n$. There are q_i equations corresponding to the eigenvalue Z_i and the j^{th} equation is of order p_{ij}. A requirement can be made for convenience $p_{i1} \geq p_{i2} \geq..\geq p_{iqi}$. One can observe that $m_i = p_{i1} + p_{i2} +..+ p_{iqi}$. The eigenvalues of the characteristic polynomial are the desired ones for the closed loop system and they satisfy the following

$$CP(z_i) = 0 \ , \quad \text{for } i = 1,2,....s \quad (3.31)$$

due to the multiplicity of z_i

$$\frac{d^k z(z_i)}{dz^k} = 0 \ , \quad \text{for } k = 1,2,....m_i\text{-}1 \quad (3.32)$$

Equations (3.31) and (3.32) yield a set of n independent vector equations as in equation (3.20). With this set of equations, K is found in the following manner :

$$\mathrm{Det}[I_r - K(z_i I_n - A)^{-1}B] = 0 \quad (3.33)$$

$$[I_r - K(z_i I_n - A)^{-1}B]f_{ij}^{(L-1)} = 0 \quad (3.34)$$

$$f_{ij}^{(L-1)} = K\ (z_i I_n - A)^{-1}Bf_{ij}^{(L-1)} \quad (3.35)$$

$$j = 1,2,...,q_i, \qquad \text{for } L = 1,2,...,p_{ij}$$

The superscript L is used to differentiate between the p_{ij} free parameter vectors corresponding to each one of the q_i equations. To

simplify the calculations, the parameter vectors $f^{(L-1)}{}_{ij}$ with subscript $L \neq 1$ are assumed to be zero vectors.

Equation (3.35) specifies only q_i vector equations and the additional $(m_i - q_i)$ are specified by (3.32) as follows :

$$\frac{Kd^k (z_i I_n - A)^{-1} B f_{ij}^{(L-1)}}{dz^k} = 0, \quad k = 1,2,\ldots p_{ij} \tag{3.36}$$

With some manipulation it can be shown that

$$K[D(z_1), D(z_2), \ldots, D(z_s)] = [F_1, F_2, \ldots, F_s] \tag{3.37}$$

Where $D(z_i)$ and F_i are :

$$\begin{aligned} F_i = [f_{i1}^{(0)}, \ldots, f_{i1}^{(p_{i1}-1)}, \ldots, f_{i2}^{(0)}, \ldots\ldots, f_{i2}^{(p_{i2}-1)}, \\ \ldots\ldots, f_{iq_i}^{(0)}, \ldots, f_{iq_i}^{(p_{iq_i}-1)}] \end{aligned} \tag{3.38}$$

But using the simplifying assumption : $f^{(L-1)}{}_{ij} = 0$, for $L \neq 1$ (3.38) becomes

$$F_i = [\underbrace{f_{i1}^{(0)}, 0, \ldots, 0}_{p_{i1}}, \underbrace{f_{i2}^{(0)}, 0, \ldots, 0}_{p_{i2}}, \ldots, \underbrace{f_{iq_i}^{(0)} 0, \ldots, 0}_{p_{iq_i}}] \tag{3.39}$$

and

$$D(z_i) = S(z_i) f_{i1}^{(0)}, \ldots, \frac{d^{p_{i1}-1}}{dz^{p_{i1}-1}} (S(Z_i) f_{i1}^{(0)}), \ldots\ldots$$

$$\ldots S(z_i)f^{(0)}_{iq_i},\ldots,\frac{d^{p_{iq_i}-1}\,(S(z_i)f_{iq_i})}{dz^{p_{iq_i}-1}} \tag{3.40}$$

The columns of $D(z_i)$ are in fact the eigenvectors and generalized eigenvectors of the closed-loop system, except for an added scalar multiplication factor [12]. To simplify the mathematical expressions, the scalar factor is taken out of the generalized eigenvector equations simply by letting the free parameter vectors $f^{(L-1)}{}_{ij} = 0$, for $L \neq 1$. With some algebraic manipulations it can be shown that the eigenvectors and generalized eigenvectors are of the form:

$$v_{ij}^{(L-1)} = (-1)^{L-1}(z_iI_n - A)^{-L}\,Bf_{ij}^{(0)} \tag{3.41}$$

for $i = 1,2,\ldots s$, for $j = 1,2,..q_i$, for $L = 1,2,\ldots p_{i1}-1$

The vectors of (3.41), for $L = 1$, are the eigenvectors of A_c and for $L = 2,3,\ldots,p_{i1}-1$ are the generalized eigenvectors of A_c. Eigenvectors and generalized eigenvectors are used to calculate the modal matrix V. The free parameter vectors of F are determined, as in the case of distinct eigenvalues, under the sole condition that V must be invertible [12,13]. The following controller can then be found :

$$K = FV^{-1} \tag{3.42}$$

The closed loop-system matrix A_c is similar to a Jordan form with a structure that is not unique. The non-uniqueness is attributable to the freedom available in choosing the number of equations, q_i, associated with each eigenvalue. The q_i equations imply q_i Jordan blocks, of order p_{ij} where $j = 1,2,..q_i$. An interesting result [12] is that the minimum polynomial of A_c is of the form:

$$\mu(z) = (z-z_1)^{p_{11}}(z-z_2)^{p_{21}}\ldots(z-z_s)^{p_{s1}} \tag{3.43}$$

3. Assignment of all Zero Eigenvalues to the Closed-loop system : In the design of deadbeat controllers, all zero eigenvalues are to be assigned to the closed-loop system. The theory of this section is based on the results of the general case, with the basic difference that the eigenvectors contain an added scalar multiplication factor.

The closed loop characteristic polynomial for an all zero mode assignment is $CP(z) = z^n$. The algebraic multiplicity of the zero mode is n. As in the previous sections, the controller K is in the parametric form:

$$K = FV^{-1} \tag{3.44}$$

For convenience the subscript (i), previously used to distinguish between roots, is dropped. The new notation for the matrix F formed of the free parameter vectors is:

$$F = [F_1, F_2, \ldots, F_q] \tag{3.45}$$

$$F_j = [f_j^{(0)}, f_j^{(1)}, \ldots, f_q^{(p_j-1)}] \tag{3.46}$$

$$V = [V_1, V_2, \ldots, V_q] \tag{3.47}$$

$$V = [v_j^{(0)}, v_j^{(1)}, \ldots, v_j^{(p_j-1)}] \tag{3.48}$$

For $j = 1,2,\ldots,q$

The equations for the eigenvectors and generalized eigenvectors are [13] :

$$v_j^{(0)} = -A^{-1}Bf_j^{(0)}$$

$$v_j^{(1)} = -A^{-2}Bf_j^{(0)} - ABf_j^{(1)}$$

$$v_j^{(2)} = -A^{-3}Bf_j^{(0)} - A^{-2}Bf_j^{(1)} - A^{-1}Bf_j^{(2)}$$

$$\vdots \tag{3.49}$$

$$v_j^{(p_j-1)} = -A^{-p_j}Bf_j^{(0)} - \ldots\ldots - A^{-1}Bf_j^{(p_j-1)}$$

The free parameter vectors that make the (rxn) matrix F are, as in the previous cases, found arbitrarily under the sole condition that V must be invertible.

Having a freedom available of selecting the modal matrix implies a freedom in selecting a Jordan normal form of A_c. The Jordan form J of A_c contains q Jordan blocks and each block is of order $p_j, j = 1,2,\ldots,q$. The minimum polynomial is found from equation (3.43) to be :

$$\mu(z) = z^{p_1} \tag{3.50}$$

There are many interesting properties in the study of the non-unique structure of the deadbeat controller. These properties are studied in detail in the next section.

D. PROPERTIES OF DEADBEAT CONTROLLERS

The closed loop system of (3.3) has the following properties [11]:

1. The closed-loop matrix A_c is a nilpotent matrix of degree p_1. p_1, the index of nilpotency [13] is equal to the number of time steps required to drive the system from any arbitrary initial condition to the origin of the state space.

2. The eigenvalues of A_c are zero, the eigenvectors span the null space $Ker(A_c)$, and the generalized eigenvectors span the range space of A_c [13].

3. The matrix A_c is similar to the Jordan form matrix composed of q nilpotent blocks J_{p_j} of order p_j, for $j = 1,2,\ldots,q$.

Discussion of Properties : Property (i). The feedback controller of (3.2) takes a minimum of p, and at most n time steps in driving the system to the origin, which can be shown by looking at the system of (3.1) at the p_1th time step ($p < p_1 < n$) :

$$x(p_1) = A^{p_1}(0) + [B,AB,\ldots,A^{p_1-1}B][u(p_1),\ldots,u(0)]^T \qquad (3.51)$$

$$\text{let } C_{p_1} = [B,AB,\ldots\ldots,A^{p_1-1}B] \qquad (3.52)$$

If the controllability matrix has full rank for a controllability index p, a unique control vector can be obtained [15] :

$$[u(p_1-1),\ldots,u(0)] = -C^{-1}{}_{p_1}A^{p_1}x(0) \qquad (3.53)$$

In other words, letting $p_1 = p$, there exist n independent columns in C_p such that a vector $-A^{p_1}x(0) \in R^r$ is in the range space of a control vector.

To show that at most n steps are required to drive the nth order system to the origin, the characteristic equation (3.5) and the Cayley-Hamilton theorem [15,18] are used :

$$CP(A_c) = A_c^n = 0 \qquad (3.54)$$

therefore,

$$A_c^{(n+L)} = 0, \quad \text{for } L = 1,2,\ldots \qquad (3.55)$$

Equation (3.55) implies that at most n time steps are required to reach the origin. A matrix A_c is termed nilpotent if for some positive integer, p_1, $A^{p1}{}_c = 0$ and $A^{(p1-1)}{}_c \neq 0$. This is the case here and can be seen by

$$x(p_1) = A_c^{p1} x(0) = 0 \tag{3.56}$$

Property (ii). It has been shown that $A^{p1}{}_c = 0$ for some $p_1 < \infty$. Now suppose that z_i is an eigenvalue of A_c with associated eigenvector v_i, then

$$A_c^{p1} v_i = z_i^{p1} v_i \tag{3.57}$$

Since eigenvectors are assumed to have nonzero (unit) length, all the eigenvalues must be equal to zero. It may be surmised, that the effect of the feedback controller is to shift the eigenvalues of A to the origin [13].

To demonstrate that the eigenvectors of A_c span the Ker(A_c) and than the generalized eigenvectors of A_c span its range space, the following is needed.

Definition. The simplicity of any full rank (not necessarily square) polynomial matrix is defined as the number of nonunity invariant polynomials. A matrix is simple if and only if it has simplicity one [15]. If a matrix A_c has multiple eigenvalues, it may or may not be simple. If the eigenvalues are distinct, A_c is always simple.

A matrix A_c having repeated eigenvalues z_i with multiplicity m_i is simple if

$$\mathrm{rank}[z_j - A_c] = n - m_j \tag{3.58}$$

and not simple if

$$\operatorname{rank}[z_j - A_c] > n - m_j \tag{3.59}$$

When the matrix is not simple it can not be diagonalized because one can not find n independent eigenvectors. In such a case, the next best structure is the Jordan form. The Jordan form is defined by the eigenvectors and generalized eigenvectors. Given an eigenvector $v^{(0)}{}_j$, the concomitant generalized eigenvectors can be expressed as

$$A_c v_j = z_j v_j + v_{j-1}, \quad \text{for } j = 2,3,\ldots,q \tag{3.60}$$

In the case under study the closed-loop system is always "not simple" with algebraic multiplicity n. The characteristic polynomial can be factored into q terms (q < r) with orders adding to n. The characteristic polynomial written in terms of the q equations is :

$$CP(Z) = z^n = z^{p_1} z^{p_2} \ldots\ldots z^{p_q} \tag{3.61}$$

Corresponding to the q terms, there are q chains of assignable eigenvectors and generalized eigenvectors which can be written in the form :

$$\begin{aligned} A_c v_j^{(0)} &= 0 \\ A_c v_j^{(L)} &= v_j^{(L-1)} \end{aligned} \tag{3.62}$$

$$\text{for } j = 1,2,\ldots,q \quad \text{and} \quad L = 1,2,\ldots,p_j - 1$$

On examination of (3.62), it may be concluded that the q eigenvectors $v^{(0)}{}_j$ span the range space of A_c. Additionally, the (n-q) eigenvectors $v^{(0)}{}_j, v^{(1)}{}_j, \ldots, v^{(pj-2)}{}_j$ span the range space of A_c. The combination of eigenvectors and generalized eigenvectors form the modal

matrix of the closed-loop system :

$$V = [\ V_1, V_2, \ldots, V_{q-1}, V_q\ ] \tag{3.63}$$

$$V_j = [\ v_j^{(0)},\ v_j^{(1)}, \ldots,\ v_j^{(p_j-1)}\ ] \tag{3.64}$$

The modal matrix V is of great importance in finding the Jordan form of the closed loop-system.

Property (iii). The closed-loop system matrix A_c can not be transformed into a diagonal form because there are less than n independent eigenvectors. However, using the eigenvectors and generalized eigenvectors, one can at best transform A_c into a Jordan canonical form. The Jordan form is found from the modal matrix of A_c as follows :

$$J = V^{-1}A_cV \tag{3.65}$$

The Jordan form of A_c has q Jordan blocks of orders p_j, where $j = 1,2,\ldots,q$.

$$J = \begin{bmatrix} Jp_1 & & & & \\ & Jp_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & Jp_q \end{bmatrix} \tag{3.66}$$

Where each Jordan block J_{pj} is a nilpotent matrix of degree p_j with ones on the first superdiagonal and zeros elsewhere.

$$Jp_j = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \tag{3.67}$$

The matrix A_c is therefore a nilpotent matrix of order p_1 provided

$$p_1 \geq p_2 \geq \ldots . \geq p_{q-1} \geq p_q \tag{3.68}$$

The Jordan form of A_c has the following properties:

1. $q \geq r$ The number of Jordan blocks in J will always be less than or equal to the number of inputs in the system.

2. The sum of the order of the q Jordan blocks is equal to the order of the system ($p_1 + p_2 + \ldots + p_q = n$). The order of each of the blocks is equal to the controllability indices.

3. A requirement is made to keep the nilpotency of A_c always equal to p_1, that is $p_1 \geq p_2 \geq \ldots \geq p_q$.

4. The order of the first Jordan block is greater than or equal to the controllability index p, and less than or equal to the number of states ($p \geq p_1 \geq n$).

5. The number of Jordan blocks of order Jp_j of all orders is equal to the nullity A_c.

6. The minimum polynomial of A_c is equal to the minimum polynomial of J ($\mu(z) = z^{p_1}$) [13].

7. The number of Jordan blocks of any order Jp_j is $2m_j - m_{j+1} - m_{j-1}$, where m_j is the nullity of $A^j{}_c$.

In the discussion above, order one means the 1x1 zero matrix [18]. The properties described here are of great importance. They are the basis for the developments to follow in this chapter, and also point out some important properties common to all deadbeat controllers.

E. PARAMETRIC DEADBEAT CONTROLLER

Fahmy and O'Reilly [13,14] have approached the problem of deadbeat control from the perspective of closed-loop eigenstructure assignment. They use the non-unique structure of the feedback matrix to find a solution for K in terms of a number free parameters. This form of K is called the parametric deadbeat controller [19,20], and is reviewed in this section.

It is shown that different Jordan forms can be assigned to the closed-loop system, and that each Jordan form can be obtained with an infinite number of controllers. Further, a comparison is made between the degrees of freedom available in K and the time required to drive all the states of the closed-loop system to the origin.

Taking the approach of eigenstructure assignment, a solution to the deadbeat controller is of the form

$$K = FV^{-1} \tag{3.69}$$

where V, the modal matrix of A_c, is defined by

$$V = [V_1, V_2, \ldots, V_q] \tag{3.70}$$

$$V_j = [v_j^{(0)}, v_j^{(1)}, \ldots, v_j^{(p_j-1)}] \tag{3.71}$$

$$\text{for } j = 1, 2, \ldots, q$$

The eigenvectors and generalized eigenvectors that form (3.71) are of the form :

$$v_j^{(0)} = -A^{-1}Bf_j^{(0)}$$

$$v_j^{(1)} = -A^{-2} Bf_j^{(0)} - A^{-1} Bf_j^{(1)}$$

$$\vdots \qquad (3.72)$$

$$v_j^{(p_j-1)} = -A^{-p_j} Bf_j^{(0)} - \dots - A^{-1} Bf_j^{(p_j-1)}$$

for $j = 1,2,\dots,q$

The q free parameter vectors are chosen arbitrarily under the condition that the eigenvectors and generalized eigenvectors that are linearly independent.

Define

$$F = [F_1, F_2, \dots, F_q] \qquad (3.73)$$

$$F_j^{(0)} = [f_j^{(1)}, f_j^{(2)}, \dots, f_j^{(p_j-1)}] \qquad (3.74)$$

From (3.69), an alternate expression is

$$F = KV \qquad (3.75)$$

$$[F_1\ F_2\ \dots\ F_q] = K[V_1\ V_2\ \dots\ V_q] \qquad (3.76)$$

or also

$$Kv_j^{(p_j-1)} = f_j^{(p_j-1)} \qquad (3.77)$$

for $j = 1, 2, \dots, q$

combining equations (3.72) and (3.77), the following expressions are obtained :

$$-KA^{-1}Bf_j^{(0)} = f_j^{(0)}$$

$$-K[A^{-2}Bf_j^{(0)} + A^{-1}Bf_j^{(1)}] = f_j^{(1)}$$

$$\vdots \tag{3.78}$$

$$-K[A^{-1}Bf_j^{(p_j-1)} + A^{-2}Bf_j^{(p_j-2)} + .. + A^{-p_j}Bf_j^{(0)}] = f_j^{(p_j-1)}$$

for $j = 1, 2, ..., q$

An alternate expression can be written as (cc. Appendix B) :

$$KA^{-1}v_j^{(0)} = [I_r + KA^{-1}B]f_j^{(1)}$$

$$KA^{-1}v_j^{(1)} = [I_r + KA^{-1}B]f_j^{(2)}$$

$$\vdots \tag{3.79}$$

$$KA^{-1}v_j^{(p_j-2)} = [I_r + KA^{-1}B]f_j^{(p_j-1)}$$

for $j = 1, 2, ..., q$

On the basis of expression (3.79), the following conclusions are made :

(i) The eigenvectors $v^{(0)}{}_j$ span the null space as shown by (3.62). It is observed that from (3.72), that these vectors belong to the range space of $A^{-1}B$. That is,

$$\text{null}[A_c] \subset \text{range}[A^{-1}B] \tag{3.80}$$

(ii) It is implied by (3.62) that the (n-q) vectors $v^{(0)}{}_j, v^{(1)}{}_j$, . . . , $v^{(pj-2)}{}_j$ span the range space of A_c

The rank of the gain matrix K of (3.69) depends upon the rank of F. The modal matrix V has rank n and F less or equal r. The number of equations in which the characteristic polynomial can be factored is q, which is the number of linearly independent vectors $f^{(i)}{}_j$ in F. Thus,

$$q \leq \text{rank}[K] = \text{rank}[F] \leq r \qquad (3.81)$$

<u>1. The Set of Numbers n_v</u> : The closed-loop system under study is assigned one of different admissible Jordan forms [13]. A set of numbers, n_v, is developed to identify each Jordan form assignable by the use of feedback.

Consider the set of p_1 integers $\{n_v\ ;\ v = 0,1,..,p_1-1\}$ defined as :

$$\begin{aligned}
n_0 &= \text{number of parameter vectors } f_j^{(0)} \\
&= \text{number of eigenvectors } v_j^{(0)} \\
n_1 &= \text{number of parameter vectors } f_j^{(1)} \qquad (3.82) \\
&= \text{number of generalized eigenvectors } v_j^{(1)} \\
n_{p_1-1} &= \text{number of parameter vectors } f_j^{(p_j-1)} \\
&= \text{number of generalized eigenvectors } v_j^{(p_j-1)}
\end{aligned}$$

<u>Examples</u>

Example 1. Admissible Jordan forms for a system with n = 3, r = 2, p = 2.

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 2,\ p_2 = 1 \\ n_0 = 2,\ n_1 = 1 \\ q = 2 \end{array} \tag{3.83}$$

$$J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 3 \\ n_0 = 1,\ n_1 = 1,\ n_2 = 1 \\ q = 1 \end{array} \tag{3.84}$$

Example 2. Admissible Jordan forms for a system with $n = 4$, $r = 3$, $p = 2$.

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 2,\ p_2 = p_3 = 1 \\ n_0 = 3,\ n_1 = 1 \\ q = 3 \end{array} \tag{3.85}$$

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = p_2 = 2 \\ n_0 = n_1 = 2 \\ q = 2 \end{array} \tag{3.86}$$

$$J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 3,\ p_2 = 1 \\ n_0 = 2,\ n_1 = n_2 = 1 \\ q = 2 \end{array} \tag{3.87}$$

$$J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 4 \\ n_0 = n_1 = n_2 = n_3 = 1 \\ q = 1 \end{array} \tag{3.88}$$

From the examples, many conclusions can be made about the set of numbers n_ν :

$$n_0 + n_1 + \dots + n_{p_1 - 1} \tag{3.89}$$

$$n_0 \geq n_1 \geq \ldots \geq n_{p_1-1} \tag{3.90}$$

$$n_0 = q \tag{3.91}$$

$$n_{p_1-1} \geq 1, \; n_{p_2-1} \geq 2, \; n_{p_q-1} \geq q-1 \tag{3.92}$$

and that

$$\mathrm{null}[J] = \mathrm{null}[A_c] = q \tag{3.93}$$

Also,

$$\begin{aligned} n_{p_1-1} &= \text{number of Jordan blocks of order } p_1 \\ n_{p_1-2} - n_{p_1-1} &= \text{number of Jordan blocks of order } (p_1-1) \\ &\;\;\vdots \\ n_1 - n_2 &= \text{number of Jordan blocks of order } 2 \\ n_0 - n_1 &= \text{number of Jordan blocks of order } 1 \end{aligned} \tag{3.95}$$

Making use of property 7 of Jordan blocks :

$$\delta = 2m_j - m_{j+1} - m_{j-1} \tag{3.96}$$

where δ is the number of Jordan blocks of order J_{pj} and m_j is the nullity of $A^j{}_c$. Using (3.95) in conjunction with (3.96), the following results :

$$\mathrm{null}[J] = \mathrm{null}[A_c] = q = n_0$$

$$\mathrm{null}[J^2] = \mathrm{null}[A_c^2] = n_0 + n_1$$

$$\vdots \qquad (3.97)$$

$$null[J^{p_1}] = null[A_c^{p_1}] = n_0 + n_1 + \ldots + n_{p_1-1}$$

Therefore, using (3.95)

$$rank[A_c] = n - n_0 = n_1 + n_2 + \ldots + n_{p_1-1}$$

$$rank[A_c^2] = n - n_0 - n_1 = n_2 + n_3 + \ldots + n_{p_1-1}$$

$$\vdots \qquad (3.98)$$

$$rank[A_c^{p_1-1}] = n - n_0 - \ldots - n_{p_1-2} = n_{p_1-1}$$

$$rank[A_c^{p_1}] = n_1 - n_0 - n_1 - \ldots - n_{p_1-1} = 0$$

This result is of great significance because it explains how the closed-loop system matrix loses its rank at each time-step. The matrix is nilpotent with index p_1.

2. Free Parameters Available in the Deadbeat Controller : The closed-loop Jordan form assigned by a deadbeat controller is non-unique. As such, some freedom is facilitated in the design of the controller, which if properly used, can be of great advantage. In designing the controller, a set of parameters may be chosen to provide all possible degrees of freedom, while removing redundancies. Equation (3.78) can be rearranged as follows

$$[I_r + KA^{-1}B]f_j^{(0)} = 0, \quad j = 1,2,\ldots,n_0 \qquad (3.99.1)$$

$$[I_r + KA^{-1}B|KA^{-2}B][f_j^{(1)}\ f_j^{(0)}]^T = 0, \quad j = 1,2,..,n_1 \tag{3.99.2}$$

$$\vdots$$

$$[I_r + KA^{-1}B|KA^{-2}B|\ldots|KA^{-p_1}B]F^{p_1} = 0, \quad j = 1,2,..,n^{p_1-1} \tag{3.99.3}$$

where,

$$F^{p_1} = [f_j^{(p_1-1)}\ f_j^{(p_1-2)} \ldots f_j^{(0)}]^T \tag{3.99.3.b}$$

for $j = 1,2,\ldots,n_{p_1-1}$

Let the minimum number N of free parameters in the parametric form (3.69) of K, equal the number of independent scalar equations in (3.99). Recall that there are q vectors $v^{(0)}{}_j$ that span the null space of A_c, and that these vectors belong to the range scope of $A^{-1}B$.

Carefully scrutinizing (3.99) for a case where $q < r$, one observes that :

(i) $$\text{rank}[I_r - KA^{-1}B]f_j^{(0)} = r - n_0 = r - q \tag{3.100}$$

There are n_0 vectors containing $(r-n_0)$ independent scalar equations. No other equation (for a particular K), can be written in this form because otherwise there would be more than the specified number of q eigenvectors.

(ii) From (3.99.2), one observes that there are $(r-n_1)$ independent scalar equations in each one of the n_1 vector equations. In this case the rank test, as for (3.99.1), may yield a higher rank than $(r-n_1)$. However, the vector $f^{(0)}{}_j$ is required to satisfy (3.99.1) and (3.99.2) which results in a possible decrease of the number of independent scalar equations.

(iii) Equation (3.99.3), yields a set of $(r-n_{p1-1})$ independent scalar equations for each one of the n_{p1-1} vector equations. The number of independent scalar equations is found as above.

The number of independent scalar equations implied by each vector equation of (3.99), is employed to yield the minimum number of free parameters in the parametric form of K. Then

$$N = n_0(r-n_0) + n_1(r-n_1) + \ldots + n_{p_1-1}(r-n_{p_1-1}) \tag{3.101}$$

using (3.89)

$$N = nr - (n_0^2 + n_1^2 + \ldots. + n_{p_1-1}^2) \tag{3.102}$$

From (3.90) and (3.102), it is observed that the number of free parameters become a minimum if the values of n_0, $n_1, .. n_{p1-1}$ are maximized in succession. In attaining the latter, the number of Jordan blocks, q, in the Jordan form of A_c is chosen as large as possible. The maximum value of q is the number of inputs to the system. This may or may not be the case in designing a minimum time deadbeat controller. Nevertheless, if $q = r$ some interesting properties are observed.

Hence (3.81) and (3.79) imply

$$q = \mathrm{rank}[K] = \mathrm{rank}[F] = r \tag{3.103}$$

$$\mathrm{null}[A_c] = \mathrm{range}[A^{-1}B] \tag{3.104}$$

On examination of the numerical examples,

$$n_0 = n_1 = \ldots. = n_{p_r-1} = r \tag{3.105}$$

which implies that the first p_r bracketed terms of (3.101) vanish. In view of this, the vector equations of (3.99.1) are written as

$$[I_r + KA^{-1}B]f_j^{(0)} = 0 \ , \quad j = 0,1,2,\ldots,r \tag{3.106}$$

but with the fact that the r vectors are linearly independent

$$[I_r + KA^{-1}B] = 0, \text{ or } \ KA^{-1}B = -I_r \tag{3.107}$$

Similarly, the vector equations of (3.99.2) using (3.107) are written

$$KA^{-2}B = 0 \tag{3.108}$$

In general it may be written

$$KA^{-L}B = 0 \ , \quad L = 1,2,\ldots,p_r \tag{3.109}$$

As a consequence, the first p_r vectors of each of the q - r vectors (3.78) disappear. The whole last $(r-n_{pr})$ sets $(j = n_{pr+1}, n_{pr+2}, \ldots, r)$ disappear because each of them contains just p_r vector equations; with some manipulation, it can be shown (cc. Appendix C.), that n_{pr} sets with $(n-rp_r)$ vector equations are left :

$$\begin{aligned} &KA^{-(p_r+1)}Bf_j^{(0)} = 0 \\ &K[A^{-(p_r+1)}Bf_j^{(1)} + A^{-(p_r+2)}Bf_j^{(0)}] = 0 \\ &\quad\cdot \\ &\quad\cdot \\ &K[A^{-(p_r+1)}Bf_j^{(p_j-p_r-1)} + A^{-(p_r+2)}Bf_j^{(p_j-p_r-2)} \\ &\qquad\qquad + \ldots + A^{-p_j}Bf_j^{(0)}] = 0 \end{aligned} \tag{3.110}$$

$$\text{for } j = 1,2,\ldots,n_{p_r} \quad \text{and} \quad n_{p_r} = r = q.$$

The resultant feedback matrix K is independent of the last p_r parameter vectors in each submatrix F_i of F. Even though the solution is independent of these vectors, they are required by the algorithm to provide an invertible modal matrix of A.

The matrix F containing $(p_j - rp_r)$ effective vectors and rp_r auxiliary vectors can be written as

$$F_j = [\underbrace{f_j^{(0)}, f_j^{(1)}, \ldots, f_j^{(p_j-p_r-1)}}_{(p_j-p_r)} | \underbrace{f_j^{(p_j-p_r)}, \ldots, f_j^{(p_j-1)}}_{p_r}]$$

$$\text{for } j = 1,2,\ldots,r \tag{3.111}$$

Defining :

$$\begin{aligned} n_{p_r} &= \text{Number of effective parameter vectors } f_j^{(0)} \\ n_{p_r+1} &= \text{Number of effective parameter vectors } f_j^{(1)} \\ &\cdot \\ &\cdot \\ &\cdot \\ n_{p_1-1} &= \text{number of effective parameter vectors } f^{(p_1-p_r-1)} \\ j &= 1,2,\ldots,r \end{aligned} \tag{3.112}$$

The number of free parameters available in F is a minimum when q=r. In fact, if additional constraints are met, the number of free parameters may be zero. When this occurs, the deadbeat controller is unique.

3. Minimum-time and Unique Deadbeat Controller : Tou, Farison and Fu [8] first noticed that the multi-input control goal may be accomplished in $p = n/r$ time-steps, where p is assumed to be an integer. This claim is only valid for a restricted class of systems for which

$$\text{rank}[A^{-1}B, A^{-2}B, \ldots, A^{-p}B] = n, \quad p = n/r \tag{3.113}$$

The statement of (3.63) is valid if each $A^{-p_1}B$ ($p_1 = 1,2,\ldots,p$), contains r linearly independent columns, i.e. they span an r-dimensional space such that the direct sum of these p subspaces constitutes the required n-dimensional subspace R^n.

In this special case, $q = r$ and n/r = integer $= p_1 = p_2 = \ldots = p_r$. The input vector u(i) required to drive the system to the origin is unique because the controllability indices are equal. The closed-loop system is assigned r Jordan blocks of order p. The numbers forming the set $\{n_v\}$ are all equal to r, so that the number of free parameters in F is zero, further indicating that the controller is unique.

4. Structure of the Parametric Matrix F : The problem at hand is one of locating the N free parameters in the parametric matrix F. The remaining (nr-N) elements in F are taken as constants, which may be chosen as zeros and ones. Each effective parameter vector is assigned part of the N free parameters, in such a way that the vector equations of (3.78) yield N independent scalar equations. The first parameter vector $f^{(0)}{}_j$ in the submatrix F_j is assigned $(r-n_{pj-1})$ free parameters, which is also the number of independent scalar equations in the last vector equation of (3.78). Likewise, the second vector $f^{(1)}{}_j$ is assigned $(r-n_{pj-2})$ parameters and so forth, until the last parameter vector $f^{pj-1}{}_j$ is assigned $(r-n_0)$ parameters. In a general form, the free parameters are assigned as follows :

$$r - n_{p_j-i-1} = \text{number of free parameters in } f_j^{(i)} \tag{3.114}$$

$$j = 1,2,\ldots,q \; ; \quad i = 0,1,2,3,\ldots,p_j-1$$

A parametric matrix F, where the free parameters are assigned as described by (3.114), may fail to yield a subclass of solutions K. This happens in the special situation when, in (3.79), the parameters in the parameter vector $f^{(i)}{}_j$ are to be multiplied by null columns in the matrices $[I_r+A^{-1}B]$, $KA^{-L}B$, thus rendering inconsistent equations. A solution when this problem is encountered is derived through use of the same number N of free parameters but differently arranged in the structure of F.

This assignment of free parameters to the parametric matrix is summarized in the next section and illustrated by the numerical examples.

5. Selection of Free Parameters : The freedom afforded by state feedback beyond pole placement is described as that of assigning generalized eigenvectors from specific subspaces. This characterization has been used to design state feedback controllers with desirable properties. A relationship has been developed for the deadbeat controller in terms of the available free parameters. The free parameters selected, provide an analytical expression for the deadbeat controller of minimum Frobenius norm [13,17,18], while minimizing the control effort.

The Frobenius norm of a matrix K is defined as

$$\|K\|_F = \sqrt{(\sum_{i,j} m^2{}_{i,j})} \tag{3.115}$$

Two methods can be utilized to select the parameters of the deadbeat controller to minimize the Frobenius norm [13] :

(i) Finding a deadbeat controller in terms of the free parameters available, and then its parametric Frobenius norm. Calculus techniques may be used in this task.

(ii) A computational algorithm for assigning numerical values to the free parameters in F can be developed. In each iteration the minimum Frobenius norm for the deadbeat controller is calculated and compared (cc. Appendices D and E).

The two ways of selecting the free parameters available in the parametric controller of minimum Frobenius norm are accurate. However, depending on the order of the system, they may take unreasonable computational time.

6. Design of the Parametric Deadbeat Controller : The structure of the parametric deadbeat controller of (3.69) varies according to the Jordan form assigned to closed-loop system. Once the parametric matrix is designed, the modal matrix is obtained in terms of the free parameters. The N free parameters in K can be chosen to improve the response of the system.

The parametric deadbeat controller may be designed following this procedure :

(i) The system description should satisfy the requirements that the matrix A must be invertible, B should of full rank, and the pair (A,B) ought be completely controllable with controllability p.

(ii) An admissible Jordan form to be assigned to the closed-loop system is selected. The order of the Jordan blocks must be such that $p_1 \geq p_2 \geq \ldots \geq p_q$, where the order of p_1 is equal to the number of steps required to drive all the states of the system to the origin. The set of numbers n_v is found as described in section 3.2.

(iii) The number of free parameters available in F is determined by

$$N = n_0(r-n_0) + n_1(r-n_1) + \ldots + n_{p_1-1}(r-n_{p_1-1}) \qquad (3.116)$$

(iv) The free parameters are assigned to the parametric matrix F using (3.114). The vectors corresponding to indices $i = 0$ are selected in such a way that if put together they would form a submatrix equal to $[I_q | 0_{r-q}]^T$. Those of indices $i \neq 0$ are such that if put together they form a submatrix $[0_{rx(n-q)}]$. The free parameters are assigned to the lowest elements of the effective parameter vectors, where

$$r-n_{p_j}-i-1 = \text{number of free parameters in } f_j^{(i)} \qquad (3.117)$$

(v) The matrix K is determined from (3.69) and the modal matrix V from (3.70).

(vi) The deadbeat controller obtained from (3.70) is in terms of the N free parameters. The N free parameters could be chosen as zero (to simplify the design), or selected in conjunction with the criteria of section 3.5.

7. Examples

Example 1 : Consider the system

$$x(i+1) = Ax(i) + Bu(i) \qquad (3.118.a)$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \qquad (3.118.b)$$

$$xo = [\ 5 \quad 3 \quad -3.5 \quad -4\ ]^T \tag{3.118.c}$$

The controllability index is 2. There are four Jordan forms that can be assigned to the closed-loop system and each one of them has a corresponding set of numbers n_v. It is of interest to explore the minimum time, minimum number of free parameters deadbeat controller :

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} p_1 = 2,\ p_2 = p_3 = 0 \\ q = 3 \\ n_0 = 3,\ n_1 = 1 \end{array} \tag{3.119}$$

The number of free parameters available in F is determined from (3.116) as

$$N = 3(3-3) + 1(3-1) = 2 \tag{3.120}$$

and they are assigned to the parametric matrix F by use of (3.114). That is,

$$F = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ H_1 & 0 & 1 & 0 \\ H_2 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} f_1^{(0)} & f_1^{(1)} & f_2^{(0)} & f_3^{(0)} \end{array} \end{array} \tag{3.121}$$

In the parametric matrix of (3.121), H_1 and H_2 are the free parameters. The eigenvectors and generalized eigenvectors are

$$\begin{aligned} v_1^{(0)} &= -A^{-1}Bf_1^{(0)} \\ v_1^{(1)} &= -A^{-1}Bf_1^{(1)} - A^{-2}Bf_1^{(0)} \end{aligned} \tag{3.122}$$

$$v_2^{(0)} = -A^{-1}Bf_2^{(0)}$$

$$v_3^{(0)} = -A^{-1}Bf_3^{(0)}$$

The modal matrix then becomes

$$V = \begin{bmatrix} -H_1+H_2-1 & 2H_1-1 & -1 & 1 \\ H_1+H_2-1 & -H_1-H_2 & 1 & 1 \\ H_2 & H_1-2H_2 & 0 & 1 \\ -2H_2 & H_2 & 0 & -2 \end{bmatrix} \tag{3.123}$$

The feedback gain matrix is given by (3.69) in terms of the free parameters H_1 and H_2. By (3.69)

$$K = \frac{1}{4H_1-6H_2} \begin{bmatrix} -2H_1+3H_2 & -2H_1+3H_2 & -2H_1+4H_2-2 & -3H_1+5H_2-1 \\ -2H_1+3H_2 & 2H_1-3H_2 & 2H_1+6H_2-2 & H_1+3H_2-1 \\ 0 & 0 & 2H_2 & -2H_1+4H_2 \end{bmatrix} \tag{3.124}$$

The free parameters H_1 and H_2 which minimize the Frobenius norm of K can be found as

$$[K]_F^2 = 1 + \frac{22H_1{}^2+106H_2{}^2-32H_1H_2+4H_1-56H_2+10}{16H_1{}^2+36H_2{}^2-48H_1H_2} \tag{3.125}$$

The first derivatives of (3.125) with respect to H_1 and H_2 are set to zero to obtain

$$17H_1{}^2 + 82H_1H_2 - 22H_1 - 42H_2 + 15 = 0 \tag{3.127}$$

The solution of (3.126) and (3.127) yields the minimizing values for the Frobenius norm of the deadbeat controller. The values for H_1 and H_2 are

$$H_1 = -.1875 \qquad H_2 = .3438 \tag{3.128}$$

The minimum time, minimum Frobenius norm controller is

$$K = \begin{bmatrix} -.5 & -.5 & .0888 & -.4556 \\ -.5 & .5 & .1110 & .0555 \\ 0 & 0 & -.2445 & -.6222 \end{bmatrix} \tag{3.129}$$

The deadbeat response of the system to the initial conditions is shown in Figure 3.3. It may be seen that the controller of (3.129) drives the initial conditions x_0 to the origin of the state space in $p = 3$ time-steps. In other words,

$$x(3) = (A + BK)^p x_0 = 0 \tag{3.130}$$

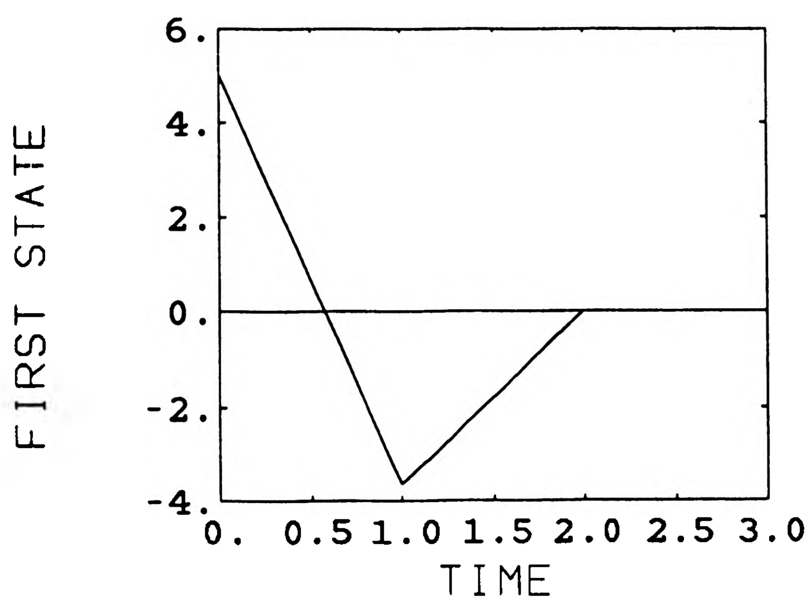

Figure 4. Initial conditions driven to the origin of the state space by a deadbeat controller.

a. First state.

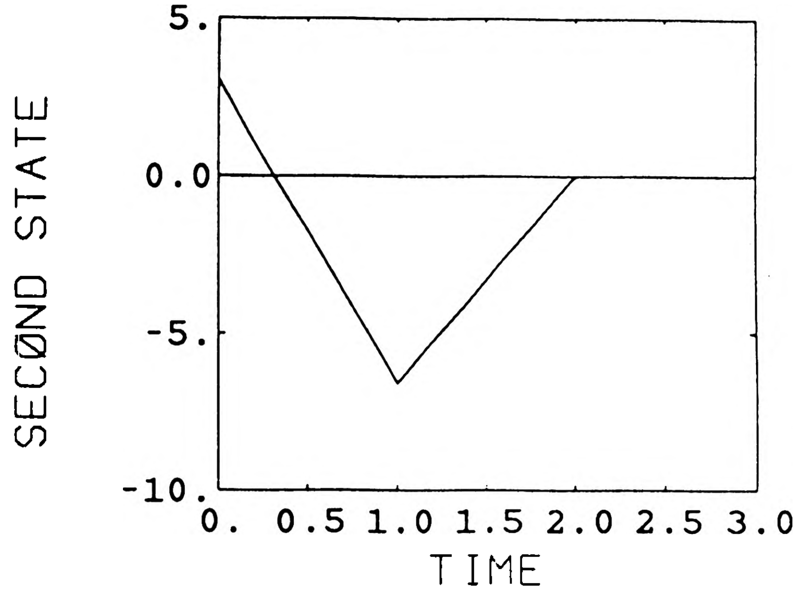

Figure 4 (cont.) b. Second state driven to the origin of the state space.

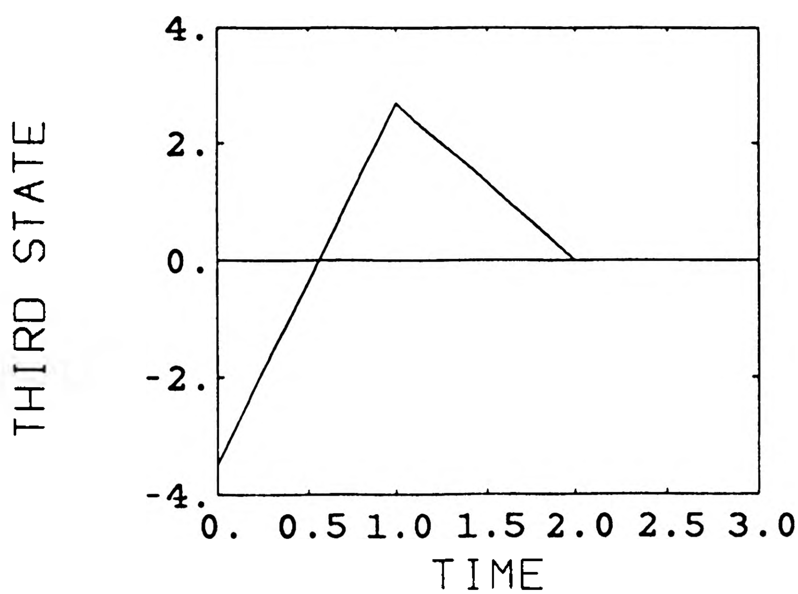

Figure 4 (cont.) c. Third state driven to the origin of the state space.

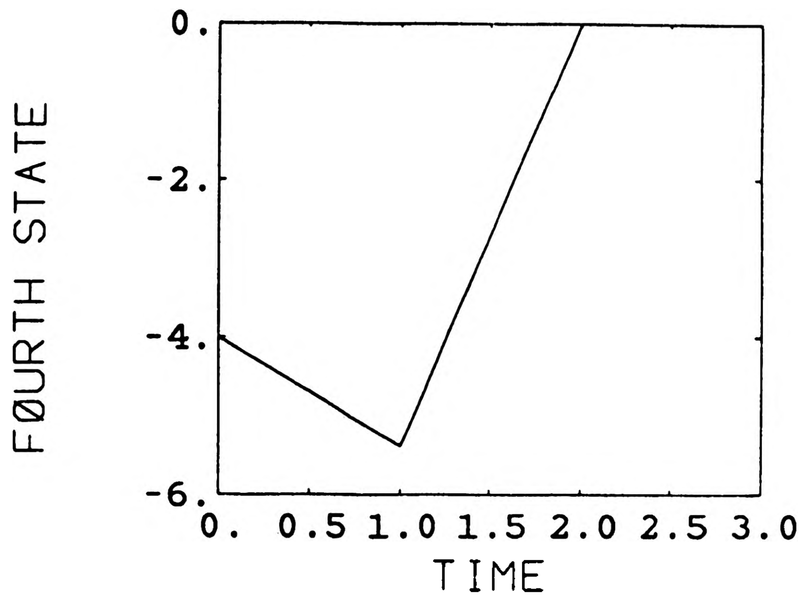

Figure 4 (cont.) d. Fourth state driven to the origin of the state space.

Example 2 : Consider the system,

$$x(i+1) = Ax(i) + Bu(i) \tag{3.131.a}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ .00384 & -.0784 & .54 & -1.6 & 2.1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{3.131.b}$$

$$xo = [\ 5 \quad 3 \quad 1 \quad -2.5 \quad -4.5\ ]^T \tag{3.131.c}$$

The rank of the controllability matrix

$$C = [\ B \mid AB \mid A^2B\ ] = \text{rank } 5 \tag{3.132}$$

implies that p = 3. The Jordan form that drives all the states of the

system in the minimum number of time steps is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} p_1 = 3 \ ; \ p_2 = p_3 = 1 \\ q = 3 \\ n_0 = 3 \ ; \ n_1 = 1 \ ; \ n_2 = 1 \end{array} \tag{3.133}$$

The number of free parameters available is

$$N = n_0(r-n_0) + n_1(r-n_1) + \ldots + n_{p_1-1}(r-n_{p_1-1}) \tag{3.134}$$

$$= 3(3-3) + 1(3-1) + 1(3-1) = 2 + 2 = 4$$

The parametric matrix is

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ H_1 & H_3 & 1 & 0 & 0 \\ H_2 & H_4 & 0 & 1 & 1 \end{bmatrix} \tag{3.135}$$

$$\begin{array}{ccccc} f_1^{(0)} & f_1^{(1)} & f_1^{(2)} & f_2^{(0)} & f_3^{(0)} \end{array}$$

and the modal matrix is

$$V = [\ v_1^{(0)} \ \ v_1^{(1)} \ \ v_1^{(2)} \ \ v_2^{(0)} \ \ v_3^{(0)}\ ] \tag{3.136}$$

where,

$$v_1^{(0)} = -A^{-1}Bf_1^{(0)}$$

$$v_1^{(1)} = -A^{-1}Bf_1^{(1)} - A^{-2}Bf_1^{(0)}$$

$$v_1^{(2)} = -A^{-1}Bf_1^{(2)} - A^{-2}Bf_1^{(1)} - A^{-3}Bf_1^{(0)} \tag{3.137}$$

$$v_2^{(0)} = -A^{-1}Bf_2^{(0)}$$

$$v_3^{(0)} = -A^{-1}Bf_3^{(0)}$$

To determine the free parameters that yield the controller of minimum Frobenius norm (example 1), is a long and complicated task. However, they can be selected using a trial and error computational algorithm (cc. Appendix D), but the time associated with this task is considerable. Depending on the accuracy desired, the number of iterations may be very large. To simplify the selection, a limited number of iterations can be performed in looking for a solution near the minimum Frobenius norm. Then

$$\begin{aligned} H_1 &= 5, \quad H_2 = 5 \\ H_3 &= 5, \quad H_4 = -5 \end{aligned} \tag{3.138}$$

The resultant deadbeat controller is

$$K = \begin{bmatrix} -.0038 & .0736 & -.4414 & 1.6 & -1.1 \\ 0.0 & -.0048 & .1130 & 1.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & 0.0 \end{bmatrix} \tag{3.139}$$

Having a Frobenius norm equal to 2.64. To verify that it forces all the states from the arbitrary initial conditions to the origin in 3 time-steps one may observe that

$$x(3) = (A + BK)^3 x_o = 0 \tag{3.140}$$

The closed-loop system has a deadbeat response to the initial conditions as shown in figure 3.4.

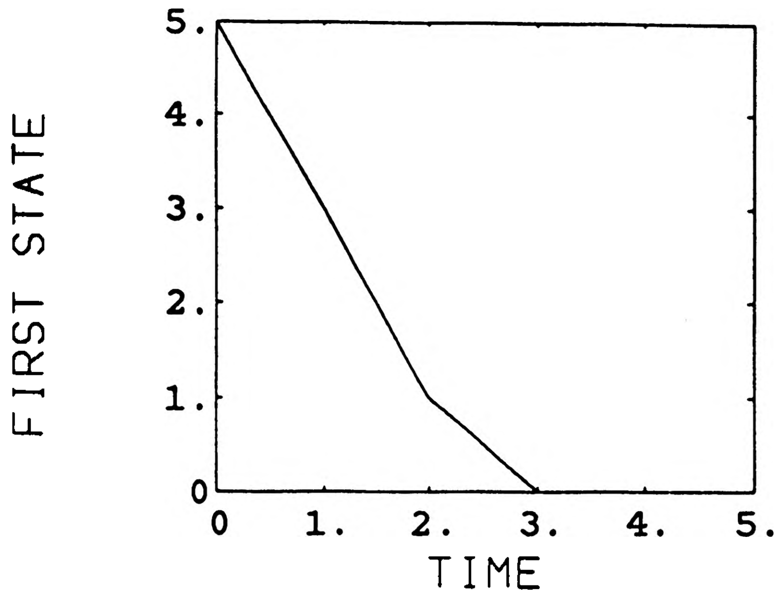

Figure 5. Initial conditions of the system being driven to the origin of the state space.

a. First state.

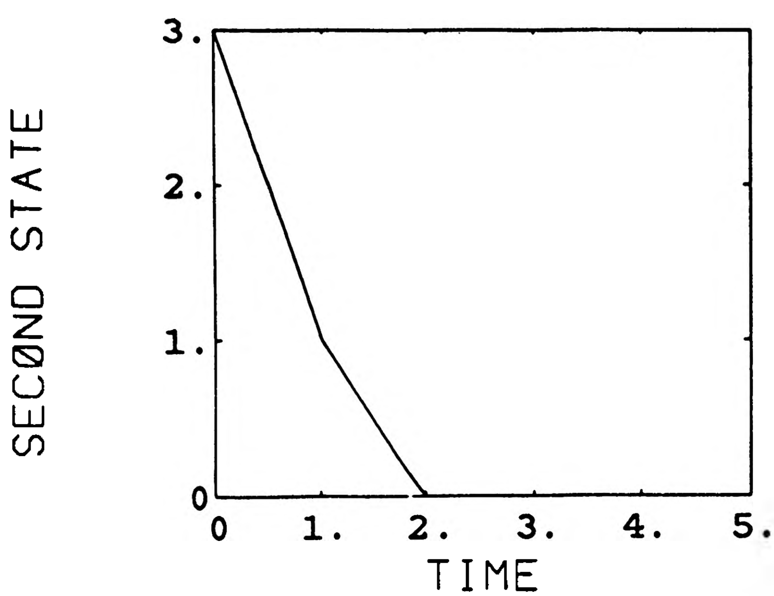

Figure 5 (cont.) b. Second state being driven to the origin.

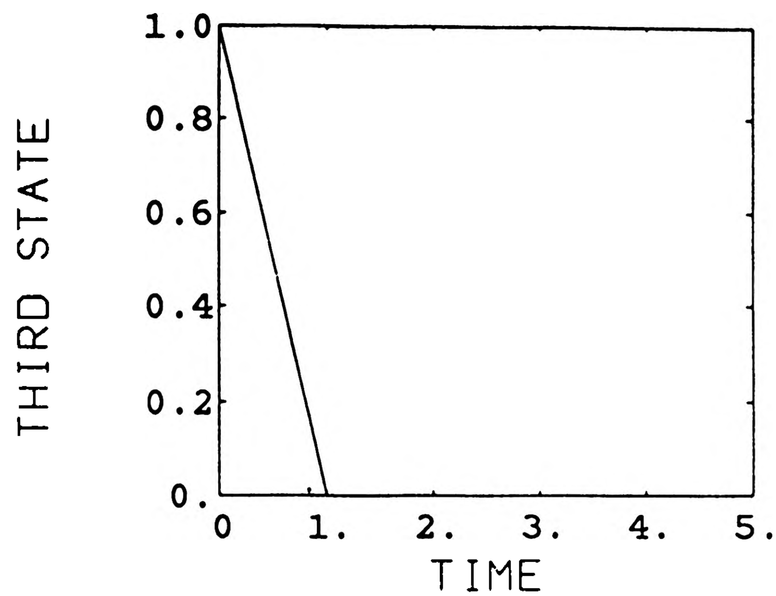

Figure 5 (cont.) c. Third state of the system being driven to the origin.

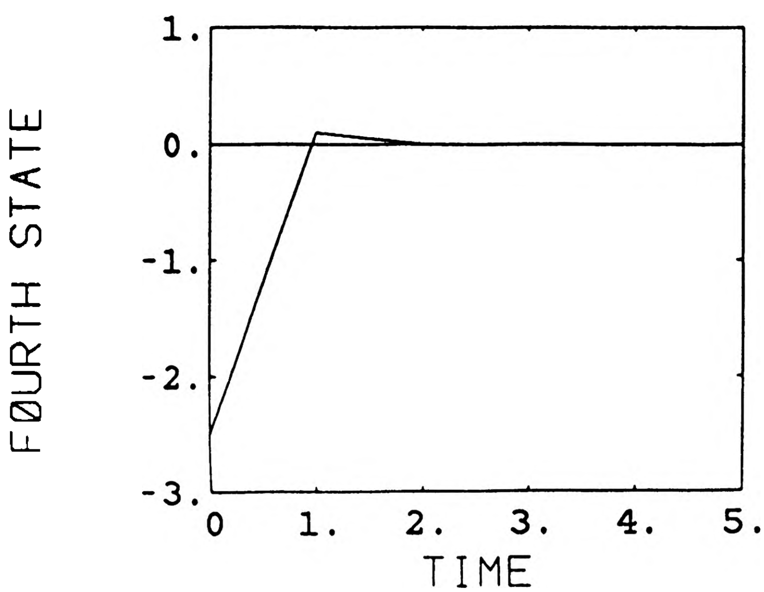

Figure 5 (cont.) d. Fourth state being driven to the origin.

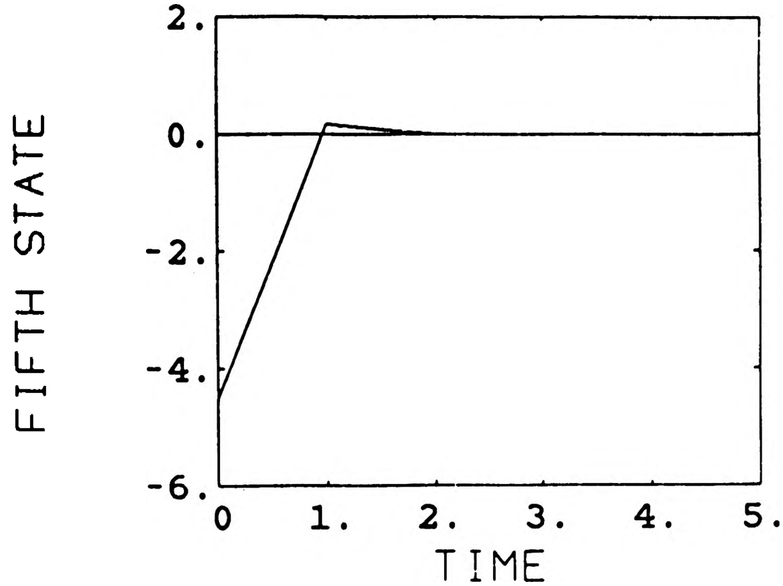

Figure 5 (cont.) e. Fifth state of the system being driven to the origin.

On examination of the graphs, it is observed that the all the states of the system are driven to the origin in at most three time steps. The deadbeat response does not contain abrupt oscillations.

IV. DESIGN OF PARAMETRIC DEADBEAT CONTROLLERS USING ORDER-REDUCTION TECHNIQUES

A. INTRODUCTION

There has been a lot of research done in determining a constant feedback gain matrix that can drive the modes of a given linear multivariable discrete time system to preassigned locations. The algorithms for the design of such controller are based on the nth order original system and with the assumption that all the open-loop eigenvalues need to be relocated.

In the area of large scale control systems, techniques have been developed to find the feedback matrix that assigns only d of n eigenvalues to a new location [16]. Using this methodology, the remaining (n-d) modes of the original system are not disturbed, and are passed on to the resultant feedback system. These techniques can also be applied to obtain a near optimal solution, i.e. shifting only the controllable and dominant modes of the original system to the desired location. In a case where all the eigenvalues are to be relocated, the approach can be applied by grouping the eigenvalues into appropriate sets and then finding a controller in a recursive way [21]. Designing a controller using order reduction techniques may be of great advantage from a computational point of view. In this chapter, a method is presented to design deadbeat controllers using reduced order models.

B. EIGENVALUE ASSIGNMENT VIA ORDER-REDUCED MODELS

1. Aggregated Reduced Order Models : A reduced order model with associated aggregation matrix can be chosen to approximate the original nth-order system. Due to the fact that the parametric deadbeat controller design is based on eigenvalue assignment, the reduced-order model must contain the dominant eigenvalues of the original system [16]. The dominant modes are those which contribute to the system behavior over a

longer period of time. The approach used to select the reduced order model is similar to the one presented in chapter II.C.

Consider a linear multivariable discrete time system described by the following difference equation:

$$x(i+1) = Ax(i) + Bu(i) \tag{4.1}$$

where $x \in R^n$ and $u \in R^r$. It is of interest to replace the large model description of (4.1) by a satisfactory aggregated model given by

$$z(i+1) = Fz(i) + Gu(i) \tag{4.2}$$

where $z \in R^d$ vector and $d < n$. The link between the linear discrete time models of (4.1) and (4.2) is established by a linear transformation of the form :

$$z(i) = Lx(i) \tag{4.3}$$

L is an dxn constant aggregation matrix of rank d. Using (4.3) the equivalence between the models (4.1) and (4.2) is achieved provided that the conditions

$$FL = LA \tag{4.4}$$

$$G = LB \tag{4.5}$$

$$z(0) = Lx(0) \tag{4.6}$$

are satisfied. Since the aggregation matrix is assumed to be of full rank, it possesses a pseudo-inverse [17,18] and therefore a least-squares solution to (4.4). That is

$$F = LA^T[LL^T]^{-1} \tag{4.7}$$

By a suitable choice of the aggregation matrix L, it is possible to ensure that the eigenvalues of F are a subset of the eigenvalues of A. Let L be chosen as

$$L = M_0 P M^{-1} \tag{4.8}$$

where

$$P = [I_d \mid 0] \tag{4.9}$$

and

$$M_0 = P M P^T \tag{4.10}$$

I_d is a dxd identity matrix and M the modal matrix of A. The first d columns of M correspond to the d dominant eigenvalues of A to be shifted. This ensures that the eigenvalues of F are identical to those d eigenvalues if A that need to be relocated.

In chapter II.C it is demonstrated that if one lets

$$J_1 = P M^{-1} A M P^T \tag{4.11}$$

$$\Gamma_1 = P M^{-1} B \tag{4.12}$$

Then the matrixes F and G satisfying all the requirements of (4.4), (4.5) and (4.7) are given by :

$$F = M_0 J_1 M^{-1}{}_0 \tag{4.13}$$

$$G = M_0 \Gamma_1 \tag{4.14}$$

<u>2. Eigenvalue Assignment Using The Reduced Order Models</u> : In this section a procedure is described for determining the state feedback gain

matrix of a linear control system for which only d, of the total n, system eigenvalues need to be relocated. The algorithm is based on constructing an aggregated model of order d for which a state feedback controller is desired. The controller is designed to shift a set of d eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ of the original system to specified new locations. The resultant closed-loop system should contain the desired eigenvalues, and the remaining (n-d) roots should appear unaffected.

Consider a linear discrete-time shift invariant state controllable nth order system represented by

$$x(i) = Ax(i) + Bu(i) \tag{4.15}$$

Let the simplified dth order dynamic model be represented by

$$z(i+1) = Fz(i) + Gu(i) \tag{4.16}$$

It is generally possible to synthesize a stationary state feedback control law for the reduced model

$$u(i) = Kz(i) \tag{4.17}$$

where the gain matrix K is to be evaluated in such manner that the d eigenvalues of the closed-loop system

$$z(i+1) = F_c z(i) \;, \quad \text{where } F_c = F + GK \tag{4.18}$$

are assigned to the desired locations or regions. Substitution of (4.3) in (4.17) yields

$$u(i) = KLx(i) \tag{4.19}$$

The feedback control law, designed for the reduced order model, can be applied to the original system of (4.15) to obtain

$$x(i+1) = A_c x(i), \quad \text{where } A_c = (A + BKL) \tag{4.20}$$

To describe the eigenspectrum of the resultant closed-loop system of (4.20), theorem 4.1 is applied. The eigenvalues of the matrix A_c are the sum of the eigenvalues of the matrix F_c and the (n-d) undisturbed eigenvalues of the matrix A [21]. This can is demonstrated in theorem 4.1.

Theorem 4.1 : The eigenvalues of the matrix A_c are the sum of the eigenvalues of the matrix F_c and the (n-d) undisturbed eigenvalues of the matrix A.

Proof: Using the similarity transformation

$$x(i) = Mw(i) \tag{4.21}$$

where M is the modal matrix of A. The original system of (3.15) is then transformed into the Jordan canonical form

$$w(i+1) = Jw(i) + \Gamma u(i) \tag{4.22}$$

where

$$J = M^{-1}AM, \quad \Gamma = M^{-1}B \tag{4.23}$$

A substitution of (4.19) and (4.21) into (4.22) yields a resultant feedback system

$$w(i+1) = (J + \Gamma KLM)w(i) \tag{4.24}$$

The similarity transformation preserves the eigenvalues [15,17,21]. Therefore, the eigenspectra of the systems of (4.20) and (4.24) are identical. Let

$$J_c = J + \Gamma KLM \tag{4.25}$$

Making a substitution of (4.8) in (4.25) results in

$$J_c = J + \Gamma KM_0P \tag{4.26}$$

which can be partitioned to obtain

$$J_c = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} [I_r \mid 0] \tag{4.27}$$

$$J_c = \begin{bmatrix} J_1-D_1 & 0 \\ -D_2 & J_2 \end{bmatrix} \tag{4.28}$$

where

$$J_1 = PJP^T = \mathrm{diag}(\lambda_1,\ldots,\lambda_d) \tag{4.29}$$

$$J_2 = \mathrm{diag}(\lambda_{d+1},\ldots,\lambda_n) \tag{4.30}$$

$$D_1 = \text{Top (dxd) submatrix of } \Gamma KM_0 = P\Gamma KM_0 \tag{4.31}$$

$$D_2 = \text{Bottom (n-d)xd} \tag{4.32}$$

The eigenvalues of J_c in (4.28) are given by

$$\sigma(J_c) = \det(zI_d - J_1 + D_1)\det(zI_{(n-d)} - J_2) \tag{4.33}$$

$$\sigma(J_c) = \det(zI_d - J_1 + D_1)\prod_{i=d+1}^{n}(z-\lambda_i) = 0 \tag{4.34}$$

The second term of (4.34) results in the set of eigenvalues $\{\lambda_{d+1},\ldots,\lambda_n\}$, meaning that this set is unaffected by the use of the

feedback gain matrix KL.

The remaining term

$$\det(zI_d - J_1 + D_1) = 0 \tag{4.35}$$

Substitution of (4.31) in (4.35) results in

$$\det(zI_d - J_1 + P\Gamma KM_0) = 0 \tag{4.36}$$

From (4.5), (4.7) and (4.8) it can be shown that, for a diagonal or Jordan canonical form J,

$$F = m_0 J_1 M_0 \qquad G = M_0 \Gamma_1 \tag{4.37}$$

By use of the feed-back law of (4.17), the closed-loop equation for the lower-order system becomes

$$z(i+1) = (m_0 J_1 M_0{}^{-1} + M_0 \Gamma_1 K) z(i) \tag{4.38}$$

$$z(i+1) = M_0 (J_1 + \Gamma_1 K M_0) M_0{}^{-1} \tag{4.39}$$

with eigenvalues given by

$$\det(M_0)\det(zI_d - J_1 + \Gamma_1 K M_0)\det(M_0{}^{-1}) = 0 \tag{4.40}$$

$$\det(z_d - J_1 + \Gamma_1 K M_0) = 0 \tag{4.41}$$

The similarity of (4.36) and (4.41) indicates the eigenvalues of the matrix (A + BKL) are the sum of the eigenvalues of (F + GK) and the undisturbed eigenvalues of the matrix A.

The reduced-order system is controllable if and only if the original system is controllable. This is due to the fact that the reduced

order system contains only a subset of the eigenvalues of the original model.

3. Recursive Algorithm : The eigenspectrum of the original closed-loop system of (4.20) consists of the eigenspectrum of the reduced order closed-loop system of (4.18) and the set of eigenvalues remaining undisturbed. There is no restriction on d, the order of the reduced model. In fact, the proposed method of eigenvalue assignment can be applied to the particular situation in which all the modes are to be shifted. In such case the technique can be applied in a recursive manner, relocating appropriate groups of eigenvalues in each iteration.

Consider the original system

$$x(i+1) = Ax(i) + Bu(i) \tag{4.42}$$

containing a set of n eigenvalues. Let the eigenvalues be divided into two groups: $\{\lambda_1,\lambda_2,\ldots\lambda_d\}$ and $\{\lambda_{d+1},\lambda_{d+2},\ldots\lambda_n\}$. By constructing a d^{th} order model and applying the technique presented here, the eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_d\}$ can be moved to desired new locations or regions.

The control vector is given by

$$u(i) = K_1L_1x(i) + u_1(i) \tag{4.43}$$

which yields a resultant system

$$x(i+1) = (A + BK_1L_1)x(i) + Bu_1(i) \tag{4.44}$$

At this point one is in a position to construct a model to place all, or some, of the remaining eigenvalues to specified new locations. By constructing a $(n-d)^{th}$ order model of (4.44) the eigenvalues $\{\lambda_{d+1},\lambda_{d+2},\ldots\lambda_n\}$ can be shifted to desired new locations or regions. The control vector $u_1(i)$ is given by

$$u_1(i) = K_2L_2x(i) \tag{4.45}$$

The resultant closed-loop system is

$$x(i+1) = [A + B(K_1L_1 + K_2L_2)]x(i) \tag{4.46}$$

and the feedback matrix is given by

$$K = K_1L_1 + K_2L_2 \tag{4.47}$$

This procedure is repeated (m-1) times to yield the feedback control law

$$u_{m-2} = K_{m-1}L_{m-1}x(i) + u_{m-1}(i) \tag{4.48}$$

where the overall feedback gain matrix K_R is given by:

$$K_R = K_1L_1 + K_2L_2 + \dots + K_mL_m \tag{4.50}$$

The number of groups is selected in accordance to the computational requirements.

C. DEADBEAT CONTROL VIA ORDER-REDUCED MODELS

1. Problem Formulation and Discussion : The parametric deadbeat controller of chapter III is designed principally by the approach of closed-loop eigenstructure assignment. In selecting the free parameters available to yield a minimum Frobenius norm controller, one may use a mathematical approach (example 3.1) or a computational algorithm. Depending on the order of the system and the type of deadbeat controller desired, the selection of the free parameters could be a rather complicated task.

An alternate way of driving all the eigenvalues of a linear discrete multivariable system to the origin is via order reduced models. In fact, a parametric deadbeat controller of minimum Frobenius norm can be designed for an order-reduced model that shifts only d, out of the total n, eigenvalues to the origin. As before, having no restrictions on the order of the reduced model allows a recursive algorithm such as the one of IV.B.3 to be used. The deadbeat controller designed in a recursive manner has a higher Frobenius norm that the optimum solution obtained from the large system. Even with this loss, the performance of the controller is still acceptable. On the same token, the computational efforts are greatly reduced.

Consider the system described by the difference equation of (4.42) having the eigenspectrum divided into m appropriate groups

$$(\lambda_1, \ldots, \lambda_{d_1+1}, \ldots, \lambda_{d_2+d_1}, \ldots, \lambda_{n-d_m}, \ldots, \lambda_n)$$

of sizes d_1, $d_2, \ldots, d_m$ respectively, and satisfying $d_1 + d_2 + \ldots + d_m = n$. A satisfactory aggregated model is given by

$$z(i+1) = Fz(i) + Gu(i) \tag{4.52}$$

where $z(i) \in R^d$ and $u(i) \in R^r$. The model inherits a prespecified set of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_{d1})$ of the original system. The reduced order model is controllable provided the original system is controllable. The F and G as in guarantees that the matrix F is invertible. Hence the controllability matrix

$$[A^{-1}B \; A^{-2}B \; \ldots \; A^{-p_1}B] \;, \quad p \le p_1 \le d_1 \tag{4.53}$$

is of full rank. If a linear state-feedback control law

$$u(i) = K_{d_1} z(i) \tag{4.54}$$

is applied to the reduced order model, a closed loop system is obtained in the form

$$z(i+1) = F_c z(i) \ , \quad F_c = F + GK_{d_1} \tag{4.55}$$

and

$$z(i) = F_c^i z(0) \tag{4.56}$$

gives a closed-loop response starting from an initial state z(0). The deadbeat control problem for the reduced model is that of determining an rxd real constant feedback matrix K_{d1} such that the d eigenvalues of the closed-loop system of (4.55) are placed at the origin. A parametric solution to the deadbeat controller K_{d1} is obtained in a similar fashion to the one of chapter III.E.6. Selecting the free parameters available to obtain a minimum Frobenius norm controller is a much easier task if d_1 is chosen as small as possible. The reason for this being that one deals with a smaller matrices and fewer free parameters.

The deadbeat controller K_{d1} is given by

$$K_{d_1} = F_{db_1} V_{db_1}^{-1} \tag{4.57}$$

where F_{db1} is the parametric matrix and V_{db1} the modal matrix of the reduced order model. These matrices are obtained following the procedure of III.E.6. The deadbeat controller is then applied to the original system using the aggregated matrix defined by (4.8). That is

$$u(i) = K_{d1} L_1 x(i) + u_1(i) \tag{4.58}$$

The corresponding system is

$$x(i+1) = A_1x(i) + Bu_1(i)$$
$$A_1 = A + BK_{d_1}L_1 \tag{4.59}$$

The system of (4.59) contains d_1 eigenvalues at the origin and $(n-d_1)$ at the original location. A new reduced model of order d_2 can be constructed to shift the eigenvalues $\{\lambda_{d1+1},\ldots,\lambda_{d2}\}$ to the origin. The procedure to find the deadbeat controller that drives the set of eigenvalues d_2 is similar to the one for the set of eigenvalues d_1. Once a controller K_{d2} and aggregation matrix L_2 are obtained, the resultant system containing the set of d_1+d_2 eigenvalues at the origin is given by

$$x(i+1) = (A_1 + BK_{d_2}L_2)x(i) + Bu_2(i) \tag{4.60}$$

In general, once the first controller is designed, the algorithm is repeated (m-1) times until the n eigenvalues of the original system of (4.42) are driven to the origin. With some manipulation it can be shown that the resultant closed-loop system is given by

$$x(i+1) = A_{cR}x(i), \qquad A_{cR} = (A + BK_R) \tag{4.61.a}$$

where the deadbeat controller resultant of the m iterations is

$$K_R = K_{d_1}L_1 + K_{d_2}L_2 + \ldots + K_{d_m}L_m \tag{4.61.b}$$

with similar properties to the parametric controller designed from the original system. In fact, the structure of the deadbeat controller is, as for the large system, non-unique. The designer has the freedom of selecting the admissible Jordan form that is similar to the resultant closed-loop system matrix. The particular Jordan form is assigned with a set of m controllers, some of which are written in terms of free parameters. In the case of having free parameters available in the

reduced-order controller, they are chosen to yield the minimum Frobenius norm controller.

2. Design Procedure : An algorithm for the design a deadbeat controllers of minimum Frobenius norm via order-reduction techniques is summarized as follows :

(i) Free parameters are usually assigned to the left most vectors of the parametric matrix. Therefore, it is advisable to select the simplified models with orders

$$1 \le d_1 \le d_2 \le \ldots \le d_m \tag{4.62}$$

in an effort to simplify the computations involved in selecting the free parameters.

(ii) The controller resultant of the m iterations is required to drive the system to the origin in p_1 time steps, $(p \le p_1 \le n)$. Therefore, the Jordan forms assigned to each reduced model should be of order $J_{d1}, J_{d2}, \ldots, J_{dm}$. The Jordan form J_{di}, for $i = 1,2,\ldots m$, of F_c has q Jordan blocks $(1 \le q \le r)$. The q Jordan blocks J_{diLj}, for $j = 1,2,\ldots,q$, are of order $d_i L_j$. In order to meet the time-step requirement let :

$$d_1L_1 + d_2L_1 + \ldots + d_mL_1 = p_1 \tag{4.63}$$

(iii) The reduced order model of (4.52) required for each iteration is selected from (4.8), (4.13) and (4.16).

(iv) A deadbeat controller satisfying the feedback control law of (4.54) is designed for minimum Frobenius norm following the procedure of III.E.6. Then, it is applied to the original system using the aggregated matrix L of (4.8). The resultant system takes the form

$$x(i+1) = (A + BK_{d_1}L_1)x(i) + u_1(i) \tag{4.64}$$

(v) Repeating the procedure (m-1) times as described in IV.C one obtains the resultant deadbeat controller K_R of (4.58). The closed-loop system obtained after the last iteration is of the form

$$x(i+1) = A_{cR}x(i), \quad A_{cR} = A + BK_R \tag{4.65}$$

where the matrix A_{cR} is a nilpotent matrix of order p_1 having all of its eigenvalues at the origin. This procedure is demonstrated with numerical examples in the next section.

3. Numerical Example : Consider the system

$$x(i+1) = Ax(i) + Bu(i) \tag{4.66.a}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.00384 & -0.0784 & 0.540 & -1.6 & 2.1 \end{bmatrix} \tag{4.66.b}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{4.67}$$

Here the system being studied has $n = 5$, $r = 3$, $p = 3$, having eigenvalues $\sigma(A) = 0.8, 0.6, 0.4, 0.2, 0.1$. The simplified models selected are of orders $d_1 = 1, d_2 = 1$ and $d_3 = 3$, to drive the eigenvalues in decreasing order of dominance to the origin. The modal matrix of the

matrix A is

$$M = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.8 & 0.4 & 0.4 & 0.2 & 0.1 \\ 0.64 & 0.36 & 0.16 & 0.04 & 0.01 \\ 0.512 & 0.216 & 0.064 & 0.008 & 0.001 \\ 0.4096 & 0.1296 & 0.0256 & 0.0016 & 0.0001 \end{bmatrix} \quad (4.68)$$

Let the matrix

$$P = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0] \quad (4.69)$$

and

$$M_0 = PMP^T = 1 \quad (4.70)$$

The reduced order system containing the eigenvalue $\sigma(z) = 0.8$ is obtained by use of (4.11), (4.12), (4.13) and (4.14). With some manipulation, the expressions take the form

$$F = M_0PM^{-1}AMP^TM^{-1}{}_0 = 0.8 \quad (4.71)$$

$$G = M_0PM^{-1}B = [29.7619 \quad -8.9286 \quad 7.7381] \quad (4.72)$$

and the aggregation matrix is obtained from (4.8) as

$$L = M_0PM^{-1} = [0.143 \quad -2.738 \quad 16.667 \quad -38.691 \quad 29.762] \quad (4.73)$$

Having selected the reduced order system

$$z(i+1) = Fz(i) + Gu(i) \quad (4.74)$$

the deadbeat controller is designed according to the theory of III.E.6. The Jordan form to be assigned is $J_{d1} = [0]$ where $J_{d1L1} = 1$ and $n_0 = 0$.

The number of free parameters available is

$$N = n_0(r - n_0) = 1(3 - 1) = 2 \tag{4.75}$$

which implies that a parametric matrix of the form

$$F_{db1} = [1 \; H_1 \; H_2]^T \tag{4.76}$$

and the modal matrix of the simplified closed-loop system

$$V_{db1} = -F^{-1}GF_{db1} \tag{4.77}$$

are used to find the deadbeat controller of minimum Frobenius norm. The free parameters H_1 and H_2 are obtained from a trial and error computational algorithm that assigns different values to the parameters and checks for the Frobenius norm of the controller. In doing so, the selected values

$$H_1 = -0.3, \quad H_2 = 0.3 \tag{4.78}$$

are used to obtain the deadbeat controller

$$K_{d1} = [-0.0230 \quad 0.0069 \quad -0.0069]^T \tag{4.79}$$

By use of the aggregated matrix L_1, the closed-loop system matrix is

$$A_2 = A + BK_{d_1}L_1 \tag{4.80}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.0010 & 0.0189 & -0.1151 & 1.2671 & -0.2055 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0006 & -0.0154 & 0.1564 & -0.7096 & -0.7096 \end{bmatrix} \tag{4.81}$$

Where A_2 has eigenvalues $\sigma(A_2) = 0.6, 0.4, 0.2, 0, 2, 0$. A second iteration is performed, based on the new system matrix, to drive the dominant eigenvalue $\sigma(A_2) = 0.6$. The modal matrix of A_2 is

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.6 & 0.40 & 0.2 & 0.1 & 0 \\ 0.36 & 0.16 & 0.04 & 0.01 & 0 \\ 0.216 & 0.064 & 0.008 & 0.001 & 0.0008 \\ 0.1296 & 0.0256 & 0.0016 & 0.0001 & 0 \end{bmatrix} \tag{4.82}$$

The corresponding aggregated system is given by

$$F_2 = 0.6 \tag{4.83}$$

$$G_2 = [47.6598 \quad 10.7021 \quad 19.8916] \tag{4.84}$$

$$L_2 = [0.0288 \quad -0.8847 \quad 9.1895 \quad -36.9578] \tag{4.85}$$

Designing the deadbeat controller of minimum Frobenius norm for the aggregated system, the Jordan form $J_{d2} = [0]$ is assigned to the corresponding closed-loop system. Again, the Jordan form is of order one and having $J_{d2L1} = 1$, $n_0 = 1$. The number of free parameters available in the parametric matrix

$$N = n_0(r - n_0) = 1(3 - 1) = 2 \tag{4.86}$$

are assigned in the following manner

$$F_{db_2} = [1 \quad H_1 \quad H_2]^T \tag{4.87}$$

and the corresponding modal matrix V_{db_2} is

$$V_{db_2} = -F^{-1}{}_2 G_2 F_{db_2} \tag{4.88}$$

Using a trial and error computational algorithm as in the first iteration, the selected free parameters are $H_1 = 0.2$, and $H_2 = 0.4$. The deadbeat controller of minimum Frobenius norm is

$$K_{d2} = [-0.0104 \quad -0.0021 \quad -0.0042]^T \tag{4.89}$$

Applying the deadbeat controller to the large system, the corresponding closed-loop matrix is

$$A_3 = A_2 + BK_{d_2}L_2 \tag{4.90}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.0011 & 0.0226 & -0.1533 & 1.4207 & -0.4035 \\ -0.0002 & 0.0055 & -0.0573 & 0.2304 & 0.7029 \\ 0.0001 & -0.0007 & 0.0037 & -0.0953 & 0.6229 \end{bmatrix} \tag{4.91}$$

with eigenvalues $\sigma(A_3) = 0.4, 0.2, 0.1, 0, 0$.

In the third iteration, the order of the simplified system is chosen as 3. The reason being that The eigenvalues $\sigma(A_3) = 0.4$, 0.2 and 0.1 need to be shifted to the origin in one iteration. Here, one may take a closer look at the second step of the design procedure.

The modal matrix corresponding to A_3 is

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0.4 & 0.2 & 0.1 & 1 & 0 \\ 0.16 & 0.04 & 0.01 & 0 & 0 \\ 0.064 & 0.008 & 0.001 & -0.0163 & 0.0008 \\ 0.0256 & 0.0016 & 0.0001 & -0.0014 & 0 \end{bmatrix} \tag{4.92}$$

The third order aggregated system is formed by

$$L_3 = \begin{bmatrix} 1.3 & -24 & 353.8 & -1674.1 & 2337.5 \\ 0.1 & -1.5 & 26.3 & -105.8 & 136.5 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{4.93}$$

$$F_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .008 & -0.14 & 0.7 \end{bmatrix} \tag{4.94}$$

$$G_3 = \begin{bmatrix} 2337.5 & 663.4 & 1017.1 \\ 136.5 & 30.6 & 57.0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.95}$$

A minimum time minimum number of free parameters controller is designed so that the Jordan form

$$J_{d_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} J_{d_3 L_1} = 1 \\ n_0 = 3 \end{array} \tag{4.96}$$

is assigned to the reduced-order closed-loop system. The number of free parameters available is

$$N = n_0(r-n_0) = 3(3-3) = 0 \tag{4.97}$$

Therefore, the structure of the parametric matrix is

$$F_{db_3} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} f_1^{(0)} & f_2^{(0)} & f_3^{(0)} \end{matrix} \end{array} \tag{4.98}$$

where the vectors of the parametric matrix are used to find the modal matrix of the simplified closed-loop system. The vectors that form the

modal matrix are

$$v_1^{(0)} = -F_3^{-1}G_3f_1^{(0)} \tag{4.99}$$

$$v_2^{(0)} = -F_3^{-1}G_3f_2^{(0)} \tag{4.100}$$

$$v_3^{(0)} = -F_3^{-1}G_3f_3^{(0)} \tag{4.101}$$

and the modal matrix is

$$V_{db_3} = [v_1^{(0)} \quad v_2^{(0)} \quad v_3^{(0)}] \tag{4.102}$$

which results in the deadbeat controller given by

$$K_{d_3} = F_{db_3}V_{db_3}^{-1} = \begin{bmatrix} 0.0028 & -0.0474 & 0.2099 \\ 0.0024 & -0.0492 & 0.3337 \\ -0.008 & 0.14 & -0.70 \end{bmatrix} \tag{4.103}$$

The deadbeat controller designed for the simplified model is fed back to the large system by use of the aggregation matrix. That is

$$A_4 = A_3 + BK_{d_3}L_3 \tag{4.104}$$

where

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0008 & -0.0163 & 0 & 0 \\ 0 & 0 & -0.0014 & 0 & 0 \end{bmatrix} \tag{4.105}$$

It can be shown that the matrix A_4 is a nilpotent matrix of order

three. This is expected since the Jordan forms assigned to the simplified models yielded $p_1 = 3$ for (4.63). Further, All of the eigenvalues of A_4 are located at the origin.

The deadbeat controller resultant of the three iterations is

$$K_R = K_{d_1}L + K_{d_2}L_2 + K_{d_3}L_3 \tag{4.106}$$

and

$$K_R = \begin{bmatrix} -0.0038 & 0.0776 & -0.5251 & 1.6 & -1.1 \\ 0 & 0.0008 & -0.0163 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{4.107}$$

Having a Frobenius norm $[K_R]_F = 2.66$. Recall that the Frobenius norm of the controller designed for the original system is $[K]_F = 2.64$. A comparison of the performance of the deadbeat controller designed from the large system to one designed using the reduced models is shown in figure 6.

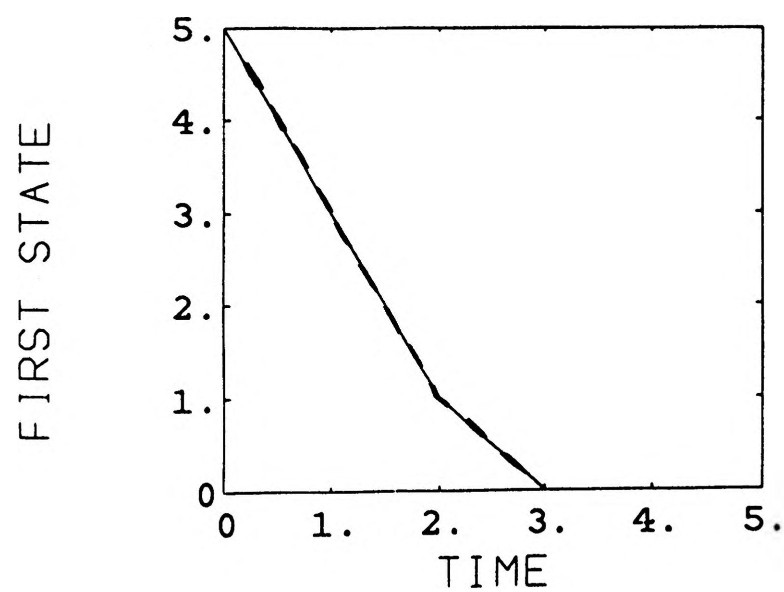

Figure 6. Initial conditions forced to the origin by a deadbeat controller designed from the large system (solid) and order reduced system using iterations (dashed).

a. First state.

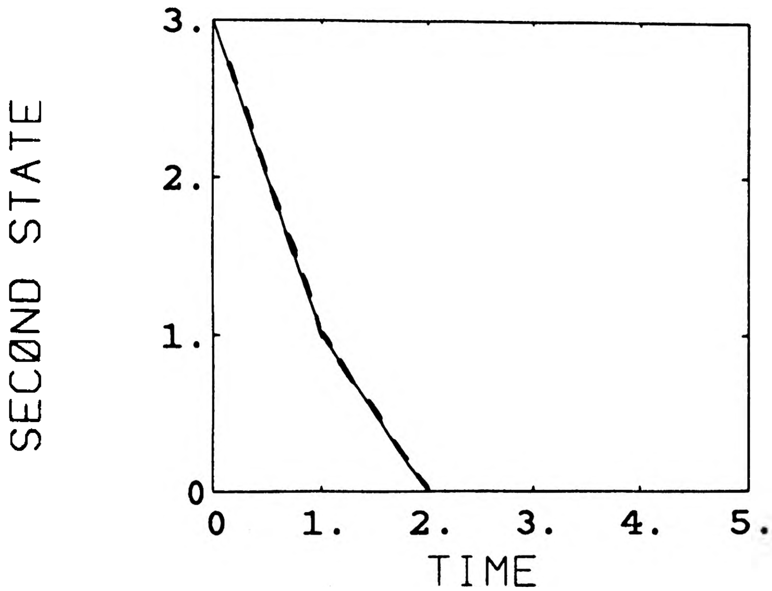

Figure 6 (cont.) b. **Second state driven to the origin of the state space.**

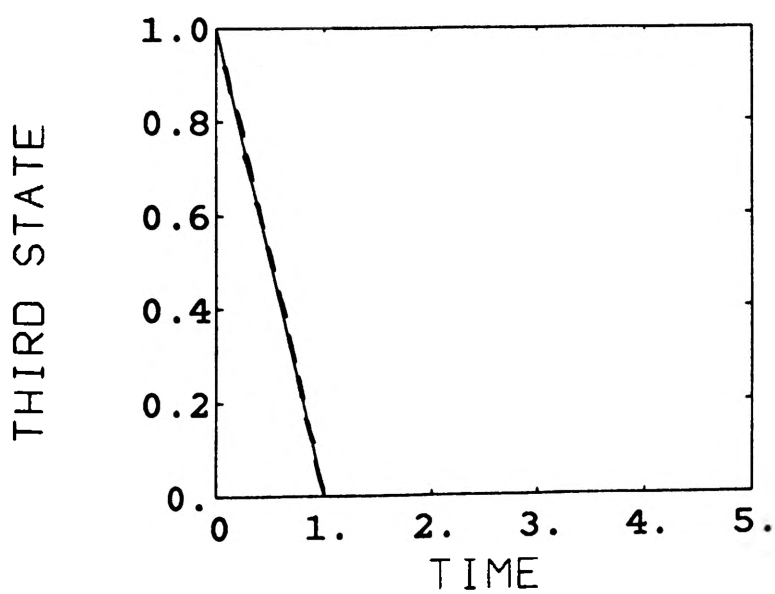

Figure 6 (cont.) c. **Third state driven to the origin of the state space.**

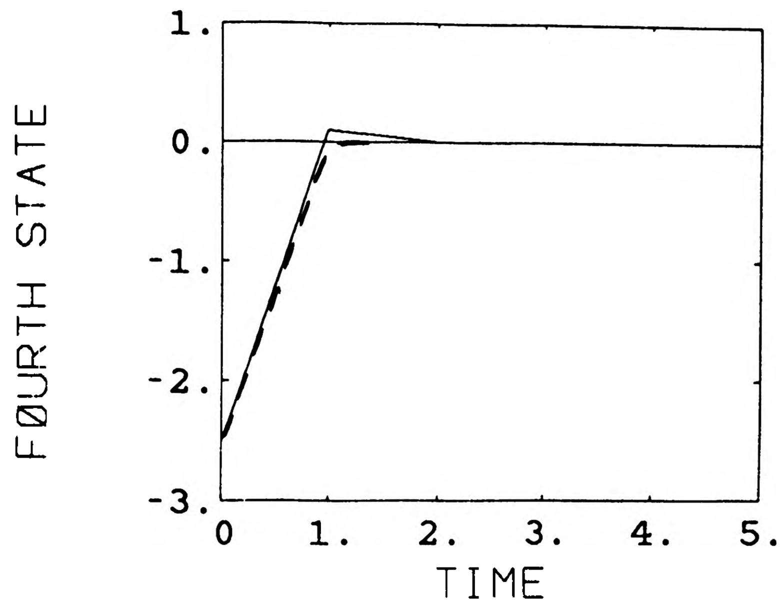

Figure 6 (cont.) d. Fourth state driven to the origin of the state space.

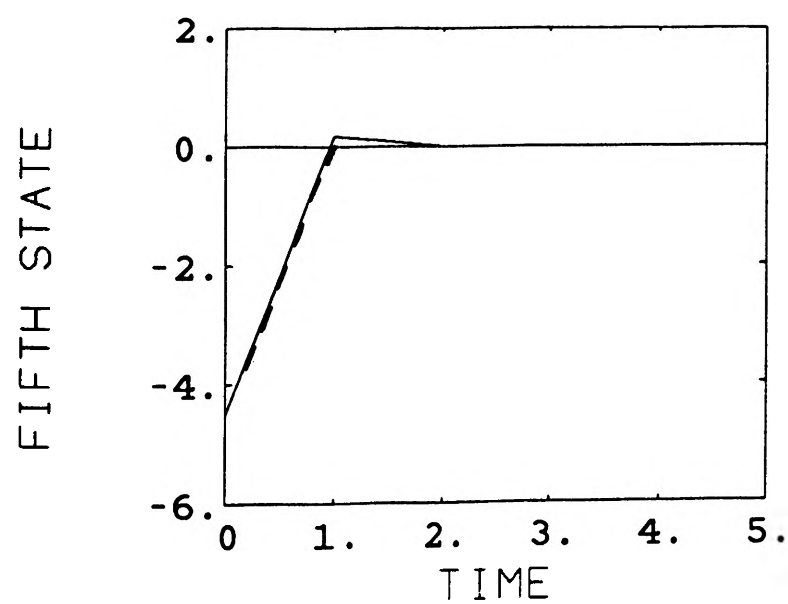

Figure 6 (cont.)e. Fifth state driven to the origin of the state space.

Comparing the closed-loop system responses using the two controllers it is observed that they are almost the same. In fact, the controller designed using order reduced models has a smoother response. Also, the computational savings in selecting the parameters that yield a minimum Frobenius norm controller are significant (cc. Appendix E).

V. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

A. CONCLUSIONS

This thesis addresses the design of multivariable deadbeat controllers based on an eigenvalue assignment approach. The non-uniqueness of the structure of the feedback matrix is utilized to develop a parametric solution. A proper selection of the parameters yields a minimum Frobenius norm controller.

Two criteria used to select the parameters are a mathematical approach, and a trial and error iterative algorithm. It is shown that depending on the order of the system, the computational requirements for the selection of parameters are considerably large.

The applications of eigenvalue assignment using reduced order models to the design of minimum Frobenius norm controllers is presented. In achieving the latter, the controller effort is reduced to a minimum. Multivariable controllers obtained from the original system and reduced order models are utilized to achieve a deadbeat response. A comparison of the responses demonstrates a close approximation.

Reduced order models are proven to be an important tool for the design of multivariable deadbeat controllers. Freedom exists in selecting the group of eigenvalues to be forced to the origin. This procedure is performed recursively. Hence, there are significant computational savings in choosing the parameters that yield a minimum Frobenius norm controller.

B. SUGGESTIONS FOR FURTHER WORK

In designing a deadbeat controller using reduced order models, a group of eigenvalues is forced to the origin by a controller. The

procedure is repeated until all the eigenvalues of the closed-loop system are driven to the origin. In each iteration, the technique requires the use of a modal matrix. Finding a modal matrix for a system matrix containing repeated eigenvalues is not always trivial. A program for this purpose can be of great use.

A flexible program can be written for the purpose of selecting parameters which yield a minimum Frobenius norm controller. It is recommended that such a program be heuristic and written in a language like Turbo-Prolog. The reason for this being that Turbo-Prolog eases symbolical mathematical operations.

A challenging software project would be the implementation of the entire algorithm for the design of deadbeat controllers of minimum Frobenius norm using order reduction techniques. Such an algorithm would be required to select the appropriate groups of eigenvalues to be placed recursively at the origin. The selection of the groups should yield a minimum-time deadbeat controller with a minimum number of parameters. All mathematical computations required by the algorithm presented in this thesis should be included.

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VITA

Joaquin Alberto Zuniga was born on June 23, 1963 in Cucuta, Colombia. He received his primary education at the Colegio Salesianto. His secondary schooling was divided between : Colegio La Salle, Seminario Diocesano San Jose de Cucuta, and Ft. Zumwalt Senior High School. He commenced his college career at Southeast Missouri State University - Cape Girardeau, whence he transferred to the University of Missouri - Rolla. He graduated from U.M.R. in July 1985 with a Bachelor of Science degree in Electrical Engineering.

Joaquin decided to continue his studies in the Graduate School at U.M.R. where he has been enrolled since his graduation. He has been a Teaching Assistant for four semesters, during which time he has taught two different junior/senior year Electrical Laboratories.

APPENDIX A

EIGENVECTORS OF THE CLOSED LOOP SYSTEM

$S(z_i)f_I$ are the assigned closed-loop eigenvectors v_i defined by the eigenvalue-eigenvector equations.

$$A_c v_i = z_i v_i \quad \text{for } i = 1,2,\ldots,n. \tag{A.1}$$

Proof : Let

$$v_i = S(z_i)f_i = (z_i I_n - A)^{-1}Bf_i \tag{A.2}$$

and

$$A_c = A + BK \tag{A.3}$$

then

$$A_c v_i = (A + BK)(z_i I_n - A)^{-1}Bf_i \tag{A.4}$$

$$= A(z_i I_n - A)^{-1}Bf_i + BK(z_i I_n - A)^{-1}Bf_i \tag{A.5}$$

Using (3.21)

$$f_i = K(z_i I_n - A)^{-1}Bf_i \tag{A.6}$$

then

$$A_c v_i = A(z_i I_n - A)^{-1}Bf_i + Bf_i \tag{A.7}$$

$$= [A(z_i I_n - A)^{-1} + I_n]Bf_i \tag{A.8}$$

$$= [A(z_i I_n - A)^{-1} + (z_i I_n - A)(z_i I_n - A)^{-1}]Bf_i \tag{A.9}$$

$$= (A + z_i I_n - A)(z_i I_n - a)^{-1}Bf_i \tag{A.10}$$

$$= z_i I_n (z_i I_n - A)^{-1} B f_i = z_i (z_i I_n - A)^{-1} B f_i \qquad \text{(A.11)}$$

$$= z_i v_i \qquad \text{(A.12)}$$

therefore

$$A_c v_i = z_i v_i \qquad \text{(A.13)}$$

APPENDIX B

EIGENVECTOR EXPANSION OF THE RANGE SPACE OF A_c

To show that the eigenvectors and generalized eigenvectors $v^{(0)}{}_j$, $v^{(1)}{}_j, \ldots, v^{(pj-1)}{}_j$ span the range space of the closed-loop system.

From (3.78)

$$-K[A^{-2}Bf_j^{(0)} + A^{-1}Bf_j^{(1)}] = f_j^{(1)} \tag{B.1}$$

$$-KA^{-1}[A^{-1}Bf_j^{(0)} + Bf_j^{(0)}] = f_j^{(1)} \tag{B.2}$$

and (3.72)

$$v_j^{(0)} = [-A^{-1}Bf_j^{(0)}] \tag{B.3}$$

then

$$-KA^{-1}[-v_j^{(0)} + Bf_j^{(1)}] = f_j^{(1)} \tag{B.4}$$

$$KA^{-1}v_j^{(0)} = [I_r + KA^{-1}B]f_j^{(1)} \tag{B.5}$$

In the same manner, given the following :

$$-K[A^{-3}Bf_j^{(0)} + A^{-2}Bf_j^{(1)} + A^{-1}Bf_j^{(2)}] = f_j^{(2)} \tag{B.6}$$

$$v_j^{(1)} = -A^{-2}Bf_j^{(0)} - A^{-1}Bf_j^{(1)} \tag{B.7}$$

$$KA^{-1}v_j^{(1)} = [I_r + KA^{-1}B]f_j^{(2)} \tag{B.8}$$

Thus for a generalized system, the proceeding is true.

$$-K[A^{-1}Bf_j^{(p_j-1)} + A^{-2}Bf_j^{(p_j-2)} +..+ A^{-p_j}Bf_j^{(0)}] = f_j^{(p_j-1)} \tag{B.9}$$

and

$$v_j^{(p_j-2)} = [-A^{-p_j+1}Bf_j^{(0)} - \ldots - A^{-1}Bf_j^{(p_j-2)}] \tag{B.10}$$

$$-KA^{-1}[-v_j^{(p_j-2)} + Bf_j^{(p_j-1)}] = f_j^{(p_j-1)} \tag{B.11}$$

therefore,

$$KA^{-1}v_j^{(p_j-2)} = [I_r + KA^{-1}B]f_j^{(p_j-1)} \tag{B.12}$$

Summarizing

$$KA^{-1}v_j^{(0)} = [I_r + KA^{-1}]f_j^{(1)} \tag{B.13}$$

$$KA^{-1}v_j^{(1)} = [I_r + KA^{-1}B]f_j^{(2)} \tag{B.14}$$

$$KA^{-1}v_j^{(p_j-2)} = [I_r + KA^{-1}B]f_j^{(p_j-2)} \tag{B.15}$$

for $j = 1,2,\ldots,q$

APPENDIX C

A SPECIAL CASE OF PARAMETRIC VECTORS

Parametric Vectors for the case $q = r$:

An alternate espression for (3.78) is developed for the case in which the number of Jordan blocks of A_c is the same as the number of inputs to the system.

Given :

$$KA^{-1}B = I_r \tag{C.1}$$

$$KA^{-L}B = 0, \qquad L = 1,2,\ldots,p_r \tag{C.2}$$

and let

$$L_c = [I_r + KA^{-1}B] \tag{C.3}$$

then (3.78) can be arranged in the form

$$\begin{aligned}
&[L_c f_j^{(0)}] = 0 \\
&[L_c f_j^{(0)} + KA^{-2}Bf_j^{(0)}] = 0 \\
&\quad \cdot \\
&\quad \cdot \\
&[L_c f_j^{(p_r-1)} + KA^{-2}Bf_j^{(p_r-2)} + \ldots + KA^{-p_r}Bf_j^{(0)}] = 0 \\
&[L_c f_j^{(p_r)} + KA^{-2}Bf_j^{(p_r-1)} + \ldots + KA^{-(p_r+)}Bf_j^{(0)}] = 0 \\
&\quad \cdot \\
&\quad \cdot
\end{aligned} \tag{C.4}$$

$$[L_c f_j^{(p_j-1)} + KA^{-2}Bf_j^{(p_j)} + \ldots + KA^{-(p_j+1)}Bf_j^{(p_j-p_r-1)}$$

$$+ \ldots + KA^{-p_j}Bf_j^{(0)}] = 0$$

where the first p_r vector equations disappear.

$$[KA^{-(p_r+1)}Bf_j^{(0)}] = 0$$

$$K[A^{-(p_r+1)}Bf_j^{(1)} + A^{-(p_r+2)}Bf_j^{(0)}] = 0$$

$$\cdot \qquad \text{(C.5)}$$

$$\cdot$$

$$K[A^{-(p_r+1)}Bf_j^{(p_j-p_r-1)} + \ldots + A^{-p_j}Bf_j^{(0)}] = 0$$

for $j = 1,2,\ldots,n_{pr}$, and $n_{pr} = r = q$.

APPENDIX D

PROGRAM 1

CTRL-C program used to select the parameters that yield a minimum Frobenius norm controller.

```
//[minnorm,LL1,LL2,LL3,LL4] = Forbini(Mini,Maxi,Stepi,norm0,A,B)
L1 = -Inv(A)*B;
L2 = -inv(A)*inv(A)*B;
L3 = -inv(A)*inv(A)*inv(A)*B;
Minnorm = norm0;
For I = Mini:stepi:Maxi;...
 For J = Mini:Stepi:Maxi;...
  For W = Mini:Stepi:Maxi;...
   For L = Mini:Stepi:Maxi;...
        f10 = [1,I,J]';...
        f11 = [0,W,L]';...
        f12 = [0,0,0]';...
        f20 = [0,1,0]';...
        f30 = [0,0,1]';...
       F = [f10,f11,f12,f20,f30];...
        v10 = L1*f10;...
        v11 = L1*f11 + L2*f10;...
        v12 = L2*f11 + L3*f10;...
        v20 = L1*f20;...
        v30 = L1*f30;...
       v = [v10,v11,v12,v20,v30];...
    if v <> 0 , k=f*inv(v);...
          norma = 0;...
          For I1 = 1:3;...
           For J1 = 1:5;...
             norma = norma + K(I1,J1)*k(I1,J1);...
           End;...
          End;...
          norma = sqrt(norma);...
          If norma < minnorm , minnorm = norma,...
                    LL1 = I,...
                    LL2 = J,...
                    LL3 = W,...
                    LL4 = L,...
                    End;...
    End;...
    End;...
   End;...
  End;...
 End;
```

APPENDIX E

PROGRAM 2

CTRL-C program used to select the parameters that yield the minimum Frobenius norm controller for the reduced order model.

```
//[minnorm,L1,L2] = Forbini(Mini,Maxi,Stepi,norm0,F,G)
Minnorm = norm0;
 For J = Mini:Stepi:Maxi;...
     For L = Mini:Stepi:Maxi;...
          f10 = [1,J,L]';...
         FF = [f10];...
          v10 = -Inv(F)*G*f10
         v = [v10];...
         if v <> 0 , k=FF*inv(v);...
                  norma = 0;...
                  For I1 = 1:3;...
                     norma = norma + K(I1,1)*K(I1,1);...
                  End;...
                  norma = sqrt(norma);...
              If norma < minnorm , minnorm = norma,...
                             L1 = J,...
                             L2 = L,...
                             End;...
          End;...
      End;...
    End;...
```