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Concentration of Solar Energy Using a Cassegrain Type Solar Furnace

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CONCENTRATION OF SOLAR ENERGY
USING A CASSEGRAIN TYPE
SOLAR FURNACE

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Abstract

A solar furnace consisting of a paraboloid of revolution that tracks the sun, and a hyperboloid of revolution reflector that has a focus in common with the paraboloid is analyzed using a three-dimensional ray trace to determine the image shape and size, and for the concentration to be obtained for various eccentricities e of the hyperboloid when used with an 84 foot diameter paraboloid (radar dish).

INTRODUCTION

A schematic view of the problem to be considered is shown in Figure 1. The paraboloid tracks the sun, and the sun's image is reflected to the hyperboloidal surface, and then re-reflected to the focal plane.

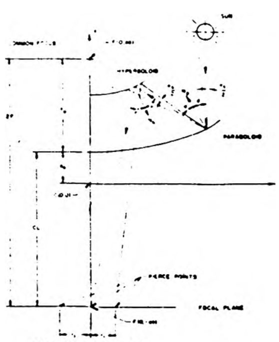

Fig. 1 Cassegrain Geometry

THREE DIMENSIONAL RAY TRACE

Vector $\bar{a}$, ray from the sun

A ray from any position on the sun may be given as

$$\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k} \tag{1}$$

and

$$A = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{2}$$

so

$$\bar{a} = \frac{\bar{A}}{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} \tag{3}$$

Equation (3) is the normal form of a ray from the sun.

Paraboloid of revolution (concentrator)

The equation of a paraboloid of revolution, see Figure 1, is

$$y = a_o + \frac{(x^2 + z^2)}{4f_o} \tag{4}$$

where

a_o = distance from the vertex of the paraboloid to the origin

f_o = focal length of the paraboloid

and in Figure 1, $r = x^2 + z^2$. We may also write

$$\phi_p = y - \frac{(x^2 + z^2)}{4f_o} = a_o \tag{5}$$

Normal to the paraboloid, $\bar{b}$

$$\nabla\phi_p = \frac{\partial\phi}{\partial x}p\bar{i} + \frac{\partial\phi}{\partial y}p\bar{j} + \frac{\partial\phi}{\partial z}p\bar{k} = \frac{-2x\bar{i}}{4f_o} + \bar{j} - \frac{2z\bar{k}}{4f_o}$$

$$= \bar{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k} \tag{6}$$

and

$$B = \sqrt{B_1^2 + B_2^2 + B_3^2} \tag{7}$$

so

$$\bar{b} = \frac{\bar{B}}{B} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k} \tag{8}$$

Angle θ between $\bar{a}$, $\bar{b}$

From Figure 2, we see that

$$\bar{a} \cdot \bar{b} = ab \cos\theta = \cos\theta \tag{9}$$

so

$$\theta = \cos^{-1}[a_1b_1 + a_2b_2 + a_3b_3] \tag{10}$$

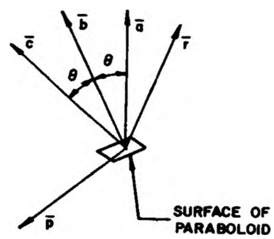

Fig. 2 Vectors at Surface of Paraboloid

Vector $\bar{p}$, normal to $\bar{a}$, $\bar{b}$

$$\bar{P} = \bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= P_1\bar{i} + P_2\bar{j} + P_3\bar{k} \tag{11}$$

and

$$P = \sqrt{P_1^2 + P_2^2 + P_3^2} \tag{12}$$

so

$$\bar{p} = \frac{\bar{P}}{P} = p_1\bar{i} + p_2\bar{j} + p_3\bar{k} \tag{13}$$

Plane ϕ_{ab} containing $\bar{a}$, $\bar{b}$

$$\phi_{ab} = p_1x + p_2y + p_3z = d_{ab} \tag{14}$$

where

$$d_{ab} = p_1x_o + p_2y_o + p_3z_o \tag{15}$$

where (x_o, y_o, z_o) are the coordinates of any point of interest on the paraboloid.

Reflected ray $\bar{c}$

The vector equations for determining $\bar{c}$ are:

$$\bar{b} \cdot \bar{c} = bc \cos\theta = \cos\theta \tag{16}$$

$$\bar{a} \cdot \bar{c} = ac \cos 2\theta = \cos 2\theta \tag{17}$$

$$\bar{p} \cdot \bar{c} = 0 \tag{18}$$

Thus

$$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k} \tag{19}$$

where

$$c_1 = \frac{D_1}{|a|}, \quad c_2 = \frac{D_2}{|a|} \quad \text{and} \quad c_3 = \frac{D_3}{|a|} \tag{20}$$

and

$$|a| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{21}$$

and D_r, $r = 1,2,3$ is determined by replacing the rth column of $|a|$ with a column of k's. Also

$$a_{11} = b_1 \quad a_{21} = a_1 \quad a_{31} = p_1 \quad k_1 = \cos\theta$$

$$a_{12} = b_2 \quad a_{22} = a_2 \quad a_{32} = p_2 \quad k_2 = \cos 2\theta$$

$$a_{13} = b_3 \quad a_{23} = a_3 \quad a_{33} = p_3 \quad k_3 = 0$$

Vector $\bar{r}$, normal to $\bar{c}$, $\bar{p}$

$$\bar{R} = \bar{c} \times \bar{p} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ c_1 & c_2 & c_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

$$= R_1\bar{i} + R_2\bar{j} + R_3\bar{k} \tag{22}$$

and

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2} \tag{23}$$

so

$$\bar{r} = \frac{\bar{R}}{R} = r_1\bar{i} + r_2\bar{j} + r_3\bar{k} \tag{24}$$

Plane ϕ_{cp} containing $\bar{c}$, $\bar{p}$

$$\phi_{cp} = r_1x + r_2y + r_3z = d_{cp} \tag{25}$$

where

$$d_{cp} = r_1x_o + r_2y_o + r_3z_o \tag{26}$$

Intersection of reflected ray $\bar{c}$ with the hyperboloid

We previously developed the equations of two planes which contain $\bar{c}$, so we may write Equations (14) and (25) alternately as

$$p_1x + p_3z = d_{ab} - p_2y \tag{27}$$

$$r_1x + r_3z = d_{cp} - r_2y \tag{28}$$

and so

$$x = \frac{D_1}{|b|}, \quad z = \frac{D_2}{|b|} \tag{29}$$

where

$$|b| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}, \quad D_1 = \begin{vmatrix} \ell_1 & b_{12} \\ \ell_2 & b_{22} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} b_{11} & \ell_1 \\ b_{21} & \ell_2 \end{vmatrix} \tag{30}$$

and where

$$b_{11} = p_1 \quad b_{21} = r_1 \quad \ell_1 = d_{ab} - p_2 y$$

$$b_{12} = p_3 \quad b_{22} = r_3 \quad \ell_2 = d_{cp} - r_2 y$$

Thus we can write

$$x = BB_1 + BB_2 y \tag{31}$$

and

$$z = BB_3 + BB_4 y \tag{32}$$

where y is known, or assumed.

Hyperboloid of revolution (cassegrain reflector)

The equation for a hyperboloid of revolution, see Figure 1, is

$$\frac{y^2}{a^2} - \frac{(x^2 + z^2)}{b^2} = 1 \tag{33}$$

where

$$F = ae = \frac{1}{2} \text{ distance between foci}$$

and

$$e = \sqrt{\frac{a^2 + b^2}{a}} > 1 = \text{eccentricity}$$

So, using Equations (31) and (32), we may write alternately

$$F(y) = \frac{y^2}{a^2} - 1 - \frac{1}{b^2}[(BB_1 + BB_2 y)^2 + (BB_3 + BB_4 y)^2] \tag{34}$$

and

$$F'(y) = \frac{2y}{a^2} - \frac{2}{b^2}[(BB_1 + BB_2 y)BB_2 + (BB_3 + BB_4 y)BB_4] \tag{35}$$

Newton's method

A recursion method for finding y accurately, when an approximate value of y is known, is

$$y_{n+1} = y_n - \frac{F(y_n)}{F'(y_n)} \tag{36}$$

Thus starting with an estimate of y, (y_n), a new value of y, $(y,_{n+1})$, is computed. The process is continued until the difference between y_{n+1} and y_n is as small as desired. Using this method, the coordinates of the pierce point of $\bar{c}$ with the hyperboloid are found, namely (x_1, y_1, z_1).

Normal to the hyperboloid, $\bar{d}$

From Equation (33), we may write

$$\nabla\phi_n = \frac{\partial\phi}{\partial x}n\bar{i} + \frac{\partial\phi}{\partial y}n\bar{j} + \frac{\partial\phi}{\partial z}n\bar{k}$$

$$= -\frac{2x\bar{i}}{b^2} + \frac{2y\bar{j}}{a^2} - \frac{2x\bar{k}}{b^2} = \bar{D}$$

$$= D_1\bar{i} + D_2\bar{j} + D_3\bar{k} \tag{37}$$

and

$$D = \sqrt{D_1{}^2 + D_2{}^2 + D_3{}^2} \tag{38}$$

so

$$\bar{d} = \frac{\bar{D}}{\bar{D}} = d_1\bar{i} + d_2\bar{j} + d_3\bar{k} \tag{39}$$

Angle γ between $\bar{c}$, $\bar{d}$

From Figure 3, we see that

$$\bar{c} \cdot \bar{d} = cd \cos\gamma = \cos\gamma \tag{40}$$

so

$$\gamma = \cos^{-1}[c_1 d_1 + c_2 d_2 + c_3 d_3] \tag{41}$$

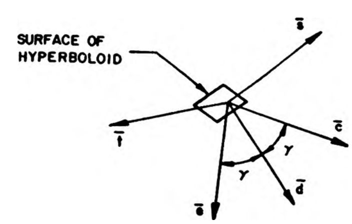

Fig. 3 Vectors at Surface of Hyperboloid

Vector $\bar{t}$, normal to $\bar{c}$, $\bar{d}$

$$\bar{T} = \bar{c} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$= T_1\bar{i} + T_2\bar{j} + T_3\bar{k} \tag{42}$$

and

$$T = \sqrt{T_1^2 + T_2^2 + T_3^2} \tag{43}$$

so

$$t = \frac{\bar{T}}{T} = t_1\bar{i} + t_2\bar{j} + t_3\bar{k} \tag{44}$$

Re-reflected ray $\bar{e}$

The vector equations for determining $\bar{e}$ are:

$$\bar{d} \cdot \bar{e} = de \cos\gamma = \cos\gamma \tag{45}$$

$$\bar{c} \cdot \bar{e} = ce \cos 2\gamma = \cos 2\gamma \tag{46}$$

$$\bar{t} \cdot \bar{e} = 0 \tag{47}$$

Thus

$$\bar{e} = e_1\bar{i} + e_2\bar{j} + e_3\bar{k} \tag{48}$$

where

$$e_1 = \frac{D_1}{|c|}, \quad e_2 = \frac{D_2}{|c|} \text{ and } e_3 = \frac{D_3}{|c|} \tag{49}$$

and

$$|c| = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

and $D_r, r = 1,2,3$ is determined, as before by replacing the rth column of $|c|$ by a column k's. Also

$c_{11} = d_1$ $\quad c_{21} = c_1$ $\quad c_{31} = t_1$ $\quad k_1 = \cos\gamma$

$c_{12} = d_2$ $\quad c_{22} = c_2$ $\quad c_{32} = t_2$ $\quad k_2 = \cos 2\gamma$

$c_{13} = d_3$ $\quad c_{23} = c_3$ $\quad c_{33} = t_3$ $\quad k_3 = 0$

Vector $\bar{s}$, normal to $\bar{e}$, $\bar{t}$

$$\bar{S} = \bar{e} \times \bar{t} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ e_1 & e_2 & e_3 \\ t_1 & t_2 & t_3 \end{vmatrix}$$

$$= S_1\bar{i} + S_2\bar{j} + S_3\bar{k} \tag{51}$$

and

$$S = \sqrt{S_1^2 + S_2^2 + S_3^2} \tag{52}$$

so

$$\bar{s} = \frac{\bar{S}}{S} = s_1\bar{i} + s_2\bar{j} + s_3\bar{k} \tag{53}$$

Plane ϕ_{cd} containing $\bar{c}$,$\bar{d}$

$$\phi_{cd} = t_1x + t_2y + t_3z = d_{cd} \tag{54}$$

where

$$d_{cd} = t_1x_1 + t_2y_1 + t_3z_1 \tag{55}$$

and where (x_1,y_1,z_1) are the coordinates of the point of intersection of the ray $\bar{c}$ with the hyperboloid.

Plane ϕ_{te} containing $\bar{t}$,$\bar{e}$

$$\phi_{te} = s_1x + s_2y + s_3z = d_{te} \tag{56}$$

where

$$d_{te} = s_1x_1 + s_2y_1 + s_3z_1 \tag{57}$$

Plane ϕ_F containing the focal plane

$$\phi_F = f_1x + f_2y + f_3z$$
$$= f_2y = y = d_f \tag{58}$$

where

$$d_f = y_2 \tag{59}$$

and y_2 is the coordinate of the focal plane.

Intersection of the re-reflected ray $\bar{e}$ with the focal plane

The controlling equations for finding the pierce of $\bar{e}$ in the focal plane are:

$$s_1x + s_2y + s_3z = d_{te} \tag{60}$$

$$t_1x + t_2y + t_3z = d_{cd} \tag{61}$$

$$y = y_2 \tag{62}$$

so

$$x = x_2 = \frac{D_1}{|d|}, \quad z = z_2 = \frac{D_2}{|d|} \tag{63}$$

where

$$|d| = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}, \quad D_1 = \begin{vmatrix} \ell_1 & d_{12} \\ \ell_2 & d_{22} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} d_{11} & \ell_1 \\ d_{21} & \ell_2 \end{vmatrix} \tag{64}$$

and where

$$d_{11} = s_1 \qquad d_{21} = t_1 \qquad \ell_1 = d_{te} - s_{2y}$$

$$d_{12} = s_3 \qquad d_{22} = t_3 \qquad \ell_2 = d_{cd} - t_{2y}$$

Thus the pierce point (x_2,y_2,z_2) of the ray $\bar{e}$ with the focal plane is now known.

Radius of the pierce point in the focal plane, r_t

$$r_t = \sqrt{x_2^2 + z_2^2} \tag{65}$$

Figures 4-8 show the image size and shape having been reflected from points on the paraboloid having radii from the axis of symmetry r_p = 2,12,22,32 and 42 feet, respectively. It is seen that the image shapes are not circular, but are not severely distorted from a circular shape. This is in contrast to the images cast by the sun, from points on the paraboloid, to a flat plate oriented perpendicular to the axis of symmetry and placed at the focus of the paraboloid at low angles of incidence.

Concentration ratio

Writing an identity for the energy arriving at the image, gives

$$\pi(r_p^2 - r_h^2)S_c\rho_p\rho_h = \pi r_{tm}^2 S_t \tag{66}$$

and defining

$$C_{av} = \frac{S_t}{S_c} = \frac{(r_p^2 - r_h^2)}{r_{tm}^2}\rho_p\rho_h \tag{67}$$

where

S_c = solar constant

S_t = average flux across the image at the focal plane

r_p = radius of parabolal considered

r_h = radius of necessary hyperboloid

$\rho_p\rho_h$ = hemispherical reflectivity of paraboloidal surface and hyperboloidal surface, respectively (averaged)

r_t = radius of ray pierce point in the focal plane measured from the focal point of the hyperboloid

r_{tm} = maximum radius of ray pierce point, considering the entire paraboloid

We may also write

$$C_{av} = C_i\eta_e \tag{68}$$

where

$$C_i = \frac{(r_p^2 - r_h^2)}{r_{tm}^2} = \text{ideal concentration (geometric)} \tag{69}$$

and

$$\eta_e = \rho_p\rho_h = \text{combined reflectivity factor} \tag{70}$$

The concentration above may be increased using a compound paraboloidal concentrator (or a cone can be used with less effectiveness) (2) according to

$$C_{cpc} = \frac{1}{\sin^2\zeta} = \text{concentration by compound paraboloidal concentrator} \tag{71}$$

and

$$C_{aug} = C_{cpc}C_{av} = \text{concentration augmented} \tag{72}$$

where from Figure 9

$$\zeta = \tan^{-1}\left(\frac{r_t + r_h}{F + y_1}\right) \tag{73}$$

Thus

$$C_{aug} = C_{cpc}C_i\eta_e = \frac{(r_p^2 - r_h^2)}{r_{tm}^2}\,\frac{\rho_p\rho_h}{\sin^2\zeta} \tag{74}$$

and

$$C_{aug_i} = C_{cpc}C_i = \frac{C_{aug}}{\eta_e} = \text{ideal augmented concentration} \tag{75}$$

Figure 10 shows a plot of the ideal concentration, C_i vs. the eccentricity e, for various values of CL, the distance from the vertix of the paraboloid to the focal plane (see Figure 1).

Figure 11 shows a plot of the ideal augmented concentration, C_{aug_i} vs. the eccentricity e, for various values of CL.

Figure 12 shows a plot of the necessary radius of the hyperboloid, r_h vs. the eccentricity e, for various values of CL.

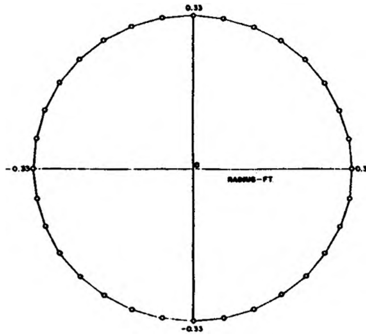

Fig. 4 Image Shape, r_p = 2 feet

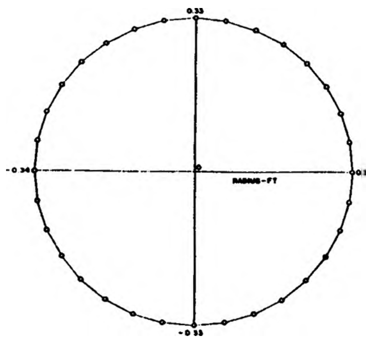

Fig. 5 Image Shape, r_p = 12 feet

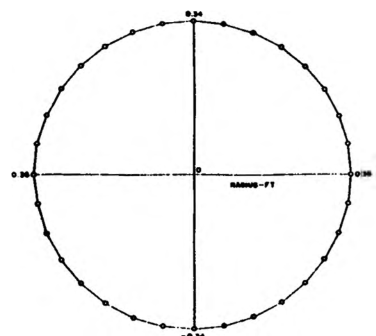

Fig. 6 Image Shape, r_p = 22 feet

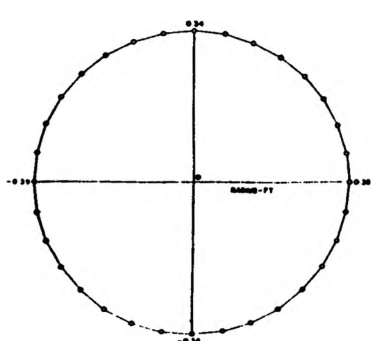

Fig. 7 Image Shape, r_p = 32 feet

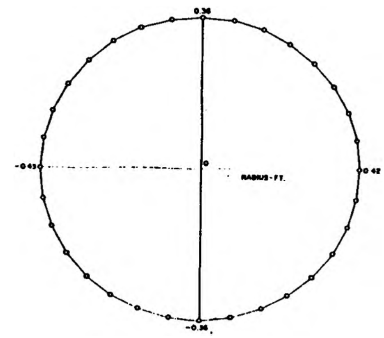

Fig. 8 Image Shape, r_p = 42 feet

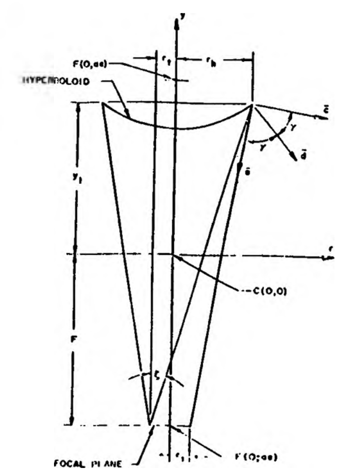

Fig. 9 Hyperboloid and Image Geometry

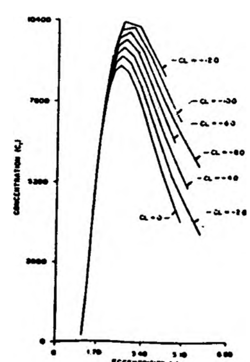

Fig. 10 Ideal Concentration vs. Eccentricity

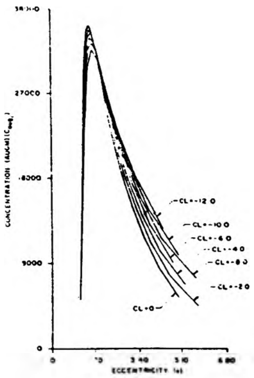

Fig. 11 Ideal Augmented Concentration vs. Eccentricity

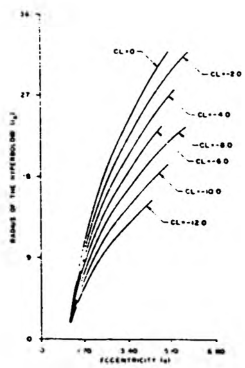

Fig. 12 Hyperboloid Radius vs. Eccentricity

NOMENCLATURE

$\bar{A},\bar{B},\bar{D},\bar{P},\bar{R}$, $\bar{T},\bar{S}$ = vectors

$\bar{a},\bar{b},\bar{c},\bar{d},\bar{e}$, $\bar{p},\bar{r},\bar{t},\bar{s}$ = normalized vectors

$|a|$, $|b|$, $|c|$, $|d|$ = determinants

a_o = paraboloid constant

a,b = hyperboloid constants

BB_1, BB_2, BB_3, BB_4 = constants

C_{av}, C_i, C_{aug}, C_{cpc}, C_{aug_i} = concentrations

CL = distance from the vertex of the paraboloid to the focal plane

d_{ab}, d_{cd}, d_{cp}, d_{te} = constants

D_r = determinant

f_o = focal length of paraboloid

e = eccentricity

F = ae

$F(y), F'(y)$ = functions of y

$\bar{i},\bar{j},\bar{k}$ = unit vectors

k_r, ℓ_r = constants

r = radius

r_h = radius of hyperboloid

r_p = radius of paraboloid

r_t = radius of ray pierce point in the focal plane measured from the focal point of the hyperboloid

r_{tm} = maximum radius of any ray pierce point considering the entire paraboloid

S_c = solar constant

S_t = average flux across the image at the focal plane

x,y,z = coordinates

α_s = angle subtended by the sun = 0.009322 radians

γ,θ,ζ = angles

∇ = gradient operator

$\phi_{ab}, \phi_{cd}, \phi_{cp}$, ϕ_{te}, ϕ_F = planes

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BIOGRAPHIES

Dr. Cobble is Professor of Mechanical Engineering at New Mexico State University. He received his Ph.D. from the University of Michigan in 1958. He has published in the areas of heat transfer, fluid mechanics and solar energy.

Dr. Hull received his Ph.D. from the University of Toledo in 1975. He is an Associate Professor of Mechanical Engineering at New Mexico State University. He has published in the areas of fatigue and failure analysis, and solar furnace design.