1882

Imaginary functions

Floyd Davis

THESIS

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DAVIS

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Columbus, Ohio, April, 1882.

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To

Professor Robert White McFarland,

an

Able Instructor,

and a man who has instilled

the Germ of Investigation in the author's mind,

These few pages are inscribed,

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Thorough Respect and Esteem.

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INTRODUCTION.

1. The imaginary expression which occurs in common Algebra, and in all the higher departments of mathematical science, has been one of the most perplexing problems which the human mind has encountered in any age. It was studied by the early mathematicians and became a field of speculation, but developed no marked results, for it was generally considered algebraically, and in that interpretation is a symbol of an impossible operation. But many of the problems arising from algebraic-geometrical analysis involving imaginaries were of such importance to mathematical and physical science that great attempts were made to establish their solution. It was known that the imaginary occurred in mathematical functions having real values, and so was supposed to have some real meaning.

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2. Thus in Dr Moivre's, Euler's, and many other theorems, the relations established involving the imaginary lie at the foundation of much mathematical science: and the value of these theorems was known to be actually real, though the imaginary could not be interpreted.

3. This great problem first had the germ of its solution with Dr. Wallis, of Oxford, and afterwards was solved by Messrs. Buée, Argand, Mourey, Gauss, and others, but the broadest interpretation remained to be discovered by Sir William Rowan Hamilton.

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square of a negative is positive, and hence we have no means of determining the sign of the square root of such an expression. The square root of α^2 is either $+\alpha$ or $-\alpha$, but what is the square root of $-\alpha^2$?

The arithmetical result cannot be determined, and so we only indicate the operation, as,

$$\pm \alpha\sqrt{-1}$$

6. By means of the above conventions and such expressions as $\alpha \pm \beta\sqrt{-1}$, we may develop other expressions, subject to the rules of algebraic and quaternion transformation.

7. But these imaginary expressions are not quantities, only symbols, which have the germ of their geometrical solution in the early part of the present century. The imaginary had been previously considered an undetermined symbol, appearing in problems that could not be interpreted. And there was no attempt

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his "Treatise of Algebra," in 1685. Therein he
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They both showed that $\beta\sqrt{-1}$ is a vector perpendicular to the initial direction vector line.

12. But Servois was the first, no doubt, who speculated in fields of research in which the slightest early anticipation of Quaternions is at present found. He endeavored to represent any point in space by an expression similar to $\alpha \pm \beta\sqrt{-1}$, thus generalizing the principles of geometry of two dimensions.

Through this induction, he reasoned by analogy and produced the expression

$$\rho' \cos.\alpha + \rho'' \cos.\beta + \rho''' \cos.\gamma$$

to represent a point in space, in which α, β, and γ are the inclinations of the three axes. He could not assign true values to ρ', ρ'', and ρ''', and this was his field of inquiry. It is now known they are the i, j, k, of Quaternions.

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impossibility vanishes from the mind, and
the imaginary becomes as clear as the subject
of ordinary Symbolic Algebra. This theory was
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Through this interpretation of imaginaries the
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By this means he established a new system
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"Oct. 15, '58.

To-morrow will be the fifteenth birthday of the Quaternions. They started into life, or light, full grown, on the 16th of Oct., 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge. That is to say, I then and there felt the galvanic circuit of thought to close; and the sparks which fell from it were the fundamental equations between i, j, k;

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exactly such as I have used them ever since.
I pulled out on the spot, a pocket-book, which
still exists, and made an entry, on which
at the very moment, I felt that it might
be worth my while to expend the labor
of at least ten (or it might be fifteen) years
to come. But then it is fair to say that
this was because I felt a problem to have
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ALGEBRAIC IMAGINARIES.

16. Every imaginary expression can be reduced to the general form, $\alpha \pm \beta\sqrt{-1}$, in which α and β are real quantities.

This is evident, inasmuch as all the real terms can be combined into one polynomial, which may be represented by α; and all the truly imaginary terms can be combined into another polynomial, which, when factored consists of a real quantity, and $\sqrt{-1}$. The real polynomial factor may be represented by β, and hence the whole imaginary expression reduces to the general form

$$\alpha \pm \beta\sqrt{-1}.$$

17. But if we have an imaginary expression of the general form, we may consider α a real quantity, and $\beta\sqrt{-1}$, imaginary. The two taken to-gether, as $\alpha \pm \beta\sqrt{-1}$, are generally considered imaginary. If $\alpha = 0$, the expression becomes truly imaginary, and equals $\beta\sqrt{-1}$; if $\beta=0$

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the expression becomes real, and equals α.

18. Every monomial imaginary can be reduced to the general form,

$$\gamma\sqrt[2\delta]{-1},$$

in which γ is a real quantity and δ is any whole number.

Let us suppose that $\lambda\sqrt[n]{-\beta}$ is a monomial imaginary, n being even. This can be reduced to the form $\lambda\sqrt[n]{\beta}\sqrt[n]{-1}$, but as $\lambda\sqrt[n]{\beta}$ is a real factor it may be denoted by γ, and the expression then equals $\gamma\sqrt[n]{-1}$.

But as an imaginary is an indicated even root of a negative quantity, we may, in place of n, substitute a quantity, 2δ, which will indicate this result.

Hence the form $\gamma\sqrt[2\delta]{-1}$.

19. When $\delta = 1$, the expression becomes $\gamma\sqrt{-1}$, which is called an imaginary of the second degree.

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If the monomial is of the form

$r\sqrt[4]{-1}$, $r\sqrt[6]{-1}$, $r\sqrt[8]{-1}$, &c., it is of the fourth, sixth, eighth, degree, &c.

20. Imaginaries are conjugate when they only differ in the sign of the coefficients of $\sqrt{-1}$; thus $\alpha+\beta\sqrt{-1}$ and $\alpha-\beta\sqrt{-1}$ are said to be conjugate imaginaries.

Hence we see that the sum and product of two conjugate imaginaries are always real.

21. The square root of the product of two conjugate imaginaries, taken with the positive sign, is called the modulus of each expression, and is of the form

$$\sqrt{(\alpha^2+\beta^2)}.$$

22. Thus from the conjugate imaginaries, $\alpha+\beta\sqrt{-1}$ and $\alpha-\beta\sqrt{-1}$, we infer that the modulus of a real quantity is the positive, numerical value of that quantity. But in order that the modulus, $\sqrt{(\alpha^2+\beta^2)}$, vanishes, α and β must each equal zero, and in this case

$\gamma\ \sqrt[4]{-1}$, $\gamma\ \sqrt[6]{-1}$, $\gamma\ \sqrt[8]{-1}$, etc., it is of the fourth, sixth, eight, degree, etc.

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both imaginary expressions vanish –
From this we see that when two imaginaries
are equal their moduli are also equal –

23 – If two imaginary expressions are equal,
the real parts must be equal and also
the coefficients of $\sqrt{-1}$ –

For suppose

$$\alpha+\beta\sqrt{-1}=\gamma+\delta\sqrt{-1} –$$

Then by transposition and factoring, we get

$$\alpha-\gamma+(\beta-\delta)\sqrt{-1}=0 –$$

$\therefore \alpha-\gamma=0$ and $(\beta-\delta)\sqrt{-1}=0$ etc.

By theory of Undetermined (no variable) Coefficients

$$\alpha=\gamma, \text{ and } \beta=\delta –$$

$a+b+4(c+d)=0$
$15-7+4(8-6)=0$
$\therefore 15=7 \quad + \quad 8=6$

Then the general equation $\alpha+\beta\sqrt{-1}=\gamma+\delta\sqrt{-1}$ may
be considered a symbolic representation, in
one statement of the equation

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24 – Take two imaginary expressions,

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and let us find their sum, difference,

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24. Take two imaginary expressions,

$$\alpha+\beta\sqrt{-1}, \text{ and } \gamma+\delta\sqrt{-1},$$

and let us find their sum, difference,

product, and quotient.
Their sum is

$$\alpha+\gamma+(\beta+\delta)\sqrt{-1}.$$

Their difference is

$$\alpha-\gamma+(\beta-\delta)\sqrt{-1}.$$

Their product is

$$(\alpha+\beta\sqrt{-1})(\gamma+\delta\sqrt{-1})=\alpha\gamma-\beta\delta+(\alpha\delta+\gamma\beta)\sqrt{-1}.$$

The quotient obtained by dividing the first by the second is

$$\frac{\alpha+\beta\sqrt{-1}}{\gamma+\delta\sqrt{-1}}.$$

Multiply both numerator and denominator by $\gamma-\delta\sqrt{-1}$; the expression becomes

$$\frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2}+\frac{\gamma\beta-\alpha\delta}{\gamma^2+\delta^2}\sqrt{-1}.$$

25. The modulus of the product or quotient of two imaginaries equals the product or quotient of their respective moduli; and the two imaginary expressions

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Their product is

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The quotient obtained by dividing the first
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$$\frac{(\alpha+\beta\sqrt{-1})}{(\gamma+\delta\sqrt{}-1)}.$$

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$$\frac{(\alpha\gamma+\delta\beta)}{\gamma 2+\delta 2}+\frac{(\gamma\beta-\alpha\delta)}{\gamma 2+\delta 2})\sqrt{-1}.$$

25. The modulus of the product or
quotient of two imaginaries equals the
product or quotient of their respective
moduli; and the two imaginary expressions

will not vanish as long as neither factor vanishes.

For, the modulus of the product of

$$\alpha + \beta\sqrt{-1}, \text{ and } \gamma + \delta\sqrt{-1},$$

by definition and Art. 24 equals

$$\sqrt{(\alpha\gamma - \beta\delta)^2 + (\alpha\delta + \gamma\beta)^2} = \sqrt{(\alpha^2+\beta^2)(\gamma^2+\delta^2)} =$$

$$\sqrt{(\alpha^2+\beta^2)} \times \sqrt{(\gamma^2+\delta^2)} \qquad (1).$$

Again, the modulus of the quotient of

$$\alpha + \beta\sqrt{-1}, \text{ and } \gamma + \delta\sqrt{-1},$$

according to Art. 24 equals

$$\sqrt{\left(\frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2}\right)^2 + \left(\frac{\gamma\beta-\alpha\delta}{\gamma^2+\delta^2}\right)^2} = \frac{\sqrt{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}}{\sqrt{(\gamma^2+\delta^2)}\times\sqrt{(\gamma^2+\delta^2)}} = \frac{\sqrt{(\alpha^2+\beta^2)}}{\sqrt{(\gamma^2+\delta^2)}} \qquad (2).$$

But $\sqrt{(\alpha^2+\beta^2)}$ and $\sqrt{(\gamma^2+\delta^2)}$ are moduli.

Equations (1) and (2) equal the product and quotient respectively of the modulus of $\alpha + \beta\sqrt{-1}$, and $\gamma + \delta\sqrt{-1}$.

Hence the theorem.

26. The even roots of imaginary expressions

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26. The even roots of imaginary expressions

are of the same general form as the expressions themselves.

Let us extract an even root of the imaginary, $\alpha \pm \beta\sqrt{-1}$; and suppose that

$$(\alpha \pm \beta\sqrt{-1})^{\frac{1}{2m}} = x \pm y\sqrt{-1} \qquad (3).$$

Then

$$(\alpha \pm \beta\sqrt{-1}) = (x \pm y\sqrt{-1})^{2m}.$$

If $m=1$, the expression

$$(\alpha \pm \beta\sqrt{-1}) = (x^2 - y^2 \pm 2xy\sqrt{-1}). \qquad (4).$$

Hence by the theory of Undetermined Coefficients

$$\alpha = x^2 - y^2 \qquad (4').$$

And

$$\beta = \pm 2xy.$$

Then by squaring and adding the last two equations, we get

$$(x^2 + y^2)^2 = \alpha^2 + \beta^2, \quad \text{or}$$

$$x^2 + y^2 = \pm\sqrt{(\alpha^2 + \beta^2)}. \qquad (5).$$

By combining equations (4') and (5) we get

$$x = \pm\sqrt{\left(\frac{\sqrt{(\alpha^2+\beta^2)} + \alpha}{2}\right)}. \qquad (6).$$

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$$y = \pm\sqrt{\left(\frac{\sqrt{(\alpha^2+\beta^2)}-\alpha}{2}\right)} \qquad (7).$$

But by supposition x and y are real, and so the sum of their squares is positive, and consequently we use the + sign before $\sqrt{(\alpha^2+\beta^2)}$.

~~The sign ± before 2xy shows that β will take the same double sign, as 2xy. Hence the expressions become identical. So for all real values of m, when even, and the general expression becomes~~

$$(\alpha \pm \beta\sqrt{-1})^{\frac{1}{2m}} = x \pm y\sqrt{-1} \qquad (8).$$

Conclusion more general than hypothesis.

27. We may find the square root of $\pm\sqrt{-1}$, by making $\alpha = 0$, and $\beta = 1$, in equations (6) and (7), and then substitute their values in equation (8).

The results are

$$\sqrt{(+\sqrt{-1})} = \pm\frac{1+\sqrt{-1}}{\sqrt{2}}, \text{ and}$$

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$$\sqrt{(-\sqrt{-1})} = \pm\frac{1-\sqrt{-1}}{\sqrt{2}}.$$

Suppose we have the equation

$$x^4 = -1.$$

Then

$$x^2 = \pm\sqrt{-1}, \text{ and } x = \pm\sqrt{\pm\sqrt{-1}}.$$

But since $x^4 = -1$, $x = \pm\sqrt[4]{-1}$.

This shows there are four, fourth roots in this equation, all of the general (?) form of -1; and they are indicated in the expression $\pm\sqrt{(\pm\sqrt{-1})}$, and in the general form $\pm\frac{1\pm\sqrt{-1}}{\sqrt{2}}$.

28. Every quantity has n, nth roots, and no more, and if n be even, two of these roots are real and the other n-2, imaginary. If n be odd, one root is real, and the other n-1, imaginary.

Let the general equation

$$x^n = p^n, \text{ or } x^n - p^n = 0,$$

Equal the quantity.

$$\sqrt{(-\sqrt{-1})} = \pm\frac{1-\sqrt{-1}}{\sqrt{2}}.$$

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Let the general equation

$$x^n = p^n, \text{ or } x^n - p^n = 0,$$

Equal the quantity.

In this problem then are two cases; one when n is even, the other when n is odd.

First, let n be even; the equation when factored is

$$(x^2-p^2)(x^{n-2}+x^{n-4}p^2+x^{n-6}p^4+\ldots\, p^{n-2})=0.$$

But either factor in this equation equals 0.

Hence

$$x^2-p^2=0, \quad \text{or} \quad x=+p \text{ and } -p.$$

Also

$$(x^{n-2}+x^{n-4}p^2+x^{n-6}p^4+\ldots\, p^{n-2})=0.$$

But there can be no real root that will satisfy this equation, for the coefficients of x are all positive and all powers of x are even; hence the n-2 roots which it contains must all be imaginary.

Next, let n be odd; then the equation when factored is

$$(x-p)(x^{n-1}+x^{n-2}p+x^{n-3}p^2+\ldots\, p^{n-1})=0.$$

But in this equation also, either factor equals 0.

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Hence

$$x - p = 0, \quad \text{or} \quad x = p.$$

Also,

$$(x^{n-1} + x^{n-2}p + x^{n-3}p^2 + \dots\dots p^{n-1}) = 0.$$

In this, as in the preceding case, there can be no real root that will satisfy this equation, for the coefficients of x are even; hence the n-1 roots which it contains must all be imaginary.

Hence the theorem.

29. Imaginary roots do not change the identity of any algebraic expression.

From the Theory of Equations we know that a quadratic expression of the form

$ax^2 + bx + c$ equals

$a(x-p)(x-q)$, when p and q are roots of the equation

$$ax^2 + bx + c = 0.$$

But if the roots are imaginary, they will

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$ax^2 + bx + c$ equals

$a(x-p)(x-q)$, when p and q are roots of the equation

$$ax^2 + bx + c = 0.$$

But if the roots are imaginary, they will

be of the form $\alpha \pm \beta\sqrt{-1}$, and the expression will become

$$a[x-(\alpha+\beta\sqrt{-1})][x-(\alpha-\beta\sqrt{-1})].$$

This reduces to

$$a[(x-\alpha)^2+\beta^2] = a(x^2-2x\alpha+\alpha^2+\beta^2)$$

which is identical with the original expression when

$$-2\alpha = \frac{b}{a}, \text{ and } \alpha^2+\beta^2 = \frac{c}{a}.$$

30. If there be an imaginary root in an equation having only real coefficients, there must also be another root forming the conjugate imaginary. Let us assume

$$\varphi(x) = (x-\alpha_1)(\alpha-\beta_1)(x-\gamma_1)\ldots\ldots(x-u)_1$$

having all the coefficients real. If one of the roots be $\alpha+\beta\sqrt{-1}$, the other must be $\alpha-\beta\sqrt{-1}$, in order that the expression be rational. The imaginary roots, if any, must occur as

$$[x-(\alpha+\beta\sqrt{-1})][x-(\alpha-\beta\sqrt{-1})] = x^2-2x\alpha+\alpha^2+\beta^2,$$

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must occur as

$$[x-(\alpha+\beta\sqrt{-1})][x-(\alpha-\beta\sqrt{-1})] = x^2-2x\alpha+\alpha^2+\beta^2,$$

which is real.

Hence the theorem.

31. But as imaginary roots may be found in $\varphi(x)$, all roots of it must satisfy the functional equation

$$\varphi(x) = 0.$$

Let this equation equal

$$x^2 - px + c = 0.$$

The roots of this equation are

$$\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - c}.$$

If we assume

$$x^2 - px + c = 0 = y,$$

the preceding roots will be beautifully illustrated as a pair of conjugate imaginaries in the Loci of Equations.

By assigning proper values to p and c, we are enabled to trace out the curve represented by the given function.

Let $p = 3$, and $c = -3$, then we will have

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the preceding roots will be beautifully illustrated as a pair of conjugate imaginaries in the Loci of Equations.

By assigning proper values to p and c, we are enabled to trace out the curve represented by the given function.

Let p = 3, and c = -3, then we will have

two real and equal, or two real and unequal roots, and one can change into the other by varying the values of p and c.

If $\frac{p^2}{4} < c$, the roots at once become imaginary and form a pair of conjugates.

But the locus of the curve does not meet the axis of abscissas as long as the roots are real.

When the curve passes below the axis of abscissas, the roots representing the points of the curve below the axis become imaginary.

32. It is shown in geometry that if a straight line intersect any curve, the number of intersections is indicated by the degree of the equation representing the curve.

If the straight line revolve so as to leave a less number of points of intersection with the curve, it is always found that two intersections first run to-gether, compounding

(23)

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TRIGONOMETRICAL IMAGINARIES.

33. De Moivre's Theorem: If n be any whole number, then

$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{\frac{p}{q}} = \cos.\tfrac{p}{q}(2n\pi+\alpha) \pm \sin.\tfrac{p}{q}(2n\pi+\alpha)\sqrt{-1}.$$

Multiply

$$\cos.\alpha \pm \sin.\alpha\sqrt{-1} \quad \text{by} \quad \cos.\beta \pm \sin.\beta\sqrt{-1}.$$

The product is

$$\cos.\alpha\cos.\beta - \sin.\alpha\sin.\beta \pm (\sin.\alpha\cos.\beta + \cos.\alpha\sin.\beta)\sqrt{-1} =$$
$$\cos.(\alpha+\beta) \pm \sin.(\alpha+\beta)\sqrt{-1}.$$

Multiply the last expression by

$$\cos.\gamma \pm \sin.\gamma\sqrt{-1}.$$

The product is

$$\cos.(\alpha+\beta)\cos.\gamma - \sin.(\alpha+\beta)\sin.\gamma \pm [\sin.(\alpha+\beta)\cos.\gamma + \cos.(\alpha+\beta)\sin.\gamma]\sqrt{-1} =$$
$$\cos.(\alpha+\beta+\gamma) \pm \sin.(\alpha+\beta+\gamma)\sqrt{-1}.$$

Continue this operation m-1 times, then make $\alpha = \beta = \gamma = $ &c., and the angular functions become, $\cos. m\alpha$, and $\sin. m\alpha$. But we have only expressed the side of the equation involving factoral angles; the other side is

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Multiply

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Continue this operation m-1 times, then make $\alpha = \beta = \gamma = \Pi$, and the angular functions become, $cos.m\alpha$, and $sin.m\alpha$. But we have only expressed the side of the equation involving factoral angles; the other side is exponential –

The expression results in the following equation:

$$\cos. m\alpha \pm \sin. m\alpha\sqrt{-1} = (\cos.\alpha \pm \sin.\alpha\sqrt{-1})^m \qquad (9).$$

But this analysis only shows equation (9) to be true when m is a positive integer –

Let us now suppose m negative, and equal to $-\varepsilon$.

Then

$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^m = \frac{1}{(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^\varepsilon} =$$

$$\frac{1}{\cos.\varepsilon\alpha \pm \sin.\varepsilon\alpha\sqrt{-1}} - (?)$$

Multiply both numerator and denominator by

$$\cos.\varepsilon\alpha \mp \sin.\varepsilon\alpha\sqrt{-1} -$$

The result is

$$\frac{\cos.\varepsilon\alpha \mp \sin.\varepsilon\alpha\sqrt{-1}}{\cos.^2\varepsilon\alpha + \sin.^2\varepsilon\alpha} = \cos.\varepsilon\alpha \mp \sin.\varepsilon\alpha\sqrt{-1} =$$

$$\cos.(-\varepsilon\alpha) \pm \sin.(-\varepsilon\alpha)\sqrt{-1} = \cos.\varepsilon\alpha \pm \sin.\varepsilon\alpha\sqrt{-1} - (?)$$

Thus equation (9) is established when m is negative.

But if we extract the m^{th} root of each

(26)

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$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^m = \frac{1}{(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^e} =$$

$$\frac{1}{(\cos.e\alpha \pm \sin.\ e\alpha\sqrt{-1}}.$$

Multiply both numerator and denominator by

$$\cos.e\alpha \pm \sin e\alpha\sqrt{-1}.$$

The result is

$$\frac{\cos.e\alpha \pm \sin.e\alpha\sqrt{-1}}{\cos.^2 e\alpha + \sin.^2 e\alpha} = \cos.e\alpha \pm \sin.e\alpha\sqrt{-1} =$$

$$= \cos.(-e\alpha) \pm \sin.(-e\alpha)\sqrt{-1} = \cos.e\alpha \pm \sin.e\alpha\sqrt{-1}.$$

Thus equation (9) is established when m is negative.

But if we extract the mth root of each

number of equation (9), the result is

$$(\cos. m\alpha \pm \sin. m\alpha\sqrt{-1})^{\frac{1}{m}} = \cos.\alpha \pm \sin.\alpha\sqrt{-1}. \qquad (10)$$

And if we suppose m equals any fraction, either positive or negative; say, $\frac{p}{q}$, then

$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{m} = (\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{p/q} = (\cos. p\alpha \pm \sin. p\alpha\sqrt{-1})^{\frac{1}{q}}.$$

It has been shown in equation (10) that this is one of the roots.

Hence the equation becomes general and can be written

$$\cos. \tfrac{p}{q}\alpha \pm \sin. \tfrac{p}{q}\alpha\sqrt{-1}.$$

Thus we have completely established equation (9) whatever be the value of m. It yet remains for us to investigate the value and generality of this equation when α assumes different values.

So long as α remains less than $\frac{\pi}{2}$, the sign-values of cos. α and sin. α remain unchanged, and also for any integral multiplier of 2π.

Hence, to make equation (9) complete, we must

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$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{m} = (\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{\frac{p}{q}} = (\cos. p\alpha \pm \sin. p\alpha\gamma - 1)\frac{1}{q}.$$

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and also for any integral multiplier of 2Π.
Hence, to make equation (9) complete, we must

in place of the varying angle α, insert $2n\pi+\alpha$, and the equation becomes

$$(\cos.\alpha \pm \sin.\alpha\sqrt{-1})^{p/q} = \cos.p/q(2n\pi+\alpha) \pm \sin.p/q(2n\pi+\alpha)\sqrt{-1}$$

which completely establishes

De Moivre's Formula.

34. The preceding method of demonstration is only one among many, and it is probably more complex than some that will be shown hereafter. The method by Vector Equations is the simplest of any yet discovered.

35. But the exponent p/q shows there are q different values to the expression

$$\cos.p/q(2n\pi+\alpha) \pm \sin.p/q(2n\pi+\alpha)\sqrt{-1}.$$

These roots are either real or of the form of the general imaginary, $\alpha \pm \beta\sqrt{-1}$.

36. This theorem can be usefully employed in extracting roots of imaginary expressions of the form of $\alpha \pm \beta\sqrt{-1}$.

Assume

in place of the varying angle α, insert
$2n\Pi + \alpha$, and the equation becomes

$$\left(\cos.\alpha \pm \sin.\alpha\sqrt{-1}\right)^{\frac{p}{q}} = cos.\frac{p}{q}(2n\Pi+\alpha) \pm sin.\frac{p}{q}(2n\Pi+\alpha)\sqrt{-1}\,.$$

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36. This theorem can be usefully employed in extracting roots of imaginary expressions of the form of $\alpha \pm \beta\sqrt{-1}$.

Assume

$$\alpha = \rho \cos.\theta, \text{ and } \beta = \rho \sin.\theta.$$

Then

$$\rho^2 = \alpha^2 + \beta^2; \; \tan.\theta = \frac{\beta}{\alpha}; \text{ and}$$

$$\alpha \pm \beta\sqrt{-1} = \rho(\cos.\theta \pm \sin.\theta\sqrt{-1}).$$

Therefore

$$(\alpha \pm \beta\sqrt{-1})^{\frac{1}{2m}} = \rho^{\frac{1}{2m}}(\cos.\theta \pm \sin.\theta\sqrt{-1})^{\frac{1}{2m}}.$$

Different roots can be extracted from this expression by assigning corresponding values to m. But by De Moivre's Theorem, one of the roots is $\cos.\frac{\theta}{2m} \pm \sin.\frac{\theta}{2m}\sqrt{-1}$; the other roots are determined from

$$\cos.\frac{p}{q}(2n\Pi + \theta) \pm \sin.\frac{p}{q}(2n\Pi + \theta)\sqrt{-1},$$

by assigning proper values to p, q, and n. The values of $(\alpha \pm \beta\sqrt{-1})^{\frac{1}{2m}}$ obtained by this method agree with the values given in Art. 26.

37\. Let it be required to express the complete values of

$$\alpha + \beta\sqrt{-1}, \text{ and } \alpha - \beta\sqrt{-1},$$

where they are derived respectively from

(29)

$$\alpha = \rho\, cos.\theta, \text{ and } \beta = \rho\, sin.\theta.$$

Then

$$\rho^{2} = \alpha^{2} + \beta^{2}; tan.\theta = \frac{\beta}{\alpha}; \text{ and}$$

$$\alpha \pm \beta\sqrt{-1} = \rho(cos.\theta \pm sin.\theta\sqrt{-1}).$$

Therefore

$$\left(\alpha \pm \beta\sqrt{-1}\right)^{\frac{1}{2m}} = \rho^{\frac{1}{2m}}(cos.\theta \pm sin.\theta\sqrt{-1})^{\frac{1}{2m}}.$$

Different roots can be extracted from this expression by assigning compounding values to m. But by De Moivre's Theorem, one of the roots is $cos.\frac{\theta}{2m} \pm sin.\frac{\theta}{2m}\sqrt{-1}$; the other roots are determined from

$$cos.\frac{p}{q}(2n\Pi + \theta) \pm sin.\frac{p}{q}(2n\Pi + \theta)\sqrt{-1},$$

by assigning proper values to p, q, and n. The values of $\left(\alpha \pm \beta\sqrt{-1}\right)^{\frac{1}{2m}}$ obtained by this method agree with the values given in Art. 26.

37\. Let it be required to express the complete values of

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where they are derived respectively from

$(\alpha+\beta\sqrt{-1})^n$, and $(\alpha-\beta\sqrt{-1})^n$.

Assume

$$\gamma=\frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}},\ \text{and}\ \delta=\frac{\beta}{\sqrt{(\alpha^2+\beta^2)}},$$

which bear the same relation as cos. and sin. of an angle.

Then

$$\gamma+\delta\sqrt{-1}=\frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}}+\frac{\beta\sqrt{-1}}{\sqrt{(\alpha^2+\beta^2)}},\ \text{and}$$

$$\gamma-\delta\sqrt{-1}=\frac{\alpha}{\sqrt{(\alpha^2+\beta^2)}}-\frac{\beta\sqrt{-1}}{\sqrt{(\alpha^2+\beta^2)}}.$$

It follows, then, that

$$\alpha+\beta\sqrt{-1}=\sqrt{(\alpha^2+\beta^2)}\,(\gamma+\delta\sqrt{-1}),\ \text{and}$$

$$\alpha-\beta\sqrt{-1}=\sqrt{(\alpha^2+\beta^2)}\,(\gamma-\delta\sqrt{-1}).$$

Hence

$$(\alpha+\beta\sqrt{-1})^n=\left[\sqrt{(\alpha^2+\beta^2)}\,(\gamma+\delta\sqrt{-1})\right]^n,\ \text{and}$$

$$(\alpha-\beta\sqrt{-1})^n=\left[\sqrt{(\alpha^2+\beta^2)}\,(\gamma-\delta\sqrt{-1})\right]^n.$$

And now if we replace γ and δ by their respective trigonometrical functions, cos. θ

(30)

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Hence

$$(\alpha+\beta\sqrt{-1})^n=[\sqrt{(\alpha^2+\beta^2)}(\gamma+\delta\sqrt{-1})]^n,\ \text{and}$$

$$(\alpha-\beta\sqrt{-1})^n=[\sqrt{(\alpha^2+\beta^2)}(\gamma-\delta\sqrt{-1})]^n.$$

And now if we replace γ and δ by their
respective trigonometrical functions, cos. θ

and sin. θ, and consider according to De Moivre's Theorem, we get

$$(\alpha+\beta\sqrt{-1})^n = (\alpha^2+\beta^2)^{\frac{n}{2}}(\cos. n\theta + \sin. n\theta\sqrt{-1}),$$

and

$$(\alpha-\beta\sqrt{-1})^n = (\alpha^2+\beta^2)^{\frac{n}{2}}(\cos. n\theta - \sin. n\theta\sqrt{-1}).$$

The n th root of each equation is respectively

$$\alpha+\beta\sqrt{-1} = \sqrt{(\alpha^2+\beta^2)}(\cos. n\theta + \sin. n\theta\sqrt{-1})^{\frac{1}{n}},$$

and

$$\alpha-\beta\sqrt{-1} = \sqrt{(\alpha^2+\beta^2)}(\cos. n\theta - \sin. n\theta\sqrt{-1})^{\frac{1}{n}}$$

38. Again, let it be required to find the complete values of

$$(\alpha+\beta\sqrt{-1})^n + (\alpha-\beta\sqrt{-1})^n, \text{ and}$$

$$(\alpha+\beta\sqrt{-1})^n - (\alpha-\beta\sqrt{-1})^n$$

From Art. 37, we get

$$(\alpha+\beta\sqrt{-1})^n + (\alpha-\beta\sqrt{-1})^n =$$

$$(\alpha^2+\beta^2)^{\frac{n}{2}}\left((\cos. n\theta + \sin. n\theta\sqrt{-1}) + (\cos. n\theta - \sin. n\theta\sqrt{-1})\right) =$$

$$2(\alpha^2+\beta^2)^{\frac{n}{2}}\cos. n\theta,$$

and

$$(\alpha+\beta\sqrt{-1})^n - (\alpha-\beta\sqrt{-1})^n =$$

and $\sin. \theta$, and consider according to De Moivre's Theorem, we get

$$(\alpha+\beta\sqrt{-1})^n = (\alpha^2+\beta^2)^{\frac{n}{2}}(\cos. n\theta + \sin. n\theta\sqrt{-1}),$$

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The nth root of each equation is respectively

$$\alpha+\beta\sqrt{-1} = \sqrt{(\alpha^2+\beta^2)}(\cos. n\theta + \sin. n\theta\sqrt{-1})^{1/n},$$

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$$2(\alpha^2+\beta^2)^{\frac{n}{2}}\cos. n\theta,$$

and

$$(\alpha+\beta\sqrt{-1})^n - (\alpha-\beta\sqrt{-1})^n =$$

$$(\alpha^2+\beta^2)^{\frac{n}{2}}\left((\cos.n\theta+\sin.n\theta\sqrt{-1})-(\cos.n\theta-\sin.n\theta\sqrt{-1})\right)=$$
$$2(\alpha^2+\beta^2)^{\frac{n}{2}}\sin.n\theta\sqrt{-1}.$$

39. If we make

$$a^{\pm\theta}=\cos.\theta\pm\sin.\theta\sqrt{-1},$$ in De Moivre's Formula,

we can derive some important exponential functions.

Then

$$a^{+\theta}=\cos.\theta+\sin.\theta\sqrt{-1},$$
$$a^{-\theta}=\cos.\theta-\sin.\theta\sqrt{-1};\ \text{and}$$
$$a^{+n\theta}=\cos.n\theta+\sin.n\theta\sqrt{-1},$$
$$a^{-n\theta}=\cos.n\theta-\sin.n\theta\sqrt{-1}.$$

From these four equations, we deduce

$$\cos.\theta=\frac{a^{+\theta}+a^{-\theta}}{2},$$
$$\sin.\theta=\frac{a^{+\theta}-a^{-\theta}}{2\sqrt{-1}};\ \text{and}$$
$$\cos.n\theta=\frac{a^{+n\theta}+a^{-n\theta}}{2},$$
$$\sin.n\theta=\frac{a^{+n\theta}-a^{-n\theta}}{2\sqrt{-1}}.$$

Hence

$$\sin.^2\theta+\cos.^2\theta=\left(\frac{a^{+\theta}-a^{-\theta}}{2\sqrt{-1}}\right)^2+\left(\frac{a^{+\theta}+a^{-\theta}}{2}\right)^2=1,$$ a well known trigonometrical relation.

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$$(a^2+\beta^2)^{\frac{n}{2}}(\cos.n\theta+\sin.n\theta\sqrt{-1})-(\cos.n\theta-\sin.n\theta\sqrt{-1}))=$$
$$2(\alpha^2+\beta^2)^{\frac{n}{2}}\sin.n\theta\sqrt{-1}.$$

39. If we make

$a^{\pm\theta}=\cos.\theta\pm\sin.\theta\sqrt{-1}$, in De Moivre's Formula,

we can derive some important exponential

functions.

Then

$a^{+\theta}=\cos.\theta+\sin.\theta\sqrt{-1}$,

$a^{-\theta}=\cos.\theta-\sin.\theta\sqrt{-1}$; and

$a^{+n\theta}=\cos.n\theta+\sin.n\theta\sqrt{-1}$,

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From these four equations, we deduce

$\cos.\theta=\frac{a^{+\theta}+a^{-\theta}}{2}$

$\sin.\theta=\frac{a^{+\theta}+a^{-\theta}}{2\sqrt{-1}}$; and

$\cos.n\theta=\frac{a^{+n\theta}+a^{-n\theta}}{2}$,

$\sin.n\theta=\frac{a^{+n\theta}+a^{-n\theta}}{2\sqrt{-1}}$.

Hence

$\sin.^2\theta+\cos.^2\theta=(\frac{a^{+\theta}+a^{-\theta}}{2\sqrt{-1}})^2+(\frac{a^{+\theta}+a^{-\theta}}{2})^2=1$, a well

known trigonometrical relation.

40. But we learn in Algebra, that

$$a^{+\theta} = 1+\theta(log.a)+\frac{\theta^2}{\lfloor 2}(log.a)^2+\frac{\theta^3}{\lfloor 3}(log.a)^3+\frac{\theta^4}{\lfloor 4}(log.a)^4+ \&c.,$$

and

$$a^{-\theta} = 1-\theta(log.a)+\frac{\theta^2}{\lfloor 2}(log.a)^2-\frac{\theta^3}{\lfloor 3}(log.a)^3+\frac{\theta^4}{\lfloor 4}(log.a)^4- \&c.$$

Therefore

$$cos.\theta = \frac{\varepsilon^{+\theta}+\varepsilon^{-\theta}}{2} = 1+\frac{\theta^2}{\lfloor 2}(log.a)^2+\frac{\theta^4}{\lfloor 4}(log.a)^4+ \&c.,$$

and

$$sin.\theta = \frac{\varepsilon^{+\theta}-\varepsilon^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}\left\{\theta(log.a)+\frac{\theta^3}{\lfloor 3}(log.a)^3+\frac{\theta^5}{\lfloor 5}(log.a)^5+ \&c.\right\}.$$

And again, if $log.a = \alpha\sqrt{-1}$, these trigonometrical functions become

$$cos.\theta = 1-\frac{\alpha^2\theta^2}{\lfloor 2}+\frac{\alpha^4\theta^4}{\lfloor 4}-\frac{\alpha^6\theta^6}{\lfloor 6}+ \&c.,$$

and

$$sin.\theta = \alpha\theta-\frac{\alpha^3\theta^3}{\lfloor 3}+\frac{\alpha^5\theta^5}{\lfloor 5}-\frac{\alpha^7\theta^7}{\lfloor 7}+ \&c.$$

If $\alpha=1$, then $log.a = \sqrt{-1}$.

By passing to exponentials

$$a=\varepsilon^{+\sqrt{-1}};\ a^{+\theta}=\varepsilon^{+\theta\sqrt{-1}};\ \text{and } a^{-\theta}=\varepsilon^{-\theta\sqrt{-1}}.$$

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$$sin.\theta = \frac{a^{+\theta}-a^{-\theta}}{2\sqrt{-1}} = \frac{\varepsilon^{+\theta\sqrt{-1}}-\varepsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}.$$

(33)

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And

$$a^{-\theta} = 1-\theta(log.a)+\frac{\theta^2}{2}(log.a)^2-\frac{\theta^3}{3}(log.a)^3+\frac{\theta^4}{4}(log.a)^4 - \text{\&c.}$$

Therefore --

$$cos.\theta = \frac{e^{+\theta}-e^{-\theta}}{2} = 1+\frac{\theta^2}{2}(log.a)^2+\frac{\theta^3}{3}(log.a)^4+\frac{\theta^4}{4}(log.a)^4 + \text{\&c.},$$

and

$$sin.\theta = \frac{e^{+\theta}-e^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}[\theta(log.a)+\frac{\theta^3}{3}(log.a)^3+\frac{\theta^5}{5}(log.a)^5 \&c.].$$

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If $\alpha=1$, then $log.a = \sqrt{-1}$.

By passing to exponentials

$$a=e^{+\sqrt{-1}};\ a^{+\theta}=e^{+\theta\sqrt{-1}};\ and\ a^{-\theta}=e^{-\theta}\sqrt{-1}.$$

Hence --

$$sin.\theta = \frac{a^{+\theta}-a^{-\theta}}{2\sqrt{-1}} = \frac{e^{+\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}.$$

and

$$\cos.\theta = \frac{a^{+\theta}+a^{-\theta}}{2} = \frac{\varepsilon^{+\theta\sqrt{-1}}+\varepsilon^{-\theta\sqrt{-1}}}{2}.$$

We have then, determinate, and indeterminate exponential expressions for $\sin.\theta$, and $\cos.\theta$.

41. But as we have shown that $\alpha^{+\theta} = \varepsilon^{+\theta\sqrt{-1}}$, we may write the exponential equation

$$\varepsilon^{+\theta\sqrt{-1}} = \cos.\theta + \sin.\theta\sqrt{-1}. \qquad (11).$$

Let us make $\theta = \frac{2m\pi}{\alpha}$, m being any even integral number; the equation becomes

$$\varepsilon^{+\frac{2m\pi}{\alpha}\sqrt{-1}} = \cos.\frac{2m\pi}{\alpha} + \sin.\frac{2m\pi}{\alpha}\sqrt{-1}.$$

By a judicious manipulation of this formula we can secure a general solution for the equation

$$x^n = \pm 1.$$

It will be found that the number of roots thus obtained will occur in rega-order, each order containing n roots, as indicated in ε^{+0}, $\varepsilon^{+\frac{2\pi}{\alpha}\sqrt{-1}}$, $\varepsilon^{+\frac{4\pi}{\alpha}\sqrt{-1}}$, ... $\varepsilon^{+\frac{2(n-1)}{\alpha}\sqrt{-1}}$.

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42. If sin. θ be divided by cos. θ, the quotient is

$$\tan.\theta = \frac{\varepsilon^{+\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{\sqrt{-1}\left(\varepsilon^{+\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{-1}}\right)}, \quad \text{or}$$

$$\tan.\theta\sqrt{-1} = \frac{\varepsilon^{+\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{\varepsilon^{+\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{-1}}} -$$

Therefore

$$\frac{1+\tan.\theta\sqrt{-1}}{1-\tan.\theta\sqrt{-1}} = \frac{\varepsilon^{+\theta\sqrt{-1}}}{\varepsilon^{-\theta\sqrt{-1}}} = \varepsilon^{+2\theta\sqrt{-1}} -$$

Take logarithms of both numbers: the result is

$$2\theta\sqrt{-1} = \log.(1+\tan.\theta\sqrt{-1}) - \log.(1-\tan.\theta\sqrt{-1}) =$$

$$2\sqrt{-1}\left(\tan.\theta - \tfrac{1}{3}\tan.^3\theta + \tfrac{1}{5}\tan.^5\theta - \&c.\right) -$$

Hence

$$\theta = \tan.\theta - \tfrac{1}{3}\tan.^3\theta + \tfrac{1}{5}\tan.^5\theta - \&c.,$$

which gives θ in terms of powers of tan. θ -

This is known as Gregory's Series -

It may be usefully employed in computing the numerical value of π, by making $\theta = \frac{\pi}{4}$ -

But this series is generally quite unsatisfactory because it does not limit the extent to which it may be relied on as arithmetically

(35)

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$$tan.\,\theta\sqrt{-1} = \frac{e^{+\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{+\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}.$$

Therefore

$$\frac{1+tan.\theta\sqrt{-1}}{1-tan.\theta\sqrt{-1}} = \frac{e^{+\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}}} = e^{+2\theta\sqrt{-1}}.$$

Take logarithms of both numbers: the result is

$$2\theta\sqrt{-1} = \log.\left(1 + tan.\,\theta\sqrt{-1}\right) - \log.\left(1 - tan.\,\theta\sqrt{-1}\right) =$$

$$2\sqrt{-1}(tan.\,\theta - 1/3tan.\,^3\theta + 1/5tan.^5\,\theta\ \&c.).$$

Hence

$$\theta = tan.\,\theta - 1/3tan.\,^3\theta + 1/5tan.\,^5\theta\ \&c.,$$

which gives θ in terms of powers of $tan.\,\theta$.

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But this series is generally quite unsatisfactory because it does not limit the extent to which it may be relied on as arithmetically

Therefore

$$\varepsilon^{+2\beta\sqrt{-1}} = \frac{1 - n\varepsilon^{-\alpha\sqrt{-1}}}{1 - n\varepsilon^{+\alpha\sqrt{-1}}}.$$

Take logarithms of both sides of the last equation, and the result is

$$2\beta\sqrt{-1} = \log.(1 - n\varepsilon^{-\alpha\sqrt{-1}}) - \log.(1 - n\varepsilon^{+\alpha\sqrt{-1}}) =$$

$$n(\varepsilon^{+\alpha\sqrt{-1}} - \varepsilon^{-\alpha\sqrt{-1}}) + \frac{n^2}{2}(\varepsilon^{+2\alpha\sqrt{-1}} - \varepsilon^{-2\alpha\sqrt{-1}}) + \frac{n^3}{3}(\varepsilon^{+3\alpha\sqrt{-1}} - \varepsilon^{-3\alpha\sqrt{-1}}) + \text{\&c.}$$

Hence

$$\beta = n \sin.\alpha + \frac{n^2}{2}\sin.2\alpha + \frac{n^3}{3}\sin.3\alpha + \text{\&c.},$$

the desired series.

This series is given in circular functions.

44. By means of this series, we are often able to solve certain triangles.

In trigonometry we have

$$\sin.B = \frac{b}{a}\sin.A = \frac{b}{a}\sin(B + C).$$

Hence by the formula,

$$B = \frac{b}{a}\sin.C + \frac{b^2}{2a^2}\sin.2C + \frac{b^3}{3a^3}\sin.3C + \text{\&c.}$$

If b be less than a the series is convergent; and if $\frac{b}{a}$ be a small fraction,

true, and a large number of terms would have to be taken to secure a close approximation.

43. If we have given

$$\sin.\beta = n\sin.(\beta + \alpha),$$

β can be expressed in terms of n and sine-functions of α, by means of exponentials.

In Art. 40, it was shown that

$$a^{+\beta} - a^{-\beta} = e^{+\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}}.$$

and by Art. 39, this equals $2\sin.\beta\sqrt{-1}$.

Replace $\sin.\beta$ by its equivalent, $n\sin.(\beta + \alpha)$, and then pass to exponentials according to Art. 39; the result is

$$e^{+\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}} = n(e^{+(\beta+\alpha)\sqrt{-1}} - e^{-(\beta+\alpha)\sqrt{-1}}).$$

Multiply the last equation by $e^{+\beta\sqrt{-1}}$; the result is

$$e^{+2\beta\sqrt{-1}} - 1 = n(e^{+(2\beta+\alpha)\sqrt{-1}} - e^{-\alpha\sqrt{-1}}),$$

or

$$e^{+2\beta\sqrt{-1}}(1 - ne^{+\alpha\sqrt{-1}}) = 1 - ne^{-\alpha\sqrt{-1}}.$$

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$$\varepsilon^{+\beta\sqrt{-1}} - \varepsilon^{-\beta\sqrt{-1}} = n\left(\varepsilon^{+(\beta+\alpha)\sqrt{-1}} - \varepsilon^{-(\beta+\alpha)\sqrt{-1}}\right).$$

Multiply the last equation by $\varepsilon^{+\beta\sqrt{-1}}$; the result is

$$\varepsilon^{+2\beta\sqrt{-1}} - 1 = n\left(\varepsilon^{+(2\beta+\alpha)\sqrt{-1}} - \varepsilon^{-\alpha\sqrt{-1}}\right),$$

or

$$\varepsilon^{+2\beta\sqrt{-1}}\left(1 - n\varepsilon^{+\alpha\sqrt{-1}}\right) = 1 - n\varepsilon^{-\alpha\sqrt{-1}}.$$

Therefore

$$e^{+2\beta\sqrt{-1}} = \frac{1-ne^{-\alpha\sqrt{-1}}}{1-ne^{+\alpha\sqrt{-1}}}.$$

Take logarithms of both sides of the last equation, and the result is

$$2\beta\sqrt{-1} = log.\left(1-ne^{-\alpha\sqrt{-1}}\right) - log.\left(1-ne^{+\alpha\sqrt{-1}}\right) =$$

$$n\left(e^{+\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right) + \frac{n^2}{2}\left(e^{+2\alpha\sqrt{-1}} - e^{-2\alpha\sqrt{-1}}\right) + \frac{n^3}{3}\left(e^{+3\alpha\sqrt{-1}} - e^{-3\alpha\sqrt{-1}}\right) + \&c.$$

Hence

$$\beta = n\sin.\alpha + \frac{n^2}{2}\sin.2\alpha + \frac{n^3}{3}\sin.3\alpha + \&c..$$

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$$B = \frac{b}{a}\sin.C + \frac{b^2}{2a^2}\sin.2C + \frac{b^3}{3a^3}\sin.3C + \&c.$$

If 1 be less than a the series is convergent; and if $\frac{b}{a}$ be a small fraction, a few terms of the series will suffice for a close degree of approximation — But as the series gives the circular measure of B, we must find the relation between circular function, and centesimal or sexigessimal functions.

Let x = number of degrees in any given angle, and θ the circular measure of the same angle — But as there are 180 sexigesimal degrees in two right angles, $\frac{x}{180}$ denotes the ratio of the given angle to two right angles — And since π denotes the circular measure of two right angles, $\frac{\theta}{\pi}$ expresses the ratio of the given angle to two right angles —

Hence

$$\frac{x}{180} = \frac{\theta}{\pi}, \quad \text{or}$$

$$\theta = \frac{x\pi}{180} \text{ —}$$

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$$\frac{x}{180} = \frac{\theta}{\Pi}, \text{ or}$$

$$\theta = \frac{x\Pi}{180}.$$

45. If we have given

$$tan.\beta = n\, tan.\alpha,$$

β can be expressed in powers of n and sine-functions of α, by means of exponentials. The given trigonometrical relation combined with Art. 42 gives

$$\frac{\varepsilon^{+\beta\sqrt{-1}} - \varepsilon^{-\beta\sqrt{-1}}}{\varepsilon^{+\beta\sqrt{-1}} + \varepsilon^{-\beta\sqrt{-1}}} = n\left(\frac{\varepsilon^{+\alpha\sqrt{-1}} - \varepsilon^{-\alpha\sqrt{-1}}}{\varepsilon^{+\alpha\sqrt{-1}} + \varepsilon^{-\alpha\sqrt{-1}}}\right).$$

Multiply numerator and denominator of the left hand member, by $\varepsilon^{+\beta\sqrt{-1}}$; and numerator and denominator of right hand member, by $\varepsilon^{+\alpha\sqrt{-1}}$; the result is

$$\frac{\varepsilon^{+2\beta\sqrt{-1}} - 1}{\varepsilon^{+2\beta\sqrt{-1}} + 1} = n\left(\frac{\varepsilon^{+2\alpha\sqrt{-1}} - 1}{\varepsilon^{+2\alpha\sqrt{-1}} + 1}\right).$$

Therefore

$$\varepsilon^{+2\beta\sqrt{-1}} = \frac{(1+n)\varepsilon^{+2\alpha\sqrt{-1}} + 1 - n}{(1-n)\varepsilon^{+2\alpha\sqrt{-1}} + 1 + n} =$$

$$\varepsilon^{+2\alpha\sqrt{-1}}\left(\frac{1 + \frac{1-n}{1+n}\varepsilon^{-2\alpha\sqrt{-1}}}{1 + \frac{1-n}{1+n}\varepsilon^{+2\alpha\sqrt{-1}}}\right).$$

(39)

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Multiply numerator and denominator of the left hand number, by $e^{+\beta\sqrt{-1}}$; and numerator and denominator of right hand number, by $e^{+\alpha\sqrt{-1}}$; the result is

$$\frac{e^{+2\beta\sqrt{-1}} - 1}{e^{+2\beta\sqrt{-1}} + 1} = n\left(\frac{e^{+2\alpha\sqrt{-1}} - 1}{e^{+2\alpha\sqrt{-1}} + 1}\right).$$

Therefore

$$e^{+2\beta\sqrt{-1}} = \frac{(1+n)e^{+2\alpha\sqrt{-1}} + 1 - n}{(1-n)e^{+2\alpha\sqrt{-1}} + 1 + n} =$$

$$e^{+2\alpha\sqrt{-1}}\left(\frac{1 + \frac{1-n}{1+n}e^{-2\alpha\sqrt{-1}}}{1 + \frac{1-n}{1+n}e^{+2\alpha\sqrt{-1}}}\right).$$

Take logarithm of both members; the result is

$$2\beta\sqrt{-1} = 2\alpha\sqrt{-1} + \log.\left(1+\frac{1-n}{1+n}\varepsilon^{-2\alpha\sqrt{-1}}\right) - \log.\left(1+\frac{1-n}{1+n}\varepsilon^{+2\alpha\sqrt{-1}}\right) =$$

$$2\alpha\sqrt{-1} - \left(\frac{1-n}{1+n}\right)\left(\varepsilon^{+2\alpha\sqrt{-1}} - \varepsilon^{-2\alpha\sqrt{-1}}\right) - \tfrac{1}{2}\left(\frac{1-n}{1+n}\right)^2\left(\varepsilon^{+4\alpha\sqrt{-1}} - \varepsilon^{-4\alpha\sqrt{-1}}\right) \ldots \&c.$$

Hence

$$\beta = \alpha - \left(\frac{1-n}{1+n}\right)\sin.2\alpha + \tfrac{1}{2}\left(\frac{1-n}{1+n}\right)^2\sin.4\alpha - \tfrac{1}{3}\left(\frac{1-n}{1+n}\right)^3\sin.6\alpha + \&c.,$$

the series sought.

46. In equation (11), let us replace α by

$$\beta+\gamma.$$

Then

$$\varepsilon^{+(\beta+\gamma)\sqrt{-1}} = \varepsilon^{+\beta\sqrt{-1}} \times \varepsilon^{+\gamma\sqrt{-1}} = \cos.(\beta+\gamma) + \sin.(\beta+\gamma)\sqrt{-1} =$$

$$(\cos.\beta + \sin.\beta\sqrt{-1})(\cos.\gamma + \sin.\gamma\sqrt{-1}) =$$

$$(\cos.\beta\cos.\gamma - \sin.\beta\sin.\gamma) + (\sin.\beta\cos.\gamma + \cos.\beta\sin.\gamma)\sqrt{-1}. \quad (12).$$

But by principles of Undetermined Coefficients

$$\cos.(\beta+\gamma) = \cos.\beta\cos.\gamma + \sin.\beta\sin.\gamma,$$

and

$$\sin.(\beta+\gamma) = \sin.\beta\cos.\gamma + \cos.\beta\sin.\gamma.$$

If we replace θ by $\beta-\gamma$, in equation (11), the result is

(40)

Take logarithm of both numbers; the result is

$$2\beta\sqrt{-1} = 2\alpha\sqrt{-1} + \log.\left(1+\frac{1-n}{1+n}e^{-2\alpha\sqrt{-1}}\right) - \log.\left(1+\frac{1-n}{1+n}e^{+2\alpha\sqrt{-1}}\right) =$$

$$2\alpha\sqrt{-1} - \left(\frac{1-n}{1+n}\right)\left(e^{+2\alpha\sqrt{-1}} - e^{-2\alpha\sqrt{-1}}\right) - 1/2\left(\frac{1-n}{1+n}\right)^2\left(e^{+4\alpha\sqrt{-1}} - e^{-4\alpha\sqrt{-1}}\right) \ldots \&c.$$

Hence

$$\beta = \alpha - \left(\frac{1-n}{1+n}\right)\sin.2\alpha + 1/2\left(\frac{1-n}{1+n}\right)^2\sin.4\alpha - 1/3\left(\frac{1-n}{1+n}\right)^3\sin.6\alpha + \&c.,$$

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$$cos.(\beta - \gamma) = cos.\beta\, cos.\gamma + sin.\beta\, sin.\gamma,$$

and

$$sin.(\beta - \gamma) = sin.\beta\, cos.\gamma - cos.\beta\, sin.\gamma.$$

These are the four fundamental formulas.
From them all other trigonometrical
formulas can be derived.
Let $\beta = \gamma$, in equation (12); then

$$cos.2\beta + sin.2\beta\sqrt{-1} = (cos.\beta + sin.\beta\sqrt{-1})^2.$$

If we continue this operation n times,
the result is

$$cos.n\beta + sin.n\beta\sqrt{-1} = (cos.\beta + sin.\beta\sqrt{-1})^n,$$

which is another demonstration for
DeMoivre's Formula.

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LOGARITHMIC IMAGINARIES.

47. In any system, the logarithm of 1 is 0, and the logarithm of 0 is $+\infty$ or $-\infty$, being $+$ if the base is less, and $-$ if the base is greater than unity.

All positive numbers between 0 and ∞, when used as bases of systems will include among the logarithms all possible numbers between $-\infty$ and $+\infty$.

It thus appears that if negative numbers have logarithms, they must be imaginary.

$$\text{Let } \varepsilon^{+y} = x.$$

Take the logarithm; then

$$\log_{(\varepsilon)} x = y.$$

But as there is only one real value of y, there can be only one arithmetical logarithm, and if we admit $\sqrt{-1}$ into the system, there may be an infinite number of logarithms, only one of which will be real:

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But as there is only one real value of y, there can be only one arithmetical logarithm, and if we admit $\sqrt{-1}$into the system, there may be an infinite number of logarithms, only one of which will be real:

the others will be of the general imaginary form, $\alpha \pm \beta\sqrt{-1}$.

48. A quantity of the form $\alpha \pm \beta\sqrt{-1}$, may have no real logarithm, and can have only one in a system whose base is $\lambda + \omega\sqrt{-1}$, unless the modulus, $\sqrt{(\alpha^2+\beta^2)}$, and the base are each equal to 1. In this case the number of real logarithms will be infinite, as is apparent. If only one real logarithm exist, it will be the ratio of the logarithm of the modulus of the quantity, and the base.

49. In equation (11), let θ be equivalent to $m\pi$; m being any even whole number. Then

$$\cos. m\pi = 1, \text{ and } \sin. m\pi = 0,$$

and the expression becomes

$$\varepsilon^{+m\pi\sqrt{-1}} = 1. \qquad (13).$$

This curious equation involves $\sqrt{-1}$ as an arithmetical impossibility, for log. 1 = 0.

the others will be of the general imaginary
form, $\alpha \pm \beta\sqrt{-1}$.

48. A quantity of the form $\alpha \pm \beta\sqrt{-1}$, may have no real logarithm, and can have only one in a system whose base is $\lambda + \omega\sqrt{-1}$, unless the modules, $\sqrt{(\alpha^2 + \beta^2)}$, and the base are each equal to 1. In this case the number of real logarithms will be infinite, as is apparent. If only one real logarithm exist, it will be the ratio of the logarithm of the [modulus?] of the quantity, and the base.

49. In equation (11), let θ be equivalent to $m\,\Pi$; m being any even whole number.
Then

$$cos.\,m\,\Pi = 1, \text{ and } sin.\,m\,\Pi = 0,$$

and the expression becomes

$$e^{+m\Pi\sqrt{-1}} = 1. \qquad (13).$$

This curious equation involves $\sqrt{-1}$ as an arithmetical impossibility,-- for log. 1 = 0.

But if we take the log. of equation (13), we have

$$m\pi\sqrt{-1} = \log.1; \text{ hence } m\pi\sqrt{-1} = 0.$$

If we ascribe any value to m according to previously mentioned conditions; say, z, we may deduce a series, true to any desired accuracy, for $\varepsilon^{+m\pi\sqrt{-1}}$.

Each series will be of the form $1+\sqrt{-1}\times 0$.

If y be the true logarithm of x, in equation $\varepsilon^{+y} = x$; and $\varepsilon^{+m\pi\sqrt{-1}} = 1$ be combined with it, the series is

$$\varepsilon^{+(y+m\pi\sqrt{-1})} = x.$$

Hence

$$\log. x = y + m\pi\sqrt{-1};$$ m being positive or negative as before.

If the real logarithm of x be denoted by log. x, and the general logarithm by Log. x; the general expression for logarithms becomes

$$\text{Log}. x = \log. x + m\pi\sqrt{-1}.$$

(44)

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$$log. x = log. x + m\Pi\sqrt{-1}.$$

So for any other number;

$$Log.\, z = log.\, z + n\pi\sqrt{-1}.$$

Then

$$Log.\, xz = log.\, xz + (m+n)\pi\sqrt{-1},$$

and

$$Log.\, \frac{x}{z} = log.\, \frac{x}{z} + (m-n)\pi\sqrt{-1}.$$

From this we readily see the relation between numbers and their logarithms. The sum or difference of two logarithms indicates that their corresponding numbers are respectively multiplied or divided.

50.

Make $\theta = \frac{\pi}{2}$ in equation (11); then

$$\varepsilon^{+\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1}.$$

Take the logarithm, and the result is

$$\frac{\pi}{2}\sqrt{-1} = log.\, \sqrt{-1},$$

which is a symbolic equation showing that imaginary arcs are logarithms. But we must not consider these functions

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But we must not consider these functions

unreal, for according to the modern interpretation of imaginaries the left hand number is simply the logarithm of a unit line perpendicular to the axis of reference. We should also remember that expressions of this kind are not made the basis of computation, and it is only in cases where the symbols are capable of being interpreted that we can conceive of the relation between them, and the specific numerical values they denote.

51. Again, in equation (11), let θ be represented by $(m+1)\Pi$; m being any even whole number, either positive or negative.

The equation becomes

$$e^{+(m+1)\sqrt{-1}\Pi} = cos.\,(m+1)\Pi + sin.\,(m+1)\Pi\sqrt{-1} = 1.$$

Hence

$$log.\,(-1) = (m+1)\Pi\sqrt{-1}, \text{ or}$$

$$\frac{log.(-1)}{\sqrt{-1}} = (m+1)\Pi.$$

(46)

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If we assume different values of m to be introduced, according to conditions, the expression becomes

$\pi, \quad 3\pi, \quad 5\pi, \quad 7\pi, \quad 9\pi, \quad \&c -$

This shows there might be an infinite number of ratios between the

$\log.(-1)$, and $\sqrt{-1}$,

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QUATERNION IMAGINARIES.

52. In the Introduction, we notice some of the historical changes which occurred in the interpretation of the imaginary, through all the stages of its development till the invention of Quaternions.

Monsieur Argand founded his interpretation on results derived from multiplication of imaginaries, as did subsequent investigators, till the time of Hamilton.

These results will now be investigated.

Let it be required to find a geometrical mean between +1 and -1.

If x denote this mean; then

$$+1 : x :: x : -1, \text{ or}$$

$$x = \pm\sqrt{-1}. \qquad (14)$$

But in this consideration we encounter a difficulty in ascribing the meaning to the notation used. If +1 denote

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But in this consideration we encounter a difficulty in ascribing the meaning to the notation used. If +1 denote

a geometrical magnitude, there can be no geometrical interpretation of -1-
But the +1 and -1 are here used only as indicators of direction, and hence they are only be considered as such-
In coördinate geometry of two dimensions, the axes may be taken as double-unit lines, to which the other transformations can be referred-

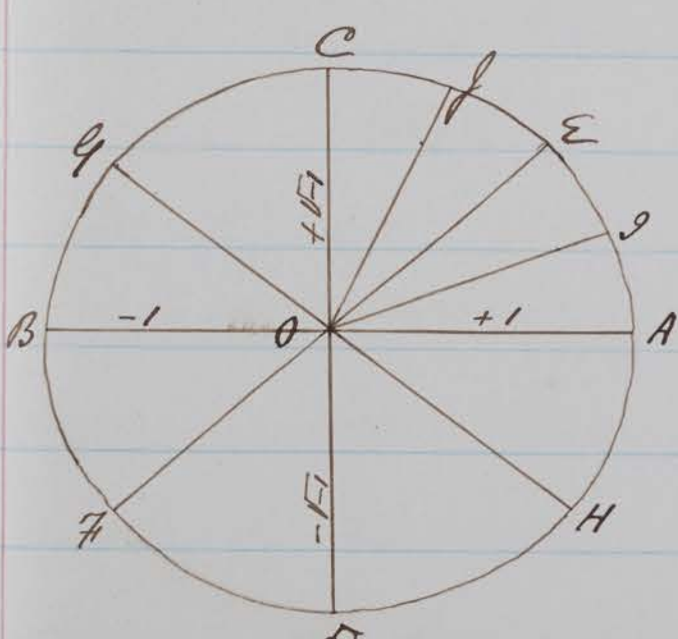

Fig. I.

In Fig. I, let O be the origin, with a radius OA= unity- Draw four diameters so as to divide the circumference into equal parts; also, bisect the arcs AE and EC-
We may denote the direction OA by +, and

(49)

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But the +1 and -1 are here used only as indications of direction, and hence they will only be considered as such.
In coordinate geometry of two dimensions, the axes may be taken as double-unit lines, to which the other transformations can be referred.

[Figure] In Fig. I, let 0 be the origin, with a radius 0A=unity. Draw from diameters so as to divide the circumference into equal parts; also, bisect the arcs AE and EC.
We may denote the direction 0A by +, and

OB by -1; hence, when OA revolves about O till
it coincides in direction with OB, the point
A describes an arc whose circular measure
is π; or in passing through an angle of
180°, the sign is changed from + to −.
And if the revolution continue from OB around
to OA, the line will revolve through another
angle of 180°, or will change the sign of
direction from − to +. But as the two
radii whose directions are indicated by
+ and − are in the same straight line,
and of equal length, the mean proportion
indicated in equation (14) must be perpen-
dicular to their common point of union,
and of the same length as either.
Hence we see that $+\sqrt{-1}$ and $-\sqrt{-1}$ simply
indicate a unit vector line perpendicular
to the axis of reference. The coefficient
1, in either case limits its length, and

(50)

0B by -1; hence, when 0A revolves about 0 till
it coincides in direction with 0B, the point
A descrives are all whose circular measure
is Π; or in passing through an angle of
180°, the sign is changed from + to -.
And if the revolution continue from 0B around
to 0A, the line will revolve through another
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Hence we see that $+\sqrt{-1}$and $-\sqrt{-1}$ simply
indicate a unit vector line perpendicular
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1, in either case limits its length, and

the sign $\pm\sqrt{-1}$ indicates its direction.
In the powers of $\sqrt{-1}$, we have $1, \sqrt{-1}, -1, -\sqrt{-1}$.
These four forms are repeated by a continued multiplication and recur in a cycle of four. The formulas,

4n, 4n+1, 4n+2, and 4n+3,

include all the changes that occur in the different positions of the right-angled vectors. And as these must coincide with the relations deduced by the mean proportion, 4n, 4n+1, 4n+2, 4n+3, must correspond respectively with

$1, \sqrt{-1}, -1$, and $-\sqrt{-1}$.

53. The expression $v = m\sqrt{-1}$ represents a vector perpendicular to the given direction line.
If $m = 1, v = \sqrt{-1}$, which, by the preceding geometric interpretation, indicates that the vector line revolves through an

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If $m = 1, v = \sqrt{-1}$, which, by the preceding geometric interpretation, indicates that the vector line revolves through an

angle of 90°, around an axis perpendicular
to the plane of rotation.

54. OC is a mean proportion between OA and OB; and OD between OB and OA.
And so again OE is a mean proportion between OA and OC; and OG between OC and OB, or between OA and OD; OF between OB and OD, or between OA and OC; OH between OD and OA, or between OF and OE.
Similarly, we might insert any number of mean proportions between two given direction vector lines.
The proportions [illegible] be thus:
OA:OI::OI:OE::OE:OJ::OJ:OC:: &c., and by this proportion the angles included between the vectors must necessarily be equal.

55. But the preceding relations between the vectors do not necessarily originate

(52)

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to the plane of rotation.

54. 0C is a mean proportion between
0A and 0B; and 0D between 0B and 0A.
And so again 0E is a mean proportion between
0A and 0C; and 0G between 0C and 0B, or
between 0A and 0D; 0F between 0B and 0D,
or between 0A and 0C; 0H between 0D and 0A,
or between 0F and 0E.
Similarly, we might insert any number
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The proportions [will?] be thus:
0A:0I::0I:0E::0E:0J::0J:0C:: &c., and
by this proportion the angle included
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55. But the preceding relations between
the vectors do not necessarily originate

from the origin 0. They can be taken from any origin, as is indicated by the general imaginary $\alpha \pm \beta\sqrt{-1}$.
Thus the general imaginary may be taken to represent the vector from the origin to the point, α, β, implicitly indicating the direction.
By operating on $\alpha \pm \beta\sqrt{-1}$, by $\sqrt{-1}$, the origin of vectors may be changed through each quadrant, and thus give a quadruple series of coördinates, but the length remains the same $= \sqrt{(\alpha^2+\beta^2)} =$ the modulus.

56. Every equation can be separated into real factors of the first or second degree, and whose roots are of the same form as $\alpha \pm \beta\sqrt{-1}$; β being positive or negative.
If the root be real, it can be represented by the ordinary graphics of Cartesian Geometry; if imaginary, by the

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If the root be real, it can be represented by the ordinary graphics of Cartesian Geometry; if imaginary, by the

preceding geometrical methods.
57. The preceding interpretation is only of a special case, and $\sqrt{-1}$ indicates an operation of rotation, or of position only.
But if α indicate any geometrical line, used as a direction line, $\pm\beta\sqrt{-1}$ represents a vector perpendicular to the direction line.
And finally all other coplanar vectors, not indicated by $\pm\alpha$, and $\pm\beta\sqrt{-1}$ must lie in one of the quadrants and be indicated by $\alpha \pm \beta\sqrt{-1}$.
But lines represented by functions containing imaginaries are as real as the direction lines themselves, and should be considered as absolute as lines indicated in a negative direction.
If we write $+\beta\sqrt{-1}$ and $-\beta\sqrt{-1}$, we have simply indicated the direction of rotation, when it is referred to the perpendicular unit

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If we write $+\beta\sqrt{-1}$ and $-\beta\sqrt{-1}$, we have simply indicated the direction of rotation, when it is referred to the perpendicular unit

vector, $\sqrt{-1}$.

58. In indicating the position of a point in a plane, we may denote the distance from the origin to the projection of the point on the abscissa, by α; the length of the projection line, by β; and the angle which the vector makes with the direction line, by θ; then instead of writing any of the four above given expressions, we may write the quaternion, $C(cos.\,\theta \pm sin.\,\theta\sqrt{-1})$.
But as $\sqrt{-1}$ turns a vector through the circular measure of $\frac{\pi}{2}$, the quaternion expression turns it through an angular part of a quadrant, represented by

$$\frac{\theta}{\frac{\pi}{2}} = \frac{2\theta}{\pi}.$$

Hence

$$C(cos.\,\theta \pm sin.\,\theta\sqrt{-1}) = (\sqrt{-1})^{\frac{2\theta}{\pi}}$$

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59. But $\sqrt{-1}$ was shown by Hamilton to be a geometrical reality, being restricted to no

particular direction in space. He showed that all directions can be represented by the imaginary, thus enabling mathematicians to represent any line or point equally well.

60. We are already using terms which should now be explained. But as the preceding discussions are only a review of those previous to the invention of Quaternions, it has been thought desirable to give an explanation of all the terms together, as long as those already used are generally understood.

Quaternions form a System of Analytical Geometry, the name of which was given by Hamilton, on account of four quantities that enter every true quaternion.

A quaternion is the product of a tensor and a versor.

A versor is the sum of a vector and a

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A quaternion is the product of a tensor
and a versor

A versor is the sum of a vector and a

scalar –
Hence we have the quantities
vector (V); tensor (T); versor (U); and scalar (S) –
But we saw in Art. 58 that a quaternion can also be represented by

$$C(\cos.\theta \pm \sin.\theta\sqrt{-1}) -$$

Hence there are two ways of representing any true quaternion:

$$Q = T \times U = C(\cos.\theta \pm \sin.\theta\sqrt{-1}) - \qquad (15) -$$

A vector is any line parallel to a given direction line, and if the line be a unit in length, it is called a unit-vector line –
A tensor is a multiplier, or literally that which stretches, one vector line till it coincides in length with another vector –
A tensor is a quantity which turns one vector, about its origin, till it

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coincides in direction with another vector, and if the angular space between the vectors be 90°, the versor is called a quadrantal versor.

But in equation (15) we saw a quaternion can be made up of a real, numerical part, and an imaginary. The numerical part is called a scalar.

61. Vector Equations:

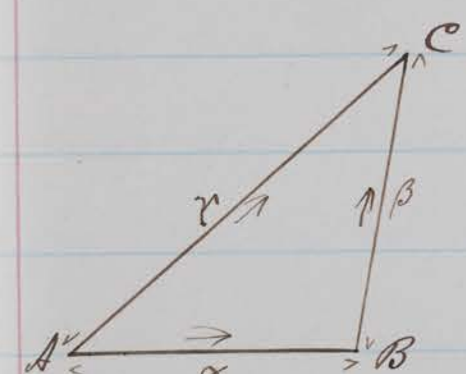

Fig. II.

In Fig. II, let ABC be any triangle. Denote the side AB by α; the side BC by β; and the side AC by γ.

Suppose the arrows indicate the direction that the sides of the triangle are generated.

If a magnitude be transferred from A to C, there are two ways by which

coincides in direction with another vector and if the angular space between the vectors be 90°, the vector is called a quadrantal versor

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[Figure] In Fig. II, let ABC be any triangle. Denote the side AB by α: the BC by β; and the side AC by γ.

Suppose the arrows indicate the direction that the sides of the triangle are generated. If a magnitude by transferred from A to C, there are two ways by which it may be accomplished; one is by going directly along the line AC; the other, by going from A to B and thence from B to C. This may be represented by $\alpha + \beta = \gamma$, or

$$\alpha + \beta - \gamma = 0,$$

which is known as the vector equation of the triangle. But the signs + and -, and =, here, do not have the same limited signification they do in Algebra.

In general language the above equation may be read, "a transformer expressed by vector α, followed by a transformer expressed by vector β, is equivalent to a transformer expressed by vector γ."

62. The fundamental formulas of trigonometry can be easily deduced by means of a vector equation.

(59)

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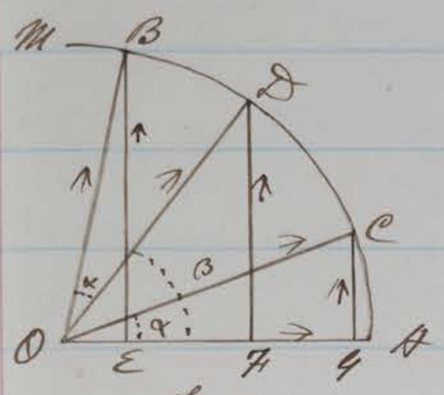

Fig. III.

In Fig. III, let AM be the arc of a circle whose radius is OA. Let the angles AOC $= \alpha$, AOD $= \beta$, and DOB $= \alpha$; then, as the sines of the angles will be perpendicular to OA, which we will use for a direction line, they will all be accompanied by $\sqrt{-1}$.

Therefore

$$OB = OE + EB = \cos.(\alpha+\beta) + \sin.(\alpha+\beta)\sqrt{-1}.$$

$$OD = OF + FD = \cos.\beta + \sin.\beta\sqrt{-1}.$$

$$OC = OG + GC = \cos.\alpha + \sin.\alpha\sqrt{-1}.$$

But we saw in Art. 54 that

OA : OC :: OD : OB, or

OA × OB = OC × OD.

The line OA is the unit radius and is the direction line; hence OA = 1.

The equation becomes

OB = OC × OD. (16)

In Fig. III, let AM be the arc of a circle whose radius is 0A. Let the angles A0C=α, ACD=β, and D0B=α; then, as the series of the angles will be perpendicular to 0A, which we will use for a direction line, they will all be accompanied by $\sqrt{-1}$.

[Figure]

Therefore

$$CB = 0E + EB = cos.(\alpha+\beta) + sin.(\alpha+\beta)\sqrt{-1}.$$

$$0D = 0F + FD = cos.\beta + sin.\beta\sqrt{-1}.$$

$$0C = 0G + GC = cos.\alpha + sin.\alpha\sqrt{-1}.$$

But we saw in Art. 54 that

0A:0C::0D:0B, or

0A x 0B=0C x 0D.

The line 0A is the unit radius and is the direction line; hence 0A = 1.

The equation becomes

0B = 0C x 0D. (16)

In this equation replace the values of OB, OC, and OD, by trigonometrical functions as found on page 60;
the result is

$$cos.(\alpha+\beta)+sin.(\alpha+\beta)\sqrt{-1}=(cos.\alpha+sin.\alpha\sqrt{-1})(cos.\beta+sin.\beta\sqrt{-1}). \quad (17).$$

By expanding and applying the principles of Undetermined Coefficients, we get

$$cos.(\alpha+\beta)=cos.\alpha\, cos.\beta-sin.\alpha\, sin.\beta,$$

and

$$sin.(\alpha+\beta)=sin.\alpha\, cos.\beta+cos.\alpha\, sin.\beta.$$

If $(\alpha+\beta)$ be replaced by $(\alpha-\beta)$, we get

$$cos.(\alpha-\beta)=cos.\alpha\, cos.\beta+sin.\alpha\, sin.\beta,$$

and

$$sin.(\alpha-\beta)=sin.\alpha\, cos.\beta-cos.\alpha\, sin.\beta.$$

If $\alpha=\beta$, in equation (17), we have

$$cos.\,2\alpha+sin.\,2\alpha\sqrt{-1}=(cos.\alpha+sin.\alpha\sqrt{-1})^2.$$

If this operation be continued m times, we get

$$cos.\,m\alpha+sin.\,m\alpha\sqrt{-1}=(cos.\alpha+sin.\alpha\sqrt{-1})^m;$$

(61)

In this equation replace the values of 0B,
0C, and 0D, by trigonometrical functions as
found on page 60;
the result is

$$cos.(\alpha+\beta)+sin.(\alpha+\beta)\sqrt{-1}=\big(cos.\alpha+sin.\alpha\sqrt{-1}\big)\big(cos.\beta+sin.\beta\sqrt{-1}\big). \quad (17).$$

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and

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If this operation be continued m times, we get

$$cos.\,m\alpha+sin.\,m\alpha\sqrt{-1}=(cos.\alpha+sin.\alpha\sqrt{-1})^{m};$$

which is another demonstration of
De Moivre's Formula.

63.

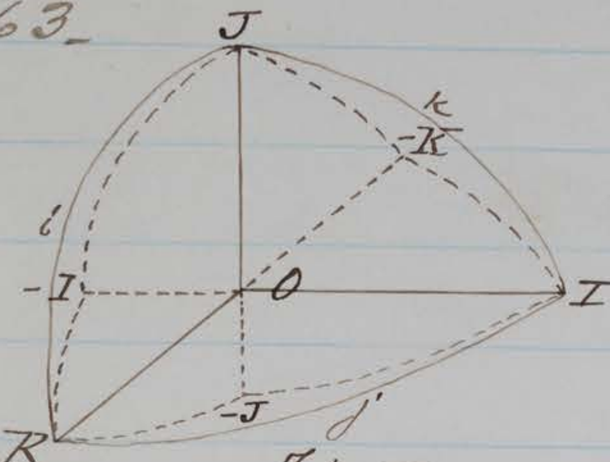

Fig. IV.

In Fig. IV, let OI, OJ, and OK be three mutually perpendicular unit-vectors. Prolongue these lines in the opposite direction, a unit's distance, and then draw arcs of circles as indicated in the figure.

Let the line OJ revolve about OI as an axis until it coincides with the line OK. So for all the other lines. The factors which turn these lines through the quadrant angle are the quadrantal versors, mentioned in Art. 60.

If we simply use the letters I, J, and K to denote the rectangular vectors

(62)

which is another demonstration of
De Moivre's Formula.
63. In Fig. IV, let 0I, 0J, and 0K be three mutually
[Figure[perpendicular unit-vectors. Prolongue these
lines in the opposite direction, a unit's distance,
and then draw arcs of circles as
indicated in the figure.
Let the line 0J revolve about 0I as an
axis until it coincides with the line 0K.
So for all the other lines. The factors which
turn these lines through the quadrant
angle are the quadrantal versors, mentioned
in Art. 60.
If we simply use the letters, I, J, and
K to denote the rectangular vectors

0I, 0J, and 0K, we have the following relations:

$\frac{K}{J} = I$, $\frac{I}{K} = J$, and $\frac{J}{I} = K$; or

$K = IJ$, $I = JK$, and $J = KI$.

If we use the negative vectors as axial lines, we have the following relations:

$\frac{J}{K} = -I$, $\frac{K}{I} = -J$, and $\frac{I}{J} = -K$; or

$J = -IK$, $K = -JI$, and $I = -KJ$.

The minus sign occurs here for two reasons; first, because the axis is considered negative, and second, because the rotation is contrary to that given in the first case, whose revolution was assumed positive.

These two cases give

$K = IJ$, $I = JK$, and $J = KI$. (18).

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$K = -JI$, $I = -KJ$, and $J = -IK$. (19).
In both of these sets of equations, we have used the unit vectors I, J, and K as axes. But these are quadrantal versors, and are generally denoted by i, j, and k.
Hence a unit vector may be employed as a quadrantal versor, having a plane perpendicular to the vectors; and the product or quotient of two perpendicular vectors is a vector perpendicular to both.
64. In algebraic multiplication, we have learned that the product does not depend upon the order of the factors, and hence is called the Commutation Principle.
But we saw in equations (18) and (19) that quaternion multiplication changes the sign of the product by changing the order of the factors, and hence is called the Non-Commutation Principle.

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65. The Associative Principle consists of maintaining the cyclical order of the factors. If $i\,j\,k$ be any known order of factors, then the associative law is

$$ijk = jki = kij = \&c.$$

But if this order be changed and the new cyclical order be used, the signs will be changed.

66. In equation (18), let J, K, and I be replaced by J, K, and I of equation (19). The result is

$$I^2 = -1,\ J^2 = -1,\ \text{and } K^2 = -1, \text{ or}$$

$$I^2 = J^2 = K^2 = i^2 = j^2 = k^2 = -1.$$

Then

$$i = j = k = \pm\sqrt{-1},$$ which is the relation between the i, j, k, of Quaternions.

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A vector designated by α is equivalent to a

line αi, i being a unit vector along the direction line; hence $(\alpha i)(\alpha i) = \alpha^2 i^2$ —— as a line $= \alpha^2$ —— as a vector. But by Art. 66, $i^2 = -1$.

Then

α^2 —— as a line $= -\alpha^2$ —— as a vector; or the square of a line is equivalent to minus the square of the corresponding vector.

68. The square of a unit vector regarded as a quaternion is sometimes called an inversor, and hence it can be written

$$\alpha^2 = -1.$$

And as $\alpha = 1$, the tensor of α, $T\alpha = 1$, or

$$T^2\alpha = 1; \text{ hence}$$

$$T^2\alpha = -\alpha^2.$$

Then any unit vector or quadrantal versor is a true representative of $\sqrt{-1}$, and has an infinite number of values, but they are all different from the symbolic scaler, $\sqrt{-1}$,

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Then any unit vector or quadrantaL versor is a true representative of $\sqrt{-1}$, and has an infinite number of values, but they are all different from the symbolic scaler, $\sqrt{-1}$,

which occurs in this convection algebraic analysis.

But to discriminate this function in Quaternions from the ordinary $\sqrt{-1}$, Hamilton called quaternions, vectors, and scalars, containing $\sqrt{-1}$, biquaternions, bivectors, and biscalars.

Therefore a biquaternion becomes

$$q = q_{,} + q_{,,}\sqrt{-1},$$

in which $q_{,}$ and $q_{,,}$ are real quaternions.

It appears that these discriminations are unnecessary, for $\sqrt{-1}$ must be involved in the production of these scalar functions.

69. If we have any quaternion, as,

$q = \frac{\beta}{\alpha} = n\pi$, n being any even integer;

then α will be parallel to β and will have the same or opposite direction.

Hence q becomes a positive or negative real number and is a scalar, but this

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scalar is one of revolution while the preceding is one of ratio.

70. These scalars and tensors can be applied to any line in space, and can occur in any quaternion.

From the nature of scalars, their product must be a scalar; and their conjugate, which considers the rotation in the opposite direction, is the identical scalar itself.

But as the imaginary, $\sqrt{-1}$, is a scalar in the biquaternion

$$q = q_{,} + q_{,,}\sqrt{-1} : \text{ then}$$

$$T(q_{,} + q_{,,})\sqrt{-1})^2 = \{S(q_{,} + q_{,,}\sqrt{-1}) + V(q_{,} + q_{,,}\sqrt{-1})\}\{S(q_{,} + q_{,,}\sqrt{-1}) - V(q_{,} + q_{,,}\sqrt{-1})\} =$$

$$Tq_{,}^2 - Tq_{,,}^2 + 2\sqrt{-1}\, Sq_{,} Kq_{,,}$$

If $q_{,} = q_{,,}$, and $Sq_{,} Kq_{,,} = 0$, the whole tensor of the biquaternion reduces to 0.

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$$q = q_{,} + q_{,,}\sqrt{-1}\text{: then}$$

$$T(q_{,} + q_{,,})\sqrt{-1})^2 = [S(q_{,} + q_{,,}\sqrt{-1}) + V(q_{,} + q_{,,}\sqrt{-1})][S(q_{,} + q_{,,}\sqrt{-1}) - V(q_{,} + q_{,,}\sqrt{-1})] =$$

$$T^2q_{,} - T^2q_{,,} + 2\sqrt{-1}Sq_{,}Kq_{,,}$$

If $q_{,} = q_{,,}$, and $Sq_{,}Kq_{,,} = 0$, the whole tensor of the biquaternion reduces to 0.

71. The reciprocal of a quaternion is the quaternion is the quaternion of its reciprocal.

Thus $q\frac{1}{q} = qq^{-1} = 1$

If $\frac{\alpha}{\beta} = q$; then $\frac{\beta}{\alpha} = \frac{1}{q} = q^{-1}$.

Suppose $-\alpha^2 = -\alpha \times \alpha = 1$, or

$$-\alpha = \frac{1}{\alpha}.$$

Then

$$\frac{1}{\sqrt{-1}} = -\sqrt{-1}$$

If we take the tensor of these equations, we see that the tensor of the reciprocal of a quaternion is the reciprocal of its tensor.

The versor has changed the direction of its angle, and to denote this negative movement, the conjugate quaternion is used.

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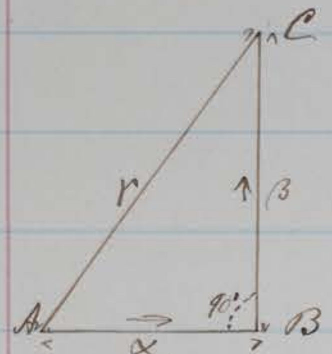

Fig. II.

Let ABC, Fig. II, be a right-angled triangle, right angled at B. Denote the sides, AB by α, BC by β, and AC by γ.
Then by Art. 61,

$\gamma = \alpha + \beta$ — as a vector.

By squaring,

$$\gamma^2 = \alpha^2 + 2\alpha\beta + \beta^2.$$

But by Art. 63, $\alpha\beta$ equals another vector perpendicular to the plane of rotation.
Hence, in the above equation,

$$2\alpha\beta = 0, \text{ and } \gamma^2 = \alpha^2 + \beta^2.$$

But in Art. 67, we saw that α^2 as a vector equals $-\alpha^2$ as a line.
Then the above equation becomes

$$-\gamma^2 = -\alpha^2 - \beta^2 \text{ — a line, or}$$

$$\gamma^2 = \alpha^2 + \beta^2.$$

Hence the theorem.

(70)

Let ABC, Fig. V, be a right-angled triangle, right angled at B. Denote the sides, AB by α, BC by β, and AC by γ.
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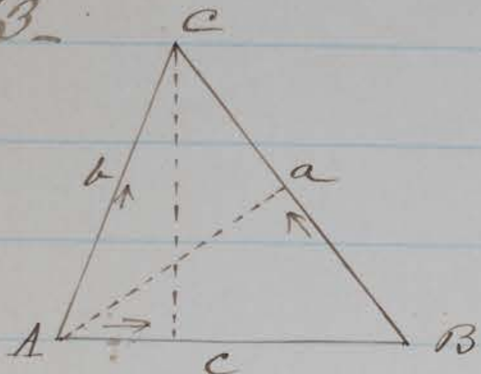
73. 

Fig. VI.

Let ABC, Fig. VI, be any triangle.

Then by Art. 61,

AC = AB + BC, or

$\overline{AC}^2 = S(AC \times AC) = S(AC)(AB+BC)$.

Hence

$b^2 = b\cos. A.c + b\cos. C.a$, or

$b = c\cos. A + a\cos. C$, a well known plane trigonometrical relation.

74. AB = AB,

AC = AB + BC,

Hence

AB × AC = AB(AB + BC).

Take the vectors, and the result is

$V(AB \times AC) = V\,AB(AB+BC) =$

$cb\sin. A = ca\sin. B$, or

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