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### THESIS

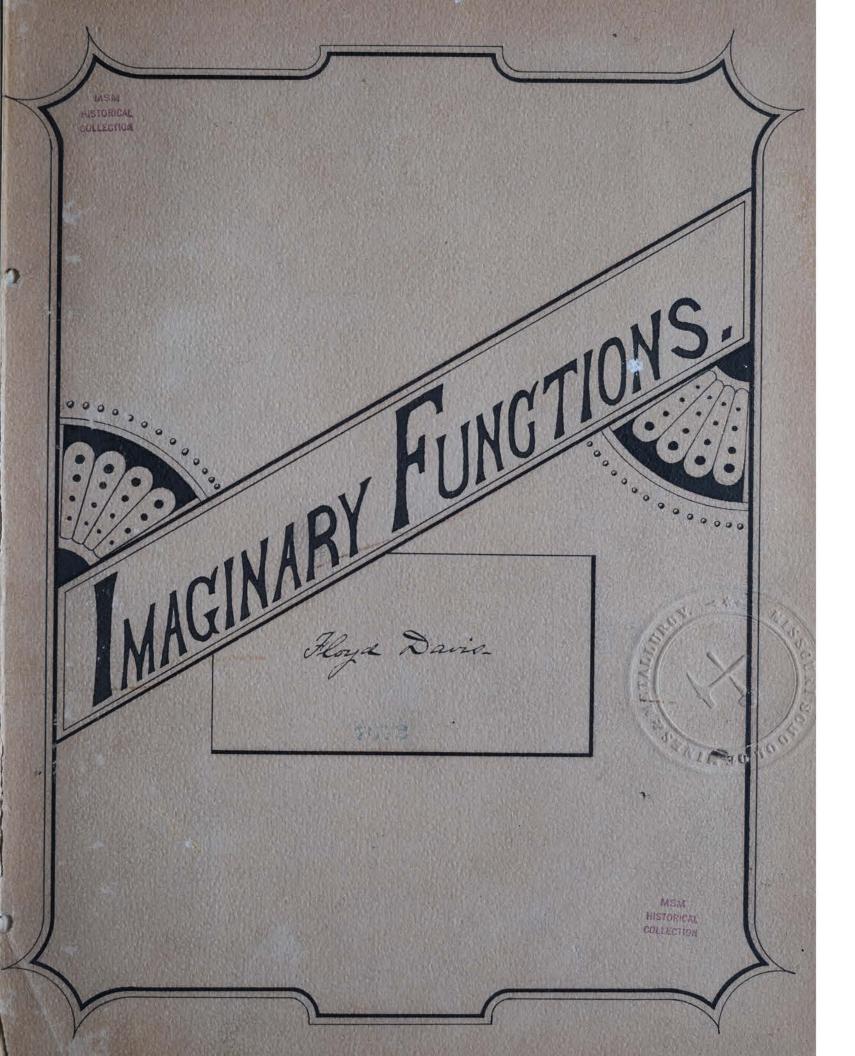
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IMAGINARY FUNCTIONS

1882

Columbus, Olio, April, 1882-

Columbus, Ohio, April, 1882.



# Imaginary Functions. Floyd Davis.

A MERARY ... Jo 88218001.08 Professor Robert White Me Facland, au Able Eustructor, and a man who has institud the Genue of Surretigation in The author's mind, These few pages are inscribed, with fulings Through Rispect and Estim.

Professor Robert White McFarland, an Able Instructor, and a man who has instilled the Germ of Investigation in the author's mind, These few pages are inscribed, with feelings of

Thorough Respect and Esteem.

То

INTRODUCTION.  $\sim \simeq \sim$ 1. The imaginary copression astarte occurs in common Algebra, and in all the higher departments of mathematical science, has been one of the most perplusing problems which the human mind has Encountered in any age. It was studied by The early mathematicians and because a first of speculation, but developed no marked resulto, for it was sumally ansidered algebraically, and in That interpretation is a symbol of an impossible operation - But many of the problems arising me algebraic- scoutriese andy sis unolong imaginaries where of such importance to mathematical and physical science that great attempts were made to establish their Solution. It was known That The imaginary occurred in mathematical functions having real values, and so was supposed to have source Mar Manning -

The imaginary expression which occurs in common Algebra, and in all the higher departments of mathematical science, has been one of the most perplexing problems which the human mind has encountered in any age. It was studied by the early mathematicians and became a field of speculation, but developed no marked results, for it was generally considered algebraically, and in that interpretation is a symbol of an impossible operation. But many of the problems arising from algebraic-geometrical analysis involving imaginaries were of such importance to mathematical and physical science that great attempts were made to establish their solution. It was known that the imaginary occurred in mathematical functions having real values, and so was supposed to have some real meaning.

## (1) Introduction

1.

(2) 2. Thus in Dr Moion's, Euler's, and many other Theorems, The relations established involing The imaginary lie at the foundation of much Matteniatical science: and The value of The Theorems was known to be actually near, Therege The imaginary could not be interpreted -3. This great problem first had the genue of its solution with Dr. Uneis, of Oxford, and afterwards was solve by Masers. Bale, Argana, Money, Janes, and others, but The broadert interpretation remained to bee discoursed by Sir William Moran Mulillon -4. Money laid The foundation on which Hamid ton Erectude The Modern Meory of Quaternions -5. Although The square rook of a negative quantity is a synton of an impossible arithunstical operation, yet it is of my great importance in mathematical conoralist By the such of signs, we leave that the

### (2)

- 2. imaginary lie at the formation of much the imaginary could not be interpreted.
- 3. This great problem first had the germ of its solution with Dr.Wallis, of Oxford, broadest interpretation remained to be
- 4.
- Although the square root of a negative 5. quantity is a symbol of an impossible arithmetical operation, yet it is of very By the rule of signs, we learn that the

Thus in DeMoivre's, Euler's and many other theorems, the relations established involving the mathematical science: and the value of these theorems was known to be actually real, though and afterwards was solved by Messrs. Buée, Argand, Mourey, Gauss, and others, but the discovered by Sir William Rowan Hamilton. Mourey laid the foundation on which Hamilton erected the Modern Theory of Quaternions. great importance in mathematical conventions.

(3)square of a negative is position, and hence we have no means of determining the sign of the square road of such an Expression -The square rook of a 2 is either + a or - a, bah what is the square work of -x 2? The arithunstical result cannot be determined, and so us only indicate the operation, as, ± x 1-1 -6 - By Means of the above conventions and such es pressions as a ± BI-T, as may develop other anpresins, subject to the rules of algebraic and quateriin transformation -7. But These imaginary copusions are not quantities, only symbols, which had The gour of their geometrical solution in The Early part of The present custing - The imaginary had been previously considered are undelering sepulal, appearing in problems That could not be interpreted - Ance There was no attempt

square of a negative is positive, and hence we have no means of determining the sign of the square root of such an expression. The square root of  $\alpha^2$  is either  $+\alpha$  or  $-\alpha$ , but what is the square root of  $-\alpha^2$ ? The arithmetical result cannot be determined, and so we only indicate the operation, as,

 $\pm \alpha \sqrt{-1}$ 

- 6. and quaternion transformation.
- But these imaginary expressions are not 7. of their geometrical solution in the early symbol, appearing in problems that could

By means of the above conventions and such expressions as  $\alpha \pm \beta \sqrt{-1}$ , we may develop other expressions, subject to the rules of algebraic

quantities, only symbols, which have the germ part of the present century. The imaginary had been previously considered an undetermined not be interpreted. And there was no attempt

(4) made to give any groundrical interpritation to these symbols Till Dr. Mallis published his Intalize of Algebra," in 1685- Therein the proposed to establish The Meaning of these functions by measuring Them on line, out of which all real quantities are measured. 8. But from the Time of Unicis Tice The beginning of the present cutury, There re-Maine a comparation quiescure in this field of instigation -9- In The gran 1805 Monsieur Buie preport au article Entitled : Memoires sur les Quantités Imaginaris," maintaining That 1-1 is a symbol of performanity to a give direction live to which it is referred -But hower, be excluded all interpretations of it as being produce by processes of Mulliptication, and consequently his method had Nerra developed any system like That plo-

(4)

made to give any geometrical interpretation to these symbols till Dr. Wallis published his "Treatise of Algebra," in 1685. Therein he proposed to establish the meaning of these functions by measuring them on line, out of which all real quantities are measured.

- 8. But from the time of Wallis till the beginning of the present century, there remained a comparative quiescence in this field of investigation.
- 9. In the year 1805 Monsieur Buée prepared an article entitled: "Memoire sur les is a symbol of perpendicularity to a given direction line to which it is referred. and consequently his method has

Quantités Imaginaires," maintaining that  $\sqrt{-1}$ But however, he excluded all interpretations of it as being produced by processes of multiplication, never developed any system like that produced

(5) duced by The product of oretir lines -10- Alust in connection with Bute's, another pauphlet was written by Monsieur Argand a-Titles : Essai sur une Manières de Representer les Quantités Suaginaires dans les Constructions Geonatriques," but was not published lite 1806 - This Theory dice not much with heady recognition until 1814, when again added Mon convincing proof to establish The principles which he had previously laid down -11- Argand was, no doubt, The first to interput The imaginary by reforming it to the multiplicalin of direction lines -This same Theory was again independently reproduced by Mr. Marrie, of England, in 1828; and shortly afterwards by Monsieur Mousey, in a work sutitled ;"Le brai Muorie des quantités Mgatife sh dis Quantités désignées suraginaires.

(5)

by the product of vector lines.

- 10. "Essai sur une Manière de Representer Géométriques" but was not published till more convincing proof to establish the principles which he had previously laid down.
- 11. of direction lines.

Almost in connection with Buée's, another pamphlet was written by Monsieur Argand entitled: les Quantités Imaginaires dans les Constructions 1806. This theory did not meet with ready recognition until 1814, when Argand added

Argand was, no doubt, the first to interpret the imaginary by referring it to the multiplication

This same theory was again independently reproduced by Mr.Warren, of England, in 1828; and shortly afterwards by Monsieur Mourey, in a work entitled: "La vraie Théorie des Quantitiés Négatives et des Quantités désigné imaginaires."

(6) They both shourd That ST-I is a victor perperdicular to The initial direction netor line -12 - But Survis was The first, no doubt, who speculated in fillds of Usearch in which The slightert Early auticipation of qualinimo is at present found - it sudiavore to represent any south in space by an repression similar to a + BFT, Thus queralizing The principles of gunutry of two dimensions-Mongh This incution, he reasoned by analogy and produced The Expression p'co. x + p"co. B + p"co. r to represent a point in space, in which &, B, and & are The inclinations of The Three as co-He could not assign True values to p', p", and p" and This was his file of inquiry - It is now know they are the i, j, k, of Quater-Miono-13- By this interpretation many idea of

They both showed that  $\beta \sqrt{-1}$  is a vector perpendicular to the initial direction vector line. But Servois was the first, no doubt, who speculated in fields of research in which the slightest early anticipation of Quaternions is at present found. He endeavored to represent any point in space by an expression similar to  $\alpha \pm \beta \sqrt{-1}$ , thus generalizing the principles of geometry of two dimensions. Through this induction, he reasoned by analogy and produced the expression  $\rho$ 'cos. $\alpha$ + $\rho$ "cos. $\beta$ + $\rho$ "'cos. $\gamma$ to represent a point in space, in which  $\alpha$ ,  $\beta$ , and  $\gamma$  are the inclinations of the three axes. He could not assign true values to  $\rho'$ ,  $\rho''$ , and  $\rho'''$ , and this was his field of inquiry. It is now known they are the i, j, k of Quaternions. By this interpretation every idea of

12. 13.

(7) unpossibility vanishes from the mind, and The imaginary becomes as clear as The subject of ordinary sepubolic Algebra - this Theory was developed into a true system, and many Malino of liver in space are diciphered, Thus forming a formumer to the Quaternion Analysis, which was soon destine to follow -Manyte this interpretation of inaquaries The 14. Quateriumo have been layely durloped -14 - At the outset, Anniellow sought to establish a system, in which he could interpret The imaginary by Excluding all augular functions -By This nears he established a new system of Mathematics conclining simplicity, Elegance, and pour - This new and comprehension and of mathunatical science has give The present cutury The greatest mathematicase impetus, The trad fields of science have sure received -His discorry was The grouchiese interpretation

## (7)

impossibility vanishes from the mind, and the imaginary becomes as clear as the subject of ordinary Symbolic Algebra. This theory was developed into a true system, and many relations of lines in space were deciphered, thus forming a forerunner to the Quaternion Analysis, which was soon destined to follow. Through this interpretation of imaginaries the Quaternions have been largely developed. At the outset, Hamilton sought to establish a system, in which he could interpret the imaginary by excluding all angular functions. By this means he established a new system of mathematics combining simplicity, elegance and power. This new and comprehensive view of mathematical science has given the present century the greatest mathematical impetus, the broad fields of science have ever received. His discovery was the geometrical interpretation

(8) of 1-1, showing That it applied to any direction in space, and is limited by no particular direction-line, which he developed into a singularly pourfue system, kunon as Calculus of Quateriino -15- Hamilton, in a letter published by The North British Review, 1866, Explains The origine of his Quaternino as follows: Och. 15, '58do-mond vice be the fiftunte bisteray of the Quateriumo - They started into life, or light free grow, on the 16th of Och, 1843, as I was walking with bady Amuillon to Dublin, and came up to Brougham Bridge, which my boys have since cauce the qua-Teruin Bridge - That is to say, I Then and These fill the salitaice circuit of Thought to close ; and The spartes which file from do Where The fundamental Equations between i, j, k;

of  $\sqrt{-1}$ , showing that it applied to any direction in space, and is limited by no particular direction-line, which he developed into a singularly powerful system, known as Calculus of Quaternions.

(8)

Hamilton, in a letter published by the 15. of his Quaternions as follows:

"Oct. 15, '58. To-morrow will be the fifteenth birthday of the Quaternions. They started into life, or light full grown, on the 16th of Oct., 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge. That is to say, I then and there felt the galvanic circuit of thought to close; and the sparks which fell from it were the fundamental equations between i, j, k,

North British Review, 1866, explains the origin

(9) Evalty such as I have there There since -I pulled out in The spoh, a portal-book, which the wists, and made an Entry, on which at the my monut, I full That it might be worth my while to Espend the labor of at least ten (or it might be fiften ) years to come - But There it is pair to day that This was because I full a problem to have been, at that very moment, soludan an intellectude want seliend - which had haunter me for at least fifteen scars befor - Less Than an hour Elapsed before I had asked and obtained leave of The Conneil of The Royal Drish Academy, of which Society ? was at that Time President, to read, at the much gueral meeting, a paper, on Quateriino, which I accordingly dich Mr. 13, 1843.

(9)

exactly such as I have used them ever since. I pulled out on the spot, a pocket-book, which still exists, and made an entry, on which at the very moment, I felt that it might be worth my while to expand the labor of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a problem to have been, at that very moment, solved -- an intellectual want relieved -- which had haunted me for at least fifteen years before. Less than an hour elapsed before I had asked and obtained leave of the Council of the Royal Irish Academy, of which Society I was at that time President, to read, at the next general meeting, a paper, on Quaternions, which I accordingly did Nov. 13, 1843."

(10) ALGEBRAIC IMAGINARIES. 16- Every imaginary expression care be reduced to the general forme, x ± BF-7, in which & and B are real quantities -This is evident, inacunde as are the real terms can les combined into one polynomial, which may be represented by x; and all the Truly imaginary Terus can les continue into auother polynomiae, which, when factore consists of a real quanlily, and V-T - The real polynomial factor May be represented by B, and have The above imaginary expression reduces to the gueral forme x = 31=1 -17. But if we have an imaginary supression of the queral forme, UR May emide & a real quantily, and BV-1, imaginary - the two Taken to-getter, as x + BI-1, are gunrally considere imaginary - If x = 0, The Expression becomes truly maginary, and Equals BU-T; if B=0

(10)

- 16. real quantities.
  - and  $\sqrt{-1}$ . The real polynomial factor

 $\alpha \pm \beta \sqrt{-1}$ 

But if we have an imaginary expression 17. of the general form, we may consider  $\alpha$  a taken together, as  $\alpha \pm \beta \sqrt{-1}$ , are generally

ALGEBRAIC IMAGINARIES.

Every imaginary expression can be reduced to the general form,  $\alpha \pm \beta \sqrt{-1}$ , in which  $\alpha$  and  $\beta$  are

This is evident, in as much as all the real terms can be combined into one polynomial, which may be represented by  $\alpha$ ; and all the truly imaginary terms can be combined into another polynomial, which, when factored consists of a real quantity, may be represented by  $\beta$ , and hence the whole imaginary expression reduces to the general form

real quantity, and  $\beta \sqrt{-1}$ , imaginary. The two considered imaginary. If  $\alpha = 0$ , the expression becomes truly imaginary, and equals  $\beta \sqrt{-1}$ , ; if  $\beta = 0$ ,

(11) The Espression becomes real, and Equals &-18. Every Monomial imaginary can be reduced to the general forme, 2 V=7. in which & is a real quantity and S is any whole number -Let us suppose That 2 V-B is a Monomial imajuan n being one - His can be reduced to The forme 2 V3 V-T, but as 2 V3 is a real factor it may be duroted by r, and the Expression The Equals rV-1\_ But as an unajuary is an indication ence look of a negative quantity, us may, in place of u, substitute a quantity, 28, which will indicate This result-Hauce the forme VV-T-19. When S= 1, the Expression becomes NV-1, which is called an imaginary of The second degree -If the monital be of the forme

### (11)

the expression becomes real, and equals  $\alpha$ . Every monomial imaginary can be reduced

18. to the general form,

 $\gamma^2 \delta \sqrt{-1}$ 

in which  $\gamma$  is a real quantity and  $\delta$  is any whole number.

n being even. This can be reduced to the may be denoted by  $\gamma$ , and the expression then equals  $\gamma \sqrt[n]{-1}$ .

19. If the monomial is of the form

Let us suppose that  $\lambda n \sqrt{-\beta}$  is a monomial imaginary, form  $\lambda n \sqrt{\beta} n \sqrt{-1}$ , but as  $\lambda n \sqrt{\beta}$  is a real factor it

But as an imaginary is an indicator even root of a negative quantity, or may, in place of n, substitute a quantity,  $2\delta$ , which will indicate this result. Hence the form  $\gamma^2 \sqrt[2]{\sqrt{-1}}$ . When  $\delta = 1$ , the expression becomes  $\gamma \sqrt{-1}$ , which is called an imaginary of the second degree.

(12) rV-T, rV-T, rV-T, ve, it is of the fourth, sister, Eight, depu, TE. 20\_ Imaginaries are enjugate when they only differ in the sign of the coefficients of V-1; Thus a+BD-I and a-BV-I are paid to be conjugate imaginarios-Hun we see that the sum and productof two conjugate imaginaries are always real -21. The square work of the product of two enjugate inaquaries, taken with the position sign, is called The Modulus of Each Expression, and is of the some V(a2+32) -22. Thus pone the conjugate imaginaries, x+BV-1 and x-BV-1, us infer that the modulus of a real quantity is The position, Munericae value of that quantity - Buch in order That The modulus, 107739, onicistus, & and 3 much Each Equal zoro, and in this care

(12)

 $\gamma^4 \sqrt{-1}$ ,  $\gamma^6 \sqrt{-1}$ ,  $\gamma^8 \sqrt{-1}$ , etc., it is of the fourth, sixth, eight, degree, etc.

20. Imaginaries are conjugate when they only differ in the sign of the coefficients of  $\sqrt{-1}$ ; conjugate imaginaries.

> Hence we see that the sum and product of two conjugate imaginaries are always real.

21. The square root of the product of two and is of the form

$$\sqrt{(\alpha^2 + \beta^2)}$$

22. Thus from the conjugate imaginaries, of a real quantity is the positive, numerical value of that quantity. But in order must each equal zero, and in this case

thus  $\alpha + \beta \sqrt{-1}$  and  $\alpha - \beta \sqrt{-1}$  are said to be

conjugate imaginaries, taken with the positive sign, is called the modulus of each expression,

 $\alpha + \beta \sqrt{-1}$  and  $\alpha - \beta \sqrt{-1}$ , we infer that the modulus that the modulus,  $\sqrt{(\alpha^2 + \beta^2)}$ , vanishes,  $\alpha$  and  $\beta$ 

(13) both imaginary supressions vanish From This un see That when two imaginaries are Equal Their Moduli are also Equal -23- If two imaginary Expressions are Equal, The real parts much les Equar and also The coefficients of V-Tfor suppre x+3V-1= +5V-1-Hue by transposition and factoring, the get x-r+(B-SJUFi = 0\_ is x-r=0 and 18-S) VFI = 0. By theory of Mutatorium Coefficience x=r, and 3= 5- 15-7+4(8-6)=0 The the guerae Equation x+ sV-1 = Y+ SV-1 may be considered a symbolic representation, in me statement of The Equation x=r, and B=S-24 - Jake Two inaginary Expressions, x+ BU=1, and r+ SU=7, and let us find their sun, difference,

both imaginary expressions vanish. From this we see that when two imaginaries are equal their moduli are also equal.

23. If two imaginary expressions are equal, the real parts must be equal and also the coefficients of  $\sqrt{-1}$ . For suppose  $\alpha + \beta \sqrt{-1} = \gamma + \delta \sqrt{-1}.$  $\alpha - \gamma + (\beta - \delta)\sqrt{-1} = 0.$  $\alpha = \gamma$ , and  $\beta = \delta$ . one statement of the equation  $\alpha = \gamma$ , and  $\beta = \delta$ . 24. Take two imaginary expressions,  $\alpha + \beta \sqrt{-1}$ , and  $\gamma + \delta \sqrt{-1}$ , and let us find their sum, difference,

Then by transposition and factoring, we get By theory of <u>undetermined</u> (no variable) <u>Coefficients</u> Then the general equation  $\alpha + \beta \sqrt{-1} = \gamma + \delta \sqrt{-1}$  may be considered a symbolic representation, in

(14) product, and quolint-Their sum is x + r + (B + S) U = 1 - 1This difference is x-r+(3-5)V=1\_ This product is  $(\alpha + \beta V = \overline{7})(r + \delta V = \overline{7}) = \alpha r - \beta \delta + (\alpha \delta + r \beta) V = \overline{7} - \beta \delta + (\alpha \delta + r \beta) V = \overline{7} - \beta \delta + \beta$ The quotient obtained by dividing the first by the second is Mutiply bitte numerator and duminator by r-SV-1; The expression becomes xr+B5 + rB-x5 V-1 -r2+ 52 + rB-x5 V-1 -25- The modulus of The product or quotient of two imaginaries Equalo The puduct or quotient of their uspection Moduli ; and the two imaginary supressions

(14)

product, and quotient. Their sum is  $\alpha + \gamma + (\beta + \delta)\sqrt{-1}$ . Their difference is  $\alpha - \gamma + (\beta - \delta)\sqrt{-1}$ .

Their product is

 $(\alpha + \beta \sqrt{-1})(\gamma + \delta \sqrt{-1}) = \alpha \gamma - \beta \delta + (\alpha \delta + \gamma \beta) \sqrt{-1}.$ The quotient obtained by dividing the first by the second is

 $\frac{\left(\alpha+\beta\sqrt{-1}\right)}{\left(\nu+\delta\sqrt{-1}\right)}.$ 

Multiply both numerator and denominator by  $(\gamma - \delta \sqrt{-1})$ ; the expression becomes  $\frac{(\alpha\gamma+\delta\beta)}{\gamma^2+\delta^2} + \frac{(\gamma\beta-\alpha\delta)}{\gamma^2+\delta^2})\sqrt{-1}.$ 

The modulus of the product or 25. quotient of two imaginaries equals the product or quotient of their respective moduli; and the two imaginary expressions

(15) ince not variable as long as mitter factor vanishus\_ For, The Modulies of the product of x+31-7, and r+SU-7, by definition and art. 24 Equalo  $V(\alpha r - \beta \delta)^{2} + (\alpha \delta + r \beta)^{2} = V(\alpha^{2} + \beta^{2})(r^{2} + \delta^{2}) =$  $V(\alpha^{2}+\beta^{2}) \times V(\gamma^{2}+\beta^{2}) - (1) -$ Again, The modulus of The quotient of a+BV-I, and r+SV-I, according to art. 24 Equalo  $\sqrt{\binom{\alpha r + \beta \delta}{r^2 + \delta^2}^2 + \binom{r \beta - \alpha \delta}{r^2 + \delta^2}^2} = \sqrt{\binom{\alpha^2 + \beta^2}{(r^2 + \delta^2)}} = \sqrt{\binom{\alpha^2 + \beta^2}{(r^2 + \delta^2)} = \sqrt{\binom{\alpha^2 + \beta^2}{(r^2 + \delta^2)}} = \sqrt$ (Z). But Va2+32) and V(r2+ 52) are Moduli -Equations (1) and (2) Equal The protect and quotient respectively of the Modulus of a + BV-T, and r+ SV-T-Huce The Theonus -26. The Erre roots of imaginary Expressions

(15)

will not vanish as long as neither factor vanishes.

For, the modulus of the pro  $\alpha + \beta \sqrt{-1}$ , and  $\gamma + \beta \sqrt{-1}$ by definition and Art. 24 eq

 $\sqrt{(\alpha \gamma - \beta \delta)^2 + (\alpha \delta)^2}$ 

 $\sqrt{(\alpha^2 + \beta^2)}$ 

Again, the modulus of the o  $\alpha + \beta^{-1}$ according to Art. 24 equals

 $\int \left(\frac{\alpha\gamma - \beta\delta}{\gamma^2 + \delta^2}\right)^2 + \left(\frac{\gamma\beta - \alpha\delta}{\gamma^2 + \delta^2}\right)^2 = -\frac{1}{2}$ 

But  $\sqrt{(\alpha^2 + \beta^2)}$  and  $\sqrt{(\gamma^2 + \delta^2)}$  are moduli.

Equations (1) and (2) equal the product and quotient respectively of the modulus  $\alpha + \beta \sqrt{-1}$  and  $\gamma + \delta \sqrt{-1}$ . of

Hence the theorem. 26. The even roots of imaginary expressions

oduct of  
- 
$$\delta\sqrt{-1}$$
,  
juals  
 $(\gamma^2 + \gamma\beta)^2 = \sqrt{(\alpha^2 + \beta^2)(\gamma^2 + \delta^2)} =$ 

$$\overline{)} x \sqrt{(\gamma^2 + \delta^2)} \quad (1)$$

$$\frac{\sqrt{(\alpha^{2}+\beta^{2})(\gamma^{2}+\delta^{2})}}{\sqrt{(\gamma^{2}+\delta^{2})}*\sqrt{(\tau^{2}+\delta^{2})}} = \frac{\sqrt{(\alpha^{2}+\beta^{2})}}{\sqrt{(\gamma^{2}+\delta^{2})}}$$
(2).

(16) are of The same gunral form as The Expressions Themselors het us Estract an Eour rook of The unajuary, at 3V-1; and suppose That  $(\alpha \pm \beta V = 1)^{\frac{1}{2}m} = \chi \pm \gamma V = 1 - (3)_{-1}$ Phin  $(\alpha \pm \beta - 7) = (\alpha \pm \gamma - 7)^{2222}$ If m= 1, The Expression (4)- $(x \pm 3V-i) = (x^2 - y^2 \pm 2xy V-i) -$ Huce by the theory of Mutitanium Enficients (4)- $\alpha = \chi^2 - \gamma^2$ Auce  $\beta = \pm 2\chi q$ -There by squaring and adding The last two Equations, us get  $(N^{2}+q^{2})^{2} = x^{2}+B^{2},$  or  $\chi^{2} + \eta^{2} = \pm V(\alpha^{2} + \beta^{2}) -$ (5)-By combining Equations (4') and (5) as get  $\chi = \pm \sqrt{\left(\sqrt{\alpha^2 + \beta^2} + \alpha\right)} - \frac{1}{2}$ (6)\_

### (16)

Are of the same general form as the expressions themselves. Let us extract an even roo imaginary,  $\alpha \pm \beta \sqrt{-1}$ ; and  $(\alpha \pm \beta \sqrt{-1})^{\frac{1}{2m}} =$ 

Then

 $(\alpha \pm \beta \sqrt{-1}) = (x)$ If m=1, the expression  $(\alpha \pm \beta \sqrt{-1}) = (x^2)^2$ Hence by the theory of Ur  $\alpha = x^2 - y^2$ And

 $\beta = \pm 2xy.$ 

Then by squaring and adding the last two equations, we get

 $(x^{2} + y^{2})^{2} = \alpha^{2}$  $x^{2} + y^{2} = \pm \sqrt{(\alpha)}$ 

By combining equations (

$$x = \pm \sqrt{\left(\frac{\sqrt{(\alpha^2 + \beta^2)}}{2}\right)}$$

ot of the  
I suppose that  
$$x \pm y \sqrt{-1}$$
 (3).

$$\pm y\sqrt{-1}$$
)<sup>2m</sup>

$$y^{2} - y^{2} \pm 2xy\sqrt{-1}$$
). (4).  
ndetermined Coefficients (4').

$$(4)^{2} + \beta^{2}$$
 or  
 $(5)^{2} + \beta^{2}$ ). (5).  
 $(4)^{2}$  and (5) we get  
 $(4)^{2} + \alpha$ ). (6).

(17) And  $y = \pm \left| \left( \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2} \right) - \alpha \right|$ (7)\_ But by supposition & and y are real, and so The sum of Their squares is posilin, and consequently we use The + signe lefore Va2+32) -The signe I before 2xy flows That is will take the same doubly signe, as 2 xy-Huer The Expressions because identical No for all real values of me, where Even, and the queroe Expression becomes (x ± 31-7) = x ± 4V-1 - (8)-Condusion more general them lypothesis. 27. Uz may fine the square rook of ± 0-1, by Making x=0, and B=1, in Equations (6) and (7), and Then substitute Their mens in Equation (8)-The results are V(+V-i) = ± 1+1-1, and

And

$$y = \pm \sqrt{\left(\frac{\sqrt{(\alpha^2 + \beta^2)} - \alpha}{2}\right)}.$$
 (7  
supposition x and y are real,  
the sum of their squares is positive,  
nsequently we use the +  
efore  $\sqrt{(\alpha^2 + \beta^2)}.$   
on  $\pm$  before 2xy shows that  $\beta$  will  
e same double sign, as 2xy.  
the expressions become identical.  
all real values of m, when even,  
e general expression becomes  
 $(\alpha \pm \beta \sqrt{-1})^{1/2m} = x \pm y \sqrt{-1}.$ 

But by and so and co sign be The sig take th Hence So for and the

27. As may find the square room of  $\pm \sqrt{-1}$ , by making  $\alpha = 0$ , and  $\beta = 1$ , in equations (6) and (7), and then substitute their values in equation (8). The results are

$$\sqrt{(+\sqrt{-1})} = \pm \frac{1+\sqrt{-1}}{\sqrt{2}}$$
, and

### (17)

(8).

(18)  $V(-V-1) = \pm \frac{1-V-1}{V_{0}} -$ Suppose us have the Equation  $\chi^{4} = -1$ -Luce  $N^2 = \pm V - i$ , and  $x = \pm V \pm V = i$ But since x = -1, x = ± U-1 -This shows there are four, pourte roots in This Equation, are of The guerae form of - 1; and They are indicative in The Expression ± V(±V=T), and in The gueroe forme + 1± 1-1 -28 - Enry quantity has 12, 1the roots, and no mon, and if n be ene, two of These worts are real and The other 11-2, imaginary - of n be odd, one rook is real, and The other n-1, inajuary -Let The gueros Equation  $x^{n} = p^{n}$  or  $x^{n} - p^{n} = 0$ , Equal the quantity-

 $\left| (-\sqrt{-1}) \right| =$ 

Suppose we have the equation

$$x^4 = -1.$$

Then

$$\alpha^2 = \pm \sqrt{-1}$$
, and  $x = \pm \sqrt{\pm \sqrt{-1}}$ .  
nce  $x^4 = -1$ ,  $x = \pm \sqrt[4]{-1}$ .  
shows there are four, fourth roots  
equation, all of of the general (?)  
of -1; and they are indicated in  
spression  $\pm \sqrt{\pm \sqrt{-1}}$ , and in the general

But si This s in this form c the ex form  $\pm \frac{1 \pm \sqrt{-1}}{\sqrt{2}}$ .

28. be odd, one root is real and the other n-1, imaginary. Let the general equation Equal the quantity.

## (18)

$$= \pm \frac{1 - \sqrt{-1}}{\sqrt{2}}.$$

Every quantity has n, nth roots, and no more, and if n be even, two of these roots are real and the other n-2, imaginary. If n

 $x^{n} = p^{n}$ , or  $x^{n} - p^{n} = 0$ ,

(19) in This problem There are two cases; one when It is some, The other acture n is odd -First, let n lie some; The Equation when factoria is  $(x^2-b^2)(x^{n-2}+x^{n-4}b^2+x^{n-6}b^4+\dots+b^{n-2})=0$ But sither factor in This Equation Equals 0 -Huce  $x^{2}-p^{2}=0, \text{ or } x=+p \text{ and } -p_{-}$ Also  $(x^{n-2} + x^{p-4})^2 + x^{p-4} + - - - - - p^{n-2}) = 0_-$ But There can be no mar rook that will satisfy this Equation, for The coefficients of x are all position and all pours of x are Ence ; hence The 11-2 worth which it an-Tains much are be majinary-Mach, like n be odd ; then The Equation when factorie is (x-p)(x+x p+x p+---- px-1)=0-But in This Equation also, Either factor Equation

### (19)

In this problem then are two cases; one when n is even, the other when n is odd. First, let n be even; the equation when factored is

 $(x^{2} - p^{2})(x^{n-2} + x^{n-4}p^{2} + x^{n-6}p^{4} + \dots p^{n-2}) = 0$ But either factor in this equation equals 0.

Hence

$$x^2 - p^2 = 0$$
, or  $x$ 

Also

 $(x^{n-2} + x^{n-4}p^{2} + x^{n-6}p^{4} + \dots p^{n-2}) = 0$ But there can be no real root that will satisfy this equation, for the coefficients of x are all positive and all powers of x are even; hence the n-2 roots which it Next, let n be odd: then the equation

contains must all be imaginary. when factored is

But in this equation also, either factor equals 0.

= +p and -p.

 $(x-p)(x^{n-1}+x^{n-2}p + x^{n-3}p^{2}+\dots p^{n-1}) = 0.$ 

(20) Huce x - p = 0, or  $x = p_-$ Also,  $(x^{n-1} + x^{n-2}b + x^{n-3}b^2 + \dots + b^{n-1}) = 0$ In This, as in The preceding case, There can be no real rook That will satisfy this Equation, for The coefficients of & are some; here The n-1 worts which it contains must all be imaginary -Stree The Theorem -29- Juajuary roots do not change The identity of any algebraic Expussion -From the Sherry of Equations us know That a quadratic sopurion of The forme ax2+bro+e Equals a (x-p)(x-q), when paul q are roots of the Equaline  $a_{10}^{2} + b_{10} + c = 0$ But if The words are imaginary, They will

Hence

Also,  $(x^{n-1} + x^{n-2}p + x^{n-3}p^2 + \dots p^{n-1}) = 0$ In this, as in the preceding case, there can be no real root that will satisfy this equation, for the coefficients of x are even; hence the n-1 roots which it contains must all be imaginary. Hence the theorem. Imaginary roots do not change the identity of any algebraic expression. From the Theory of Equations we know that a quadratic expression of the form  $ax^{2} + bx + c$  equals a(x-p)(x-q), when p and q are roots of the equation

29.

 $ax^2 + bx + c = 0.$ 

But if the roots are imaginary, they will

## (20)

x-p=0, or x=p.

(21) be of the forme a + BU-1, and The Expression will heanne a[x-(a+31-i))(x-(a-31-i))-This reduces to  $a((x-\alpha)^2+s^2) = a(x^2-zx\alpha+\alpha^2+s^2)$ which is iductical write The original apraisin ulun -2a= to, and at+32= c -30 - If Thurban insginary rook in an Equation having my real coefficients, There much also be another work forming the enjugate imaging. Let us assume  $\varphi(x) = (x - \alpha_{i})(\alpha - \beta_{i})(x - \tau_{i}) - \dots - (x - \tau_{i})_{i}$ having all the coefficients real If one of The worts be at BVFI, The other must be a - BV-1, in order that The somerin he rational - The maquiary role, if any, Mush occur as  $[x-(\alpha+\beta I-i)][x-(\alpha-\beta V-i)] = x^2 - 2 \times \alpha + \alpha^2 + \beta^2$ 

(21)

be of the form  $\alpha \pm \beta \sqrt{-1}$ , and the expression will become

$$a[x-(\alpha+\beta$$

This reduces to

$$a[(x-\alpha)^2+\beta^2]=$$

which is identical with the original expression when

$$-2\alpha = \frac{b}{a}$$
, and  $\alpha^2 + \beta^2 = \frac{c}{a}$ .

30. If there be an imaginary root in an equation having only real coefficients, there must also be another root forming the conjugate imaginary. Let us assume

 $Q(x) = (x - \alpha_1)(\alpha - \alpha_1)(\alpha$ 

having all the coefficients real. If one of the roots be  $\alpha + \beta \sqrt{-1}$ , the other must be  $\alpha - \beta \sqrt{-1}$ , in order that the expression be rational. The imaginary roots, if any, must occur as

$$(\alpha + \beta \sqrt{-1})][x]$$

 $\beta\sqrt{-1}$ ][ $x - (\alpha - \beta\sqrt{-1})$ ].

 $a(x^2-2x\alpha+\alpha^2+\beta^2)$ 

$$(-\beta_1)(x-\gamma_1)....(x-u)_1$$

 $-(\alpha - \beta \sqrt{-1})] = x^{2} - 2x\alpha + \alpha^{2} + \beta^{2}$ 

(22) which is real -Have the Theorem 31\_ But as imaginary worts may be found in Q(x), all roots of it much satisfy the functional Equation  $\varphi(x) = o -$ Let This Equation Equal  $\chi^{-} b \chi + c = o_{-}$ The roots of This Equation are 12 ± 14 - e -If we assume  $x^{2} - p_{0} + c = 0 = q_{1}$ The precising roots line be beautifully illustra the as a pair of conjugate imaginaries in The poir of Equations -By assigning proper values to p and c, us are reabling to trace out The curve represented by The give function het p= 3, auce c=-3, There we will have

which is real.

Hence the Theorem.

31. equation

> Q(x) = 0.Let this equation equal  $x^2 - px + c = 0.$ The roots of this equation are

$$\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - c}$$

If we assume

 $x^2 - px + c = 0 = y$ ,

the preceding roots will be beautifully illustrated as a pair of conjugate imaginaries in the Loci of Equations. By assigning proper values to p and c, we are enabled to trace out the curve represented by the given function. Let p = 3, and c = -3, then we will have

## (22)

## But as imaginary roots may be found in Q(x), all roots of it must satisfy the functional

(23) two real and Equal, or two real and unique rooto, and me can change into The other by varing the values of p and c. If 4 4 c, the roots at mer become imaginary and form a pair of anijngatio. But The locus of the cure does not much The acio of abocisoas as long as The roto are scal -Where The curve passes below The avis of shseissas, The woods representing The points of The curr below The abis become inaginary -32 . It is shown in growery that if a straight live intersect any curo, The mucher of intersections is indicative by The degree of The Equation representing The carreof The straight live rearter so as to leave a less number of points of intersection with The curre, it is always forma that hos adusedins first the To-getter, companding

(23)

two real and equal, or two real and unequal roots, and one can change into the other by varying the values of p and c. If  $\frac{p^2}{q} < c$ , the roots at once become imaginary and form a pair of conjugates. But the locus of the curve does not meet the axis of abscissas as long as the roots are real.

When the curve passes below the axis of abscissas, the roots representing the points of the curve below the axis become imaginary.

32. It is shown in geometry that if a straight line intersect any curve, the number of intersections is indicated by the degree of the equation representing the curve. If the straight line revolve so as to leave the curve, it is always found that two

a less number of points of intersection with intersections first run to-gether, compounding

(24) to a change of two unequal to two Equal woto, and these intersections there disappen, stenoing that the square roots are constant toto a pair of conjugate imaginaries\_

(24)

to a change of two unequal to two equal roots, and these intersections then disappear, showing that the equal roots are converted into a pair of conjugate imaginaries.

TRIGONOMETRICAL IMAGINARIES. 83. DE Moirris Thuma: If n be any whole number, Them (cor, x + lin, x V-7) = cor. = (2227+x) + sin, = (2217+x) V-1 -Muttiply cos. x + lin. x V-1 by cos. B + sice. BV-1 -The product is Cor, a cor, B- sin, a sin, B+ (sin, a cor, B+ cor, a sin, B) V-1 = Cor. (a+B) + sice. (a+B)V-1\_ Muttiply The lash Expussion by cos. r + sin. r V-1 -The product is cor. (a+B) cor. r - Sin. (a+B) sin. r + (sin. (a+B) cor. r + cor. (a+B) sin. r ) V-1 = coo. (a+3+r) + sin. (a+3+1) 1-1 -Continue this operation m-1 times, Then make x = B = r = ve, and The augular functions become, cos. ma, and sin. ma - Buch we have only Expressed The side of The Equation involoring factoral angles; The other side is

(25)

#### TRIGONOMETRICAL IMAGINARIES.

De Moivre's Theorem: If n be any whole 33. number, then

$$(\cos. \alpha \pm \sin. \alpha \sqrt{-1})^{\frac{p}{q}} = \cos \alpha$$

Multiply

 $\cos \alpha \pm \sin \alpha \sqrt{-1}$  by  $\cos \beta \pm \sin \beta \sqrt{-1}$ . The product is  $\cos \alpha \cos \beta - \sin \alpha \sin \beta \pm (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \sqrt{-1} =$ 

 $cos.(\alpha + \beta) \pm sin.(\alpha + \beta)\sqrt{-1}.$ 

Multiply the last expression by

$$cos. \gamma \pm sin. \gamma \sqrt{-1}$$

The product is

cos.  $(\alpha + \beta) \cos \gamma - \sin (\alpha + \beta) \sin \gamma \pm [\sin (\alpha + \beta) \cos \gamma + \cos (\alpha + \beta) \sin \gamma] \sqrt{-1} =$ cos.  $(\alpha + \beta + \gamma)$ .  $(\alpha + \beta + \gamma)\sqrt{-1}$ . Continue this operation m-1 times, then make  $\alpha = \beta = \gamma = \Pi$ , and the angular functions become,

 $cos. m\alpha$ , and  $sin. m\alpha$ . But we have only expressed the side of the equation involving factoral angles; the other side is

- $s.\frac{p}{a}(2n\Pi + \alpha) \pm sin.\frac{p}{a}(2n\Pi + \alpha)\sqrt{-1}.$

(26) Exponential -The appression results in The following Equation : con mat sin. mat-i = (const sin al-i) - (9). But this analycis only shows Equation (9) to be True where we is a position integer -Let us now suppose me negative, and squatto-E. There (cos. x ± lin. x V-I) = (cos. x ± lin. x V-I) = = - Cur, Ex ± sin. Ex U=1 - (?) Multiply both numerous and duminator by CUD. EX 7 Sin. EX 1-7 -The result is  $\frac{c_{UD, E} \propto \mp liu, E \propto V-1}{c_{UD, E} \propto \mp liu, E \propto V-1} = c_{UD, E} \propto \mp liu, E \propto V-1 =$ Cor. (- Ex) ± sin. (- Ex) V=1 = Cor. Ext sin. ExV-1\_?) Thus Equation (9) is Establistuce when m is negalin But if us Estrach The methe rook of Each

(26)

exponential.

The expression results in the following equation:  $\cos m\alpha \pm \sin m\alpha \sqrt{-1} = (\cos \alpha \pm \sin \alpha \sqrt{-1})^m$ . (9). But this analysis only shows equation (9) to be true when m is a positive integer. Let us now suppose m negative, and equal to -e

Then

$$\left(\cos \alpha \pm \sin \alpha \sqrt{-1}\right)^{m} = \frac{1}{\left(\cos \alpha \pm \sin \alpha \sqrt{-1}\right)^{e}} = \frac{1}{\left(\cos \alpha \pm \sin \alpha \sqrt{-1}\right)^{e}}$$

Multiply both numerator and denominator by  $\cos e\alpha \pm \sin e\alpha \sqrt{-1}$ . The result is

 $\frac{\cos e\alpha \pm \sin e\alpha \sqrt{-1}}{\cos^2 e\alpha \pm \sin^2 e\alpha} = \cos e\alpha \pm \sin e\alpha \sqrt{-1} =$  $= \cos(-e\alpha) \pm \sin(-e\alpha)\sqrt{-1} = \cos(-e\alpha) \pm \sin(-e\alpha)\sqrt{-1}$ .

Thus equation (9) is established when m is negative.

But if we extract the m<sup>th</sup> root of each

(271 Munter of Equation (9), The result is (cos. Mat sin. may-1) = cos. a + sin. a V-1- (10) And if us suppose me Equals my putin, Either position or nyation; say, &, There (cos. a + sin. xV-i) = (cos. x + sin. xV-i) = (cos. pa + sin. pari) = In has here show in Equation (10) That This is one of the works -Huce The Equation becaus guillad and can be written cio. 1/2 x + sin. 1/2 x V-1 -Thus us have completely established Equation (9) whatever be the value of no. It yet remains for us to invitigate The value and querality of this Equation where & assumes different values. to long as a remains less Than "2, The signvalues of cor. & and sin. & Main unchanged, and also for any integral multiplier of 271. Hun, to make Equation (9) complete, in much

(27)

number of equation (9), the result is  $(\cos. m\alpha \pm \sin. m\alpha \sqrt{-1})^{\frac{1}{m}} = \cos. \alpha \pm \sin. \alpha \sqrt{-1}.$ (10)And if we suppose m equals any fraction, either positive or negative; say,  $\frac{p}{q-1}$ , then  $(\cos \alpha \pm \sin \alpha \sqrt{-1})^m = (\cos \alpha \pm \sin \alpha \sqrt{-1})^{\frac{p}{q}} = (\cos p\alpha \pm \sin p\alpha \gamma - 1)^{\frac{1}{q}}.$ It has been shown in equation (10) that this

is one of the roots.

Hence the equation becomes general and can be written

cos.  $\frac{p}{a}\alpha \pm \sin \frac{p}{a}\alpha \sqrt{-1}$ .

Thus we have completely established equation (9) whatever be the value of m. It yet remains for us to integrate the value and generality of this equation when  $\alpha$  assumes different values.

So long as  $\alpha$  remains less than  $\frac{\pi}{2}$ , the sign values of  $cos. \alpha$  and  $sin. \alpha$  remain unchanged, and also for any integral multiplier of  $2\Pi$ . Hence, to make equation (9) complete, we must

(28) in place of The varying angle &, insert 271 TT + and The Equation becomes (cos. a + sin. a V-T) = cos. // (2n TI + a) + sin. // (2n TI + a) V-T 7 which emplitely Establishes DE Moioris tomula -34 - The preciding method of demonstration is only one among many, and it is probably mon complese than some That will be shown burchfur. The method by Vector Equations is The simplest of any yet discound -35- But The Experient 1/2 shows There are go different values to The Expression cus. // (22171+ x) = sice. // (27171+x) 1-1 -These worts are sitter reas or of the forme of The general imaginary, at BV-1 -36 - This Theore care be usefully suployed in Estracting worts of imaginary sopuesions of the forme of at sti-i-Assure

#### (28)

in place of the varying angle  $\alpha$ , insert  $2n\Pi + \alpha$ , and the equation becomes which completely establishes. De Moivre's Formula.

- The preceding method of demonstration 34. is only one among many, and it is probably more complete than some that will be shown hereafter. The method by Vector Equations is the simplest of any yet discovered. But the exponent  $\frac{p}{a}$  shows there are q 35. different values to the expression These roots are either real or of the form
- of the general imaginary,  $\alpha \pm \beta \sqrt{-1}$ . This theorem can be usefully employed 36. in extracting roots of imaginary expressions
  - of the form of  $\alpha \pm \beta \sqrt{-1}$ . Assume

 $\left(\cos \alpha \pm \sin \alpha \sqrt{-1}\right)^{\frac{p}{q}} = \cos \frac{p}{\alpha} (2n\Pi + \alpha) \pm \sin \frac{p}{\alpha} (2n\Pi + \alpha) \sqrt{-1}$ . cos.  $\frac{p}{a}(2n\Pi + \alpha) \pm \sin \frac{p}{a}(2n\Pi + \alpha\sqrt{-1})$ .

(29) x=pcor, o, and B= psin, o-There p= x+ 32; Tau, 0 = x' and  $\alpha \pm \beta V = i = \beta(\alpha \sigma, \sigma \pm \beta i \alpha, \sigma V = i)_{-1}$ Murifor (a + 3/-1) = " = " (co. + 1in. 0/-1) = Different roots can be Estraction from this Expression by assigning componding values to m. But by Delloin's Auma, the of- The work is cos. The + sin . The VI; The other works are determined from co. 1/2 (2nTI+0) + ein. 1/2 (2nTI+0)1-1, by assigning proper values to p.g., and n. The values of (at stri) the ablaince by This mettera aque mite The values give in Chel. 26. 37. Let it he require to Express The complete values of atpri, and a-pri, when They are derive respecting from

Then

 $\rho^2 = \alpha^2 + \beta^2$  $\alpha \pm \beta \sqrt{-1} = \rho$ 

Therefore

 $(\alpha \pm \beta \sqrt{-1})^{\frac{1}{2m}} = \rho^{\frac{1}{2m}} (\cos \theta \pm \sin \theta \sqrt{-1})^{\frac{1}{2m}}.$ Different roots can be extracted from this expression by assigning compounding values to m. But by De Moivre's Theorem, one of the roots is  $cos. \frac{\theta}{2m} \pm sin. \frac{\theta}{2m} \sqrt{-1}$ ; the other roots are determined from  $\cos \frac{p}{a}(2n\Pi +$ 

by assigning proper v

The values of  $(\alpha \pm \beta \gamma)$ method agree with the

37. Let it be required to express the complete values of

> $\alpha + \beta \sqrt{-1}$ , an where they are derive

### (29)

 $\alpha = \rho \cos \theta$ , and  $\beta = \rho \sin \theta$ .

<sup>2</sup>; 
$$tan. \theta = \frac{\beta}{\alpha}$$
; and  
 $p(cos. \theta \pm sin. \theta \sqrt{-1}).$ 

$$\theta \pm sin. \frac{p}{q}(2n\Pi + \theta)\sqrt{-1},$$
  
values to p, q, and n.  
 $\sqrt{-1})^{\frac{1}{2m}}$  obtained by this  
e values given in Art. 26.

d 
$$\alpha - \beta \sqrt{-1}$$
,  
ed respectively from

(301 (x+3/=1) and (x-3/=1) " Assure  $r = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \quad aud \quad \delta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}},$ which bear The same relation as ers. and sin. of an augle. Then  $r + \delta V - i = \frac{\alpha}{V(\alpha^2 + \beta^2)} + \frac{\beta V - 1}{V(\alpha^2 + \beta^2)} + \frac{\alpha V - 1}{V(\alpha^2 + \beta^2)}$  $r - SU - i = \frac{\alpha}{V(\alpha^2 + \beta^2)} - \frac{\beta r - i}{V(\alpha^2 + \beta^2)} - \frac{\beta r - i}{V$ It follows, Thur, That x+31==1(2+32) (r+51=1), and  $\alpha - 3 \sqrt{-1} = \sqrt{\alpha^2 + \beta^2} (\tau - 5 \sqrt{-1}) -$ Hun (x+31-7)"= (V(x+32) (r+SV=1))", and (x-31=1) = (1(x+32)(r-SV=1))-And now if us replace & and I by this respection bigonometrical pullins, cor. a

(30)

$$(\alpha + \beta \sqrt{-1})^n$$
, and  $(\alpha - \beta \sqrt{-1})^n$ 

Assume

$$\gamma = \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2)}}$$
, and  $\delta = \frac{\beta}{\sqrt{(\alpha^2 + \beta^2)}}$ ,  
bear the same relation as cos. and

which b sin. of an angle.

Then

$$\gamma + \delta \sqrt{-1} = \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2)}} + \frac{\beta \sqrt{-1}}{\sqrt{(\alpha^2 + \beta^2)}}, \text{ and}$$
$$\gamma - \delta \sqrt{-1} = \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2)}} - \frac{\beta \sqrt{-1}}{\sqrt{(\alpha^2 + \beta^2)}}.$$

It follows, then, that

$$\alpha + \beta \sqrt{-1} = \sqrt{(\alpha^2 + \beta^2)} (\gamma + \delta \sqrt{-1}), \text{ and}$$
$$\alpha - \beta \sqrt{-1} = \sqrt{(\alpha^2 + \beta^2)} (\gamma - \delta \sqrt{-1}).$$

Hence

 $(\alpha + \beta \sqrt{-1})^n = [$  $(\alpha - \beta \sqrt{-1})^n = [$ 

And now if we replace  $\gamma$ respective trigonometrica

 $\beta \sqrt{-1}$ )<sup>*n*</sup>.

$$[\sqrt{(\alpha^2 + \beta^2)}(\gamma + \delta\sqrt{-1})]^n$$
, and  
 $[\sqrt{(\alpha^2 + \beta^2)}(\gamma - \delta\sqrt{-1})]^n$ .  
and  $\delta$  by their  
al functions, cos.  $\theta$ 

(31) and lin, &, and consider according to DE Moin's Skionne, un gel (x+31=1) = (x+32) = (co. no+ sin. nov-1), and (x-31-7) = (x + 32) = (cr. no-sin. no 1-7)\_ The note soal of Each Equation is respecting a+31-1 = 162+32) (cus. no + sin. no V-1) to and x-3V=1 = V(x+32) (co. no - Sin. noV=) -38. Again, let it be required to find the emplite values of (x+BV-i) + (x-BV-i) and  $(\alpha + \beta V = i)^n - (\alpha - \beta V = i)^n$ From Art. 37, uz get  $(\alpha + \beta V - i)^{m} + (\alpha - \beta V - i) =$ (x = 32) = (cos. no+ sin, nov-i) + (cos. no-sin, nov-i)) = 2 (x2+32) = co. noand  $(\alpha + \beta \sqrt{-1})^{n} - (\alpha - \beta \sqrt{-1})^{n} =$ 

(31)

and sin.  $\theta$ , and consider according to De Moivre's Theorem, we get

$$(\alpha + \beta \sqrt{-1})^n = (\alpha^2 + \beta^2)^{\frac{n}{2}}$$

$$(\alpha - \beta \sqrt{-1})^n = (\alpha^2 + \beta^2)^{\frac{n}{2}}$$

 $\overline{2}(\cos n\theta + \sin n\theta \sqrt{-1}),$ and  $(\cos n\theta + \sin n\theta \sqrt{-1}).$ The nth root of each equation is respectively  $\alpha + \beta \sqrt{-1} = \sqrt{(\alpha^2 + \beta^2)} (\cos n\theta + \sin n\theta \sqrt{-1})^{1/n},$ and  $\alpha - \beta \sqrt{-1} = \sqrt{(\alpha^2 + \beta^2)} (\cos n\theta - \sin n\theta \sqrt{-1})^{1/n} .$ 

38. Again, let it be required to find the complete values of

$$(\alpha + \beta\sqrt{-1})^{n} + (\alpha - \beta\sqrt{-1})$$
$$(\alpha + \beta\sqrt{-1})^{n} - (\alpha - \beta\sqrt{-1})$$

From Art. 37, we get

$$(\alpha + \beta \sqrt{-1})^n + (\alpha - \beta \sqrt{-1})^n =$$
$$(\alpha^2 + \beta^2)^{\frac{n}{2}} (\alpha^2 + \beta^2)^{\frac{n}{2}} (\alpha^2$$

and

$$(\alpha + \beta \sqrt{-1})^n - (\alpha - \beta \sqrt{-1})$$

<sup>n</sup>, and

 $\left(\left(\cos n\theta + \sin n\theta \sqrt{-1}\right) + \left(\cos n\theta - \sin n\theta \sqrt{-1}\right)\right) =$  $2(\alpha^2 + \beta^2)^{\frac{n}{2}} cos. n\theta$ ,

n =

(321 (x2+32)= ((co, no+sin. nov-i)- (co, no-sin. nov-i)) = 2/a2+32) = sin. nov-1\_ 39- 2f un make a = cos. o t sin. o Fi, in De Moin's Formula, un can derin some important soponentine functions -There a = cor. + sin. + V-1, a= cur, o- sin, o V-1; and a = cor. no + Sin, no V-1, 1 - ut cis. no - sin no V-1 -Ame These for Equations, us deduce Co. 0 = ato + a o sin = at - a - i and ers. no= at no + a no sin, no= at no - a - no -Hun sin 20+ cos 20 = (at - a - a) 2 (a + a - a) = 4 a unce Kum high mutricor relation

(32)

39. If we make

> $a^{\pm \theta} = \cos \theta \pm \sin \theta \sqrt{-1}$ , in De Moivre's Formula, we can derive some important exponential functions.

Then

$$a^{+\theta} = \cos \theta + \sin \theta \sqrt{-1} ,$$
  

$$a^{-\theta} = \cos \theta - \sin \theta \sqrt{-1}; a$$
  

$$a^{+n\theta} = \cos \theta + \sin \theta \sqrt{-1}; a$$
  

$$a^{-n\theta} = \cos \theta + \sin \theta \sqrt{-1}; a$$

From these four equations, we deduce

$$cos. \theta = \frac{a^{+\theta} + a^{-\theta}}{2}$$

$$sin. \theta = \frac{a^{+\theta} + a^{-\theta}}{2\sqrt{-1}}; \text{ and}$$

$$cos. n\theta = \frac{a^{+n\theta} + a^{-n\theta}}{2},$$

$$sin. n\theta = \frac{a^{+n\theta} + a^{-n\theta}}{2\sqrt{-1}}.$$

Hence

 $\sin^2 \theta + \cos^2 \theta = (\frac{a^{+\theta} + a^{-\theta}}{2\sqrt{-1}})^2 + (\frac{a^{+\theta} + a^{-\theta}}{2})^2 = 1$ , a well known trigonometrical relation.

 $(a^{2} + \beta^{2})^{\frac{n}{2}}(\cos n\theta + \sin n\theta\sqrt{-1}) - (\cos n\theta - \sin n\theta\sqrt{-1})) =$  $2(\alpha^2 + \beta^2)^{\frac{n}{2}} \sin n\theta \sqrt{-1}$ .

and 1, 1

(33) 40- But us leave in algebra, That-and a= = 1- 0 (log, a) + = (log, a) = = (log, a) + = (log, a) + = re. Musifor and= Eto+E-o = 1+ o (log.a) + ot (log.a) + re., and lin. 0 = 2+0- 2-0 = 1 { (o(log.a) + 0 (log.a) + 0 (log.a) + 15 (log.a) And yain, if log. 2 = x 1-1, These higouruitrical functions become CO. 0 = 1 - 12 + atot - x 62 + FE -1 and  $\beta_{iu}, \phi = \alpha \phi - \frac{\alpha^3 a^3}{13} + \frac{\alpha^5 a^6}{15} - \frac{\alpha^3 a^2}{17} + \sigma z_{-}$ 2 x = 1, Then log. a = 1=1 -By passing to exponentives a= E+V=; a = E+0V=; and a = E = 0-1-Hauch  $4iu, d = \frac{a^{+d} - a^{-t}}{2\sqrt{-1}} = \frac{z^{+d\sqrt{-1}} - z^{-d\sqrt{-1}}}{2\sqrt{-1}}$ 

(33)

But we learn in Algebra, that 40.

$$a^{+\theta} = 1 + \theta(\log a) + \frac{\theta^2}{2}(\log a)^2 + \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^4}{4}(\log a)^4 \&c,$$

$$a^{-\theta} = 1 - \theta(\log a) + \frac{\theta^2}{2}(\log a)^2 - \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^4}{4}(\log a)^4 - \&c.$$
ore --
$$\cos \theta = \frac{e^{+\theta} - e^{-\theta}}{2} = 1 + \frac{\theta^2}{2}(\log a)^2 + \frac{\theta^3}{3}(\log a)^4 + \frac{\theta^4}{4}(\log a)^4 + \&c.,$$

$$\sin \theta = \frac{e^{+\theta} - e^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}[\theta(\log a) + \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^5}{5}(\log a)^5\&c.].$$
pain, if  $\log a = \alpha\sqrt{-1}$ , these
metrical functions become

And

$$1 + \theta(\log a) + \frac{\theta^{2}}{2}(\log a)^{2} + \frac{\theta^{3}}{3}(\log a)^{3} + \frac{\theta^{4}}{4}(\log a)^{4} \&c,$$

$$a^{-\theta} = 1 - \theta(\log a) + \frac{\theta^{2}}{2}(\log a)^{2} - \frac{\theta^{3}}{3}(\log a)^{3} + \frac{\theta^{4}}{4}(\log a)^{4} - \&c.$$

$$= \frac{e^{+\theta} - e^{-\theta}}{2} = 1 + \frac{\theta^{2}}{2}(\log a)^{2} + \frac{\theta^{3}}{3}(\log a)^{4} + \frac{\theta^{4}}{4}(\log a)^{4} + \&c.,$$

$$= \frac{e^{+\theta} - e^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}[\theta(\log a) + \frac{\theta^{3}}{3}(\log a)^{3} + \frac{\theta^{5}}{5}(\log a)^{5}\&c.].$$

$$\log a = \alpha\sqrt{-1}, \text{ these}$$
If functions become

There

$$a^{+\theta} = 1 + \theta(\log . a) + \frac{\theta^{2}}{2}(\log . a)^{2} + \frac{\theta^{3}}{3}(\log . a)^{3} + \frac{\theta^{4}}{4}(\log . a)^{4} \&c,$$

$$a^{-\theta} = 1 - \theta(\log . a) + \frac{\theta^{2}}{2}(\log . a)^{2} - \frac{\theta^{3}}{3}(\log . a)^{3} + \frac{\theta^{4}}{4}(\log . a)^{4} - \&c.$$
Fore --
$$\cos . \theta = \frac{e^{+\theta} - e^{-\theta}}{2} = 1 + \frac{\theta^{2}}{2}(\log . a)^{2} + \frac{\theta^{3}}{3}(\log . a)^{4} + \frac{\theta^{4}}{4}(\log . a)^{4} + \&c.,$$

$$\sin . \theta = \frac{e^{+\theta} - e^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}[\theta(\log . a) + \frac{\theta^{3}}{3}(\log . a)^{3} + \frac{\theta^{5}}{5}(\log . a)^{5}\&c.].$$
gain, if  $\log . a = a\sqrt{-1}$ , these potential functions become

and

$$a^{+\theta} = 1 + \theta(\log a) + \frac{\theta^2}{2}(\log a)^2 + \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^4}{4}(\log a)^4 \&c,$$
  

$$a^{-\theta} = 1 - \theta(\log a) + \frac{\theta^2}{2}(\log a)^2 - \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^4}{4}(\log a)^4 - \&c.$$
  
efore --  

$$\cos \theta = \frac{e^{+\theta} - e^{-\theta}}{2} = 1 + \frac{\theta^2}{2}(\log a)^2 + \frac{\theta^3}{3}(\log a)^4 + \frac{\theta^4}{4}(\log a)^4 + \&c.,$$
  

$$\sin \theta = \frac{e^{+\theta} - e^{-\theta}}{2\sqrt{-1}} = \frac{1}{\sqrt{-1}}[\theta(\log a) + \frac{\theta^3}{3}(\log a)^3 + \frac{\theta^5}{5}(\log a)^5\&c.].$$
  
again, if  $\log a = \alpha\sqrt{-1}$ , these nometrical functions become

And a trigor

$$\cos \theta = 1 - \frac{\alpha^2 \theta^2}{2} + \frac{\alpha^4 \theta^4}{4} - \frac{$$

and

 $\sin \theta = \alpha \theta - \frac{\alpha^3 \theta^3}{2} + \frac{\alpha^5 \theta^5}{5} - \frac{\alpha}{5}$ If  $\alpha = 1$ , then  $\log a = \sqrt{-1}$ . By passing to exponentials

$$a = e^{+\sqrt{-1}}; a^{+\theta} = e^{+\theta\sqrt{-1}};$$

Hence --

$$\sin \theta = \frac{a^{\theta} - a^{-\theta}}{2\sqrt{-1}} = \frac{e^{\theta} - 1}{2\sqrt{-1}}$$

$$\frac{\alpha^{7}\theta^{7}}{7} \& c.$$

and  $a^{-\theta} = e^{-\theta}\sqrt{-1}$ .

 $-\theta\sqrt{-1}$ 

(34) and  $co. \phi = \frac{a^{+\phi} + a^{-\phi}}{2} = \frac{z^{+\phi} \sqrt{-i} + z^{-\phi} \sqrt{-i}}{2}$ In han Then, determinate, and indeterminate Exponential Expressions for sin , and co. o-41- But as as hun show That at = 2+017, us may write The Exponential Equation  $\varepsilon^{+ \circ V = i} = \cos \circ \circ + \epsilon i \ldots \circ v = i - (11)_{-}$ het us make o = 2 mit, me being any some integral multer; The Equation becomes  $\mathcal{E}^{+\frac{2\pi\pi\pi}{\alpha}} = \cos \cdot \frac{2\pi\pi\pi}{\alpha} + \sin \cdot \frac{2\pi\pi\pi}{\alpha} \sqrt{-1} - 1 - \frac{1}{\alpha}$ By a judicious manipulation of this formula uz can secure a gueral solution for The Equalion x = = 1-It will be found that The mulu ofroots thus obtained will occur in regaorder, Each order containing & worts, as indicalue in 2°, 2° 2°, 2° 40 , 2° 40 , ---- 2° 2° -

and

 $\cos \theta = \frac{a^{+\theta} + a^{-}}{2}$ 

we can then, determinate, and indeterminate exponential expressions for sin.  $\theta$ , and cos.  $\theta$ .

41. we may write the exponential equation  $e^{+\theta\sqrt{-1}} = \cos.\theta + \sin.\theta\sqrt{-1}.$ Let us make  $\theta = \frac{2m\Pi}{\alpha}$ , m being any even integral number; the equation becomes

 $x^n = \pm 1.$ It will be found that the number of roots thus obtained will occur in regaorder, each order containing n roots, as indicated in  $e^{+\theta}$ ,  $e^{+\frac{2\pi}{\alpha}\sqrt{-1}}$ ,  $e^{+\frac{4\pi}{\alpha}\sqrt{-1}}$ , ...,  $e^{+\frac{2(n-1)}{\alpha}\sqrt{-1}}$ .

(34)

$$\frac{-\theta}{-\theta} = \frac{e^{+\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2}.$$

But as we have shown that  $\alpha^{+\theta} = e^{+\theta\sqrt{-1}}$ , (11).  $e^{+\frac{2m\Pi}{\alpha}\sqrt{-1}} = \cos \cdot \frac{2m\Pi}{\alpha} + \sin \cdot \frac{2m\Pi}{\alpha}\sqrt{-1}.$ By a judicious manipulation of this formula We can secure a general solution for the equation

(35) 42 - If sin. A be divided by as. O, The quotient is  $Tau, \theta = \frac{\varepsilon^{+ \alpha V - 1} - \varepsilon^{- \alpha V - 1}}{V - 1 (\varepsilon^{+ \alpha V - 1} + \varepsilon^{- \alpha V - 1})}$  $\overline{tau}. \theta V \overline{-1} = \frac{\varepsilon^{+\theta V \overline{-1}} - \varepsilon^{-\theta V \overline{-1}}}{\varepsilon^{+\theta V \overline{-1}} + \varepsilon^{-\theta V \overline{-1}}}$ Munfin  $\frac{1+tau, \theta Fi}{1-tau, \theta F} = \frac{\varepsilon + \theta V}{\varepsilon - \theta F} = \varepsilon + 2\theta V - 1 - 1$ Take logarithus of both multers: The result is 201-1 = log. (1+ Tau. 01-1) - log. (1- Tau. 01-1) = 2 1-1 (tau. 0 - 1/3 tau. 0 + 5 tau. 0 - 02.) -Huner 0 = tau. 0 - /3 tau. 0 + 1/5 tau. 0 - VE., which gives a in times of pours of taw. a-This is known as Lugory's Series-The may be usefully suployed in computing The munerical value of TT, by making o = " -But This series is guerally quite unsatisfactory because it does not limit the Estent to which it may be relieve on as arithmetically

42. If sin.  $\theta$  be divided by cos.  $\theta$  the quotient is  $tan. \theta = \frac{e^{\theta \sqrt{-1}} - e^{-\theta \sqrt{-1}}}{\sqrt{-1}(e^{\theta \sqrt{-1}} + e^{-\theta \sqrt{-1}})}, \text{ or }$  $\tan \theta \sqrt{-1} = \frac{e^{+\theta \sqrt{-1}} - e^{-\theta \sqrt{-1}}}{e^{+\theta \sqrt{-1}} + e^{-\theta \sqrt{-1}}}.$ Therefore

Take logarithms of both numbers: the result is  $2\theta\sqrt{-1} = \log\left(1 + \tan\theta\sqrt{-1}\right) - \log\left(1 - \tan\theta\sqrt{-1}\right) =$  $2\sqrt{-1}(\tan \theta - 1/3\tan \theta + 1/5\tan^5 \theta \& c.).$ 

Hence

 $\theta = tan. \theta - 1/3tan. ^{3}\theta + 1/5tan. ^{5}\theta \&c.,$ which gives  $\theta$  in terms of powers of  $tan. \theta$ . This is known as Gregory's Series. It may be usefully employed in computing the numerical value of  $\Pi$ , by making  $\theta = \frac{\pi}{4}$ . But this series is generally quite unsatisfactory because it does not limit the extent to which it may be relied on as arithmetically

### (35)

 $\frac{1+tan.\theta\sqrt{-1}}{1-tan\,\theta\sqrt{-1}} = \frac{e^{+\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}}} = e^{+2\theta\sqrt{-1}}.$ 

(37) Therefore  $\frac{\varepsilon^{+2\beta}}{\varepsilon} = \frac{1 - n\varepsilon^{-\alpha}}{1 - n\varepsilon^{+\alpha}}$ Take logarithus of both sides of The last Equation, 43. and The Mouth is 2/317= Log. (1- n = ~1-7) - Log. (r- n = + al7) = n( =+ a 1-1) + n2 (=+ 2a 1-7) + n3 (=+ 3a 1-1) + r. Have B= n sin a + n sin, za + n sin, 3a + vE., The disina serios -This series is give in circular functions-44 - By mans of This series, us are often able to solve certain Triangles-La Trigonoutry UL have lin p= to sin t = to sin (B+C)-Have by the formula, B= 5 in. C+ 62 sin. 2 C+ 63 Sin. 3 C+ 02-If he he less Than a The series is conaugual; and if a be a succe prodice

true, and a large number of terms would have to be taken to secure a close approximation.

If we have given  $sin. \beta = n sin. (\beta + \alpha),$  $\beta$  can be expressed in terms of n and sinefunctions of  $\alpha$ , by means of exponentials. In Art. 40, it was shown that

and by Art. 39, this equals  $2sin.\beta\sqrt{-1}$ . and then pass to exponentials according to Art. 39; the result is

$$e^{+\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}}$$

Multiply the last equation by  $e^{+\beta\sqrt{-1}}$ ; the result is

$$e^{+2\beta\sqrt{-1}} - 1 =$$

or

$$e^{+2\beta\sqrt{-1}}(1-ne)$$

### (36)

 $a^{+\beta} - a^{-\beta} = e^{+\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}}$ Replace sin.  $\beta$  by its equivalent, n sin. ( $\beta + \alpha$ ),

 $\overline{-1} = n(e^{+(\beta+\alpha)\sqrt{-1}} - e^{-(\beta+\alpha)\sqrt{-1}}).$ 

 $n(e^{+(2\beta+\alpha)\sqrt{-1}}-e^{-\alpha\sqrt{-1}}),$ 

 $e^{+\alpha\sqrt{-1}}) = 1 - ne^{-\alpha\sqrt{-1}}.$ 

(36) The and a large mumber of Terms unice have to be taken to secure a close appropinaline 43- If us have give lin. B= 1 sin. (B+a), Beau les sepressue in terms of ne and since functions of x, by means of Exponentials\_ See art. 40, it was shown that  $a^{+\beta} - a^{-\beta} = \epsilon^{+\beta/\beta} - \epsilon^{-\beta/\beta}$ and by art. 39, This Equals 2 lin. BF-1\_ Replace sin. B by its Equivalent, a sin (B+a), and The pass to appricitions according to act, 39; The result is  $\mathcal{E}^{+(3V-1)} - \mathcal{E}^{-(3V-1)} = \mathcal{I} \mathcal{L} \left( \mathcal{E}^{+((3+\alpha)V-1)} - \mathcal{E}^{-((3+\alpha)V-1)} \right)$ Multiply the lash Equation by Et SV-1 ; the result is  $\Sigma^{+2}\beta F_{i} - I = M(\Sigma^{+}(\Sigma^{+}\beta + \alpha)) - \Sigma^{-}\alpha \sqrt{-1}),$ or E+2/31-1(1-ME+X1-7)= 1-16E-RV-7

(37)

Therefore

 $e^{+2\beta\sqrt{-1}} = \frac{1-ne^{-\alpha\sqrt{-1}}}{1-ne^{+\alpha\sqrt{-1}}}.$ 

Take logarithms of both sides of the last equation, and the result is

$$2\beta\sqrt{-1} = \log\left(1 - ne^{-\alpha\sqrt{-1}}\right) - \log\left(1 - ne^{+\alpha\sqrt{-1}}\right) = n\left(e^{+\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right) + \frac{n^2}{2}\left(e^{+2\alpha\sqrt{-1}} - e^{-2\alpha\sqrt{-1}}\right) + \frac{n^3}{3}\left(e^{+3\alpha\sqrt{-1}} - e^{-3\alpha\sqrt{-1}}\right) + \&c.$$

Hence

 $\beta = n \sin \alpha + \frac{n^2}{2} \sin 2\alpha + \frac{n^3}{3} \sin 3\alpha + \&c..$ the desired series.

This series is given in circular functions.

By means of this series, we are often 44. able to solve certain triangles. In trigonometry we have

$$sin. \beta = \frac{b}{a}sin. \theta = \frac{b}{a}sin. (B + C)$$

Hence by the formula,

$$B = \frac{b}{a}sin. C + \frac{b^{2}}{2a^{2}}sin. 2C + \frac{b^{3}}{3a^{3}}$$

If 1 be less than a the series is convergent; and if  $\frac{b}{a}$  be a small fraction,

sin. 3C + &c.

(38) a fuo termo of The sories will suffice for a close degree of approvination - But as The Series gives The circular Measure of B, m Much find The Malin between circula puction, and cuterine or serigissinae Let x = munder of agrics in any give functions aught, and & The circular measure of The Dame augle - But as There are 180 Seviger unas depues in two right augus, 780 duotes The ratio of the given angle to two sight augles - Auch since To denotes The circular Measure of two right augus, of sopresses The ratio of the give augle to two right auglio -Auce  $\frac{\chi}{180} = \frac{\Phi}{11}$  or a= xTT-

(38)

a few terms of the series will suffice for a close degree of approximation. But as the series given the circular measure of B, we must find the relation between circular function, and centesimal or sexagesimal functions.

Let x = numbers of degrees in any given angle, and  $\theta$  the circular measure of the same angle. But as there are 180 sexigesimal degrees in two right angles,  $\frac{x}{180}$  denotes the ratio of the given angle to two right angles. And since  $\Pi$  denotes the circular measure of two right angles,  $\frac{\theta}{\Pi}$  expresses the ratio of the given angle to two right angles.

Hence

$$\frac{x}{180} = \frac{\theta}{\Pi}$$
, or  
 $\theta = \frac{x}{180}$ 

<u>П</u> 80

(39) 45- If un have give Tau. B= n tau. x, B can be Expressed in pours of a and sine - functions of a, by means of reponentials. The give Trigmmetrical relation contrined with At. 42 gins  $\frac{\varepsilon^{+\beta V_{-1}} - \varepsilon^{-\beta V_{-1}}}{\varepsilon^{+\beta V_{-1}} + \varepsilon^{-\beta V_{-1}}} = m \left( \frac{\varepsilon^{+\alpha V_{-1}} - \varepsilon^{-\alpha V_{-1}}}{\varepsilon^{+\alpha V_{-1}} + \varepsilon^{-\alpha V_{-1}}} \right) -$ Multiply munerator and demoninator of The left hand multer, by 2+BFF; and muniter and decominator of right hand member, by Et XVF; The Wellt is  $\frac{\mathcal{E}^{+2\beta V-i}-1}{s+2\beta V-i} = \mathcal{N}\left(\frac{\mathcal{E}^{+2\alpha V-i}-1}{s+2\alpha V-i}\right) -$ Shirifm  $\underline{z}^{+2,3l-1} = \frac{(l+n)\underline{z}^{+2\alpha l-1}}{(l-n)\underline{z}^{+2\alpha l-1} + l+n} =$ E+2a V-1 (1+ 1+22 E-2a/-7 1+1-22 E+2a V-7) -

(39)

If we have given  $tan. \beta = n tan. \alpha$ , with Art. 42 gives  $e^{+\beta\sqrt{-1}}-e^{-\beta}$  $\rho + \beta \sqrt{-1} + \rho - \beta$ Multiply numerato left hand number, and denominator of  $e^{+\alpha\sqrt{-1}}$ ; the result

45.

 $\frac{e^{+2\beta\sqrt{-1}}-1}{e^{+2\beta\sqrt{-1}}+1} =$ Therefore

 $e^{+2\beta\sqrt{-1}} =$ 

e +2

# $\beta$ can be expressed in powers of n and sine-functions of $\alpha$ , by means of exponentials. The given trigonometrical relation combined

$$\frac{e^{\beta\sqrt{-1}}}{e^{\beta\sqrt{-1}}} = n(\frac{e^{+\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}}}{e^{+\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}}})$$
  
r and denominator of the  
by  $e^{+\beta\sqrt{-1}}$ ; and numerator  
of right hand number, by  
is

$$= n(\frac{e^{+2\alpha\sqrt{-1}}-1}{e^{+2\alpha\sqrt{-1}}+1}).$$

$$\frac{(1+n)e^{+2\alpha\sqrt{-1}}+1-n}{(1-n)e^{+2\alpha\sqrt{-1}}+1+n} = \frac{(1-n)e^{+2\alpha\sqrt{-1}}+1+n}{(1-n)e^{-2\alpha\sqrt{-1}}} = \frac{(1+1)e^{-2\alpha\sqrt{-1}}}{(1+1)e^{-2\alpha\sqrt{-1}}} = \frac{(1+1)e^{-2\alpha\sqrt{-$$

(401 Take logarithus of both members; The ment is 2/3V-1 = 2 a V-1 + log. (1+ 1-20 E- Lav-1) - log. (1+ 1-20 E+ 2aV-1) = 2 x1-1 - (1-22) (E+ 2 x1-1 - E- 2 x1-1) - 1/2 (1-22) (E+4a 1-1 - E-4x1-1) - ... or. Huce B= a - (1-22) Sin. 20 + 1/2 (1-22) Sin. 4x - /3 (1-22) Sin. 6x + 52. The series saught-46- Lu Equation (11), let us replace & by 3+1-Iku  $\varepsilon^{+(0+r)/-i} = \varepsilon^{+(3/-i)} \times \varepsilon^{+r/-i} = co.(0+r) + sin.(0+r)/-i =$ (co. B+ sin, BV-1)(co. Y+ sin. YV-1) = (as, B cos, r- sin, B sin, r) + (sin, B cos, r + cos, B sin, r) V-1- (12). But by principles of Muditimine Confficients Co. (3+r) = co. B co.r+ sin B sin ry and Riv. (B+r) = sin, Ber. r + co. B sur. r. If us replace & by B-V, in Equation (11), The usuit is

(40)

Take logarithm of both numbers; the result is  $2\beta\sqrt{-1} = 2\alpha\sqrt{-1} + \log\left(1 + \frac{1-n}{1+n}\right)$  $2\alpha\sqrt{-1}-\left(\frac{1-n}{1+m}\right)\left(e^{+2\alpha\sqrt{-1}}-e^{-2\alpha\sqrt{-1}}\right)$ Hence  $\beta = \alpha - \left(\frac{1-n}{1+n}\right) \sin 2\alpha + 1/2\left(\frac{1-\alpha}{1+n}\right)$ 

the series sought.

In equation (11), let us replace  $\alpha$  by 46.  $\beta + \gamma$ .

> Then  $e^{+(\beta+\gamma)\sqrt{-1}} = e^{+\beta\sqrt{-1}} x e^{+\gamma\sqrt{-1}} =$  $(\cos.\beta + \sin.\beta\sqrt{-1})(\cos.\gamma + \sin.\gamma)$  $(\cos.\beta\cos.\gamma - \sin.\beta\sin.\gamma) = (\sin.\beta)$ But by principles of Undetermined  $cos. (\beta + \gamma) = cos. \beta cos. \gamma +$ and

 $sin.(\beta + \gamma) = sin.\beta cos.\gamma + cos.\beta sin.\gamma.$ If we replace  $\theta$  by  $\beta - \gamma$ , in equation (11), the result is

$$e^{-2\alpha\sqrt{-1}} - \log\left(1 + \frac{1-n}{1+n}e^{+2\alpha\sqrt{-1}}\right) = \frac{1}{-1} - \frac{1}{2}\left(\frac{1-n}{1+n}\right)^2 \left(e^{+4\alpha\sqrt{-1}} - e^{-4\alpha\sqrt{-1}}\right) \dots \&c.$$

$$(\frac{n}{n})^2 \sin 4\alpha - 1/3(\frac{1-n}{1+n})^3 \sin 6\alpha + \&c.,$$

$$cos. (\beta + \gamma) + sin. (\beta + \gamma)\sqrt{-1} =$$
  
 $\sqrt{-1}) =$   
 $\beta cos. \gamma + cos. \beta sin. \gamma)\sqrt{-1}.$  (12).  
Coefficients  
 $- sin\beta sin \gamma,$ 

(41) Cer. (B-r) = co. Blo. r+sin. soin.r. and sie (B-r) = sie Bcor - cor. B sin . r-These are the four fundamenter formulas. From Them are other trigonoustriese formulas can la derind -Let B=r, in Equation (12); Thus cus, 2 3 + Sic, 2 3/-1 = (cus. B + Sic. 3/-1) -If in continue This spiration a Times, The result is cro. 1 B+ Fin. 1 BV-i = (cro. B+ Jin. BV-i) which is another dumatration for Delloure's formula

These are the four fundamental formulas. From these all other trigonometrical formulas can be derived. Let  $\beta = \gamma$ , in equation (12); then If we continue this operation n times, the result is

and

 $\cos n\beta + \sin n\beta \sqrt{-1} = (\cos \beta + \sin \beta \sqrt{-1})^n$ which is another denomination for DeMoivre's Formula.

### (41)

 $cos.(\beta - \gamma) = cos.\beta cos.\gamma + sin.\beta sin.\gamma.$ 

 $sin.(\beta - \gamma) = sin.\beta cos.\gamma - cos.\beta sin.\gamma.$ 

 $\cos 2\beta + \sin 2\beta \sqrt{-1} = (\cos \beta + \sin \beta \sqrt{-1})^2$ .

(42) LOCARITHMIC IMAGINARIES. 47- In any lysten, the logarithm of 1 is s, and The logarithm of a is + a or - a, being + if The base is less, and - if The bace is grater Than unity -All position mumbers beturne and a, when used as boses of systems wice indude among The logarithius all possible multer. beturner - a and + a -It Thus appears that if negative multers have logarithms, They much be inajuary -Leh Et = x-Take The logarithm; there log. x = 4 -But as There is only are real value of y. There can be my our arithundicol logalithun, and if us admit V-1 into The system, These may be an infinite much of logarithus, my one of which will be wal

(42)

## LOGARITHNIC IMAGINARIES.

47. than unity.

> All positive numbers between 0 and ∞, when used as bases of systems will include among the logarithms all possible numbers between  $-\infty$  and  $+\infty$ . It thus appears that if negative numbers have logarithms, they must be imaginary. Let  $e^{+y} = x$ .

Take the logarithm; then

 $log._{(e)}x = y.$ But as there is only one real value of y, there can be only one arithmetical logarithm, and if we admit  $\sqrt{-1}$  into the system, there may be an infinite number of logarithms, only one of which will be real:

In any system, the logarithm of 1 is 0, and the logarithm of 0 is  $+\infty$  or  $-\infty$ , being + if the base is less, and - if the base is greater

(43) The others will be of The general imaginany forme, at 3/-7-48- A quantity of The forme at BV-1, may have no had logarithun, and can have only one in a system where bose is 2+ 2051, muleso The modulus, 1/0775), and The base are Ead-Equal to 1 - Su This case The monther of max logarithus und les infinite, as is apparent. If only one real logarithm coint, it will be the tatis of the logarithus of The Mothelus of The quantity, and The bore -49- la Equation (11), let a les Equivalent to ma; ne being any some whole multi-There cos. Man = 1, and in. mat = 0, and The Expression becomes  $\Sigma^{+MLTIV=I} = I_{-}$ (13). This curious Equation moolors FI as an arithmetical unpossibility, for by, 1=0-

(43)

form,  $\alpha \pm \beta \sqrt{-1}$ .

48. equal to 1. In this case the number of real logarithms will be infinite, as is 49. m being any even whole number. Then

> and the expression becomes  $e^{+m\Pi\sqrt{-1}} = 1.$

This curious equation involves  $\sqrt{-1}$  as an arithmetical impossibility, -- for log. 1 = 0.

### the others will be of the general imaginary

A quantity of the form  $\alpha \pm \beta \sqrt{-1}$ , may have no real logarithm, and can have only one in a system whose base is  $\lambda + \omega \sqrt{-1}$ , unless the modules,  $\sqrt{(\alpha^2 + \beta^2)}$ , and the base are each apparent. If only one real logarithm exist, it will be the ratio of the logarithm of the [modulus?] of the quantity, and the base. In equation (11), let  $\theta$  be equivalent to  $m \Pi$ ;

 $cos. m \Pi = 1$ , and  $sin. m \Pi = 0$ ,

(13).

(44)But if us take The log. of Equation (13), as have MATIVI = log. 1; hunce MATIVI = 0\_ If in ascribe any value to me according to previously mentioned conditions; say, 2, us may deduce a derico, True to any desired accuracy, for Et Milling Each series will be of the form I+ U-1×0\_ If y be the two logarithm of x, in Equation Et = x; and Et martin = 1 he combrand with it, The series is  $\mathcal{E}^{+(9+\mathcal{W}_{\overline{i}})} = X_{-}$ Huca log. x = y+ minv=1; me being priction or negalin as lefon -If The Mal logarithm of x be devoled by log. x, and The gueras logarithe by bog. x; The queral Expression for loga -Nithus becomes Leog. x = log. x + ML TIV-I -

### (44)

If we ascribe any value to m according to previously mentioned conditions; say, z, we may deduce a series, true to any desired accuracy, for  $e^{+m\Pi\sqrt{-1}}$ . Each series will be of the form  $1 + \sqrt{-1x0}$ . If y be the true logarithm of x, in with it, the series is  $e^{+(y+m\Pi\sqrt{-1})} = x.$ 

Hence

negative as before. If the real logarithm of x be denoted by log. x, and the general logarithm by log. x, the general expression for logarithms becomes

 $log.x = log.x + m\Pi\sqrt{-1}.$ 

But if we take the log. of equation (13), we have  $m\Pi\sqrt{-1} = \log 1$ ; hence  $m\Pi\sqrt{-1} = 0$ .

equation  $e^{+y} = x$ ; and  $e^{+m\pi\sqrt{-1}} = 1$  be combined

 $log. x = y + m \Pi \sqrt{-1}$ ; m being positive or

to for any other muter ; 2009. Z = log. Z + 2 TIV-I -Thur Log. XZ = log. XZ + (aut w) TIV-1, and Log. = log. + + (11-2) TIV-1\_ From This we readily see The Matin belivere number and Their logarithuns -The sum or difference of two logarithuns indicates That This componding nautres are sespecting multiplica or dividue -50-Make o = # in Equation (11); Then ミナモー = レーー-Jake The logarithme, and the present is THE = log. FT, which is a signibilie Equation showing That unajoury aco are logarithus -But we much not consider these fulling

So for any other number;  $Log. z = log. z + n\Pi\sqrt{-1}.$ 

Then

and

From this we readily see the relation between numbers and their logarithms. The sum or difference of two logarithms indicates that their corresponding numbers are respectively multiplied or divided.

Make  $\theta = \frac{\pi}{2}$  in equation (11); then 50.  $e^{+\frac{n}{2}\sqrt{-1}} = \sqrt{-1}.$ 

Take the logarithm, and the result is  $\frac{\pi}{2}\sqrt{-1} = \log(\sqrt{-1}),$ 

which is a symbolic equation showing that imaginary arcs are logarithms. But we must not consider these functions

### (45)

 $Log.\,xz = log.\,xz + (m+n)\Pi\sqrt{-1},$ 

 $Log.\frac{x}{\pi} = log.\frac{x}{\pi} + (m-n)\Pi\sqrt{-1}.$ 

(46) unreal, for according to The modern in hespelation of maginaries The lift hand Mumber is simply the logarithm of a with live perpendicular to The anis of reference - the shore also Mucula That Expensions of This kind are not made the basis of computation, and it is only in coses where The signibolo are copable of being interpretice That ar can concern of the relation between them, and the Spicific Munerical volues They duron -51- Again, in Equation (11), let & he represeculide by (me+1)TI; me being any come whole number, Either position or negatin The Equation becomes E+(m+1)V-1 TI = curi(n+1) TI + Sin. (n+1) TI V-1 =-1-Hence log. (-1) = (n + 1) TI V-T, or <u>log.(-1)</u> = (ne+1)TI -

### (46)

unreal, for according to the modern interpretation of imaginaries the left hand number is simply the logarithm of a unit line perpendicular to the axis of reference. We should also remember that expressions of this kind are not made the basis of computation, and it is only in cases where the symbols are capable of being interpreted that we can conceive of the relation between them, and the specific numerical values they denote. Again, in equation (11), let  $\theta$  be represented by  $(m+1)\Pi$ ; m being any even whole number, either positive or negative.

51. The equation becomes

$$e^{+(m+1)\sqrt{-1}\Pi} = ce$$

Hence

$$log. (-1) = (m + 1)$$
$$\frac{log.(-1)}{\sqrt{-1}} = (m + 1)$$

 $\cos((m+1)\Pi + \sin((m+1)\Pi\sqrt{-1}) = 1.$ 

1) $\Pi\sqrt{-1}$ , or )П.

(47) of us assume different values of me to be introduced, according to conditions, The Expression becauses TI, 3TI, 5TI, TTI, 9TI, TE-This shows There might be an infinite multe of ration between The log. (-1), and V-T, which agrees with what was previous in ally

## (47)

If we assume different values of m to be introduced, according to conditions, the expression becomes  $\Pi, 3\Pi, 5\Pi, 7\Pi, 9\Pi, \&c.$ This shows there might be an infinite number of ratios between the log. (-1), and  $\sqrt{-1}$ , which agrees with what was [promised?] in Art. 43.

(48) QUATERNION IMAGINARIES. 50 52. In The Introduction, we notice some of The historicae changes which occurred in The interpretation of The imaginary, Through all The staps of its development Tile The invition of Qualimino \_ Monsiur brauce founded his interputation on results derived fre multiplication of imaginarius, as did subsequent inordigatore, Tile the line of Hamilton -These results will now be investigated bet it be require to find a geometrical Mean between + 1 and - 1\_ If & duote this mean; There +1: X:: X:-1, or  $x = \pm V - I -$ (14) But in This consideration un Euconneter a difficulty in ascribing The meaning to The Molation used - 2f + 1 durate

(48)

## QUATERNION IMAGINES.

of Quaternions. till the time of Hamilton. Let it be required to find a geometrical mean between +1 and -1. If x denote this mean; then

52.

- +1: x: : x: -1, or
- $x = \pm \sqrt{-1}$ .

But in this consideration we encounter a difficulty in ascribing the meaning to the notation used. If +1 denote

In the Introduction, we notice some of the historical changes which occurred in the interpretation of the imaginary, through all the stages of its development till the invention

Monsieur Argand founded his interpretation on results derived from multiplication of imaginaries, as did subsequent investigators, These results will now be investigated.

(14)

(49) a geometricae magnitude, There can be no geometrical interputation of -1-But The + 1 and - 1 are here used only as indicativo of direction, and have They are only be considered as such -In coordinate gunutry of two dimensions, The aver may be taken as doubl- much lino, to which the other transformations can be referred du Fig. I, let O be The origin, with a sidius OA = unity - Nraw form diameters so as to divide The ciscumperina Fig. I. into Equal parts; also, bisech The ares AE and EC. the may denote The disclime OA by +, and

(49)

a geometrical magnitude, there can be no geometrical interpretation of -1. But the +1 and -1 are here used only as indications of direction, and hence they will only be considered as such. In coordinate geometry of two dimensions, the axes may be taken as double-unit lines, to which the other transformations can be referred.

In Fig. I, let 0 be the origin, with a radius 0A=unity. Draw from diameters so as to divide the circumference into equal parts; also, We may denote the direction 0A by +, and

- [Figure]

- bisect the arcs AE and EC.

(50) OB by - ; here, when OA works about O live it concides in direction with OB, The point A describes are are atore circular measure is Ti; or in passing Through an angle of 180°, The sign is changed from + to --And if the Mothetim continue from OB around to OA, the live will know Theraugh mother augh of 180°, or will thange the sign of direction proce - to + - But as The two radii where directions are indicated by + and - are in The same straight live, and of Equal lingth, The mean proportion indicature in Equation (14) Much be perpendicular to Their common point of min, and of the same lingth as sitter -Have the see That + F-1 and - F-1 simply indicate a mich oretor line pupudicula to The asis of reference - The conficient 1, in sither ever limits its lingth, and

## (50)

0B by -1; hence, when 0A revolves about 0 till it coincides in direction with 0B, the point A descrives are all whose circular measure is  $\Pi$ ; or in passing through an angle of 180°, the sign is changed from + to -. And if the revolution continue from 0B around to 0A, the line will revolve through another angle of 180°, or will change the sign of direction from - to +. But as the two radii whose directions are indicated by + and - are in the same straight line, and of equal length, the mean proportion indicated in equation (14) must be perpendicular to their common point of union, and of the same length as either. Hence we see that  $+\sqrt{-1}$  and  $-\sqrt{-1}$  simply indicate a unit vector line perpendicular to the axis of reference. The coefficient 1, in either case limits its length, and

(57) The signe + V-1 indicates its direction -In the powers of 1-1, as have 1. 1-1, -1, -1-1-These for formes are repeated by a continued Multiplication and seen in a cycle of four - The formulas, 4n, 4n+1, 4n+2, and 4n+3, welude all The changes that occur in The different positions of The rightaughere rection - And as These much coincide with The relations deduced by The Mean ploportion, 42, 421, 42+2, 42+3, much company uspecting with 1. V=1, -1, aud-1=1-53- The Expression V= MIT hpusants a victor pupulicular to the give direction line -If m= 1, V=V=1, which, by The practing groutrieve inlupulation, indicates that The orelow live house through an

(51)

the sign  $\pm \sqrt{-1}$  indicates its direction. In the powers of  $\sqrt{-1}$ , we have  $1,\sqrt{-1}$ , -1,  $-\sqrt{-1}$ . These four forms are repeated by a continued multiplication and recur in a cycle of four. The formulas, 4n, 4n+1,4n+2, and 4n+3, include all the changes that occur in the different positions of the rightangled vectors. And as these must coincide with the relations deduced by the mean proportion, 4n, 4n+1, 4n+2, 4n+3, must correspond respectively with

53. vector perpendicular to the given direction line.

> geometric interpretation, indicates that the vector line revolves through an

 $1,\sqrt{-1}, -1, \text{ and } -\sqrt{-1}$ .

The expression  $v = m\sqrt{-1}$  represents a

If  $m = 1, v = \sqrt{-1}$ , which, by the preceding

(52) augh of 90°, around un avis perpendicular to the plane of rotation -54- De is a mean proportion between CA and OB; and OD believe OB and aA. And Do again DE is a mean proportion between Of and QC; and Og between OC and OB, or lecture Of and OD; OF between OTS and OD, or beturne Of and OC; Of beturne OD and OH, or beliving OF and DE-Similarly, un might insert any munter of mean proportions between Two give direction orchor lines -The proportions under the thus : DA:09:02:02:02:09:09:00: te, and by This proportion The augles include between The metero much massarily la Equal. 55 - But The successing relations beline The orders do not necessarily orginal-

### (52)

angle of 90°, around an axis perpendicular to the plane of rotation. 0C is a mean proportion between 0A and 0B; and 0D between 0B and 0A. And so again 0E is a mean proportion between 0A and 0C; and 0G between 0C and 0B, or between 0A and 0D; 0F between 0B and 0D, or between 0A and 0C; 0H between 0D and 0A, or between 0F and 0E. Similarly, we might insert any number of mean proportions between two given direction vector lines. The proportions [will?] be thus: 0A:0I::0I:0E::0E:0J::0J:0C:: &c., and by this proportion the angle included between the vectors must necessarily be equal.

But the preceding relations between 55.

54.

the vectors do not necessarily originate

(53) from The origin Q - They can be taken fre any origin, as is indicated by The queral inapinary & IBV=1 -Thus the queral anaginary may be toten to represent the order prove the origin to the point, a, B, implicitly indicating the direction -By operating one at BIFI, by IFI, The origin of oretors may be changed Through Each quadrant, and this give a quadruple series of coordinates, but the lugth Muains The same = V(2+32) = The Modulus-56. Erry Equation eace be separative into real factors of The first or second type, and where nots are of the same forme as at 317; B being providing or negation If the work he was, shall be represecture by The ordinary grophics of Che-Usian Geometry; of unaginary, by The

(53)

from the origin 0. They can be taken from any origin, as is indicated by the general imaginary  $\alpha \pm \beta \sqrt{-1}$ . Thus the general imaginary may be taken to represent the vector from the origin to the point,  $\alpha$ ,  $\beta$ , implicitly indicating the direction.

By operating on α ± β√−1, by √−1, the origin of vectors may be changed through each quadrant, and thus given a quadruple series of coordinates, but the length remains the same = √(α<sup>2</sup> + β<sup>2</sup>) = the modulus. Every equation can be separated into real factors of the first or second degree, and whose roots are of the same form as α ± β√−1; β being positive or negative. If the root be real, it can be represented by the ordinary graphics of Cartesian Geometry; if imaginary, by the

56.

(54) pulling quantiene mitterds -57- The preceding interpretation is my of a Special case, and V-1 indicates an operalion of rolation, or of position only-But if a undicate any geometrical line, usue as a direction line, + SV-1 Meneruls a retor perpendicular to The direction linebut finally are other coplana orclor, not indication by ± a, and ± BV= Mush lie in one of the quadrants and be indicature by a + BIFT -But lines represented by purchino containing imaginaries are as real as The direction lines Thurselors, and Should be considered as absolute as lins indicature in a Regalin direction -If us with + BFT and -BVFT, us have souply indicative the direction of solation, when it is reforme to the perpendicular mich

### (54)

57.

preceding geometrical methods. The preceding interpretation is only of a special case, and  $\sqrt{-1}$  indicates an operation of rotation, or of position only. But if  $\alpha$  indicate any geometrical line, used as a direction line,  $\pm \beta \sqrt{-1}$  represents a vector perpendicular to the direction line. And finally all other coplanar vectors, not indicated by  $\pm \alpha$ , and  $\pm \beta \sqrt{-1}$  must be in one of the quadrants and be indicated by  $\alpha \pm \beta \sqrt{-1}$ . But lines represented by functions containing imaginaries are real as the direction lines themselves, and should be considered as absolute as lines indicated in a negative direction. If we write  $+\beta\sqrt{-1}$  and  $-\beta\sqrt{-1}$ , we have simply indicated the direction of rotation, when it is referred to the perpendicular unit

(55) retor, 1-1 -58 - see indicating the position of a point in a place, us may durote The distance fre the origin to The projection of the print on the abscissa, by a; the length of The projection line, by B; and the augh which The meter makes with The direction line, by o; the instead of writing very of the for abour give repressions, in may with the quaternin, C(co.o+ sin, ov-1) -But as Fi tures a neter Through The circular measure of I, The quatinin repression lunes it through an augular part of a qualraut, represented by  $\frac{d}{4} = \frac{2d}{\pi}$ Huce C(cos. 0 + sin. 01-1) = (1-1) # 59. But V-1 was shown by Amilton to be a simulrical Mality, keining Metricha to no

vector  $\sqrt{-1}$ .

58.

In indicating the position of a point in a plane, we may denote the distance from the origin to the projection of the point of the abscissa, by  $\alpha$ ; the length of the projection line, by  $\beta$ ; and the angle which the vector makes with the direction line, by  $\theta$ ; then instead of writing any of the four above given expressions, we may write the quaternion,  $C(\cos \theta \pm \sin \theta \sqrt{-1})$ . But as  $\sqrt{-1}$  turns a vector through the circular measure of  $\frac{\pi}{2}$ , the quaternion expression turns it through an angular point of a quadrant, represented by  $\frac{\theta}{\overline{\Pi}} = \frac{2\theta}{\Pi}$ .

Hence

59.

## (55)

 $C(\cos \theta \pm \sin \theta \sqrt{-1}) = (\sqrt{-1})^{\frac{2\theta}{\pi}}.$ But  $\sqrt{-1}$  was shown by Hamilton to be a geometrical reality, being restricted to no

(56) particular direction in space - He should that all directions can be reputer by The imaginary, Thus subling mathematicia To represent any line or point Equally all-60. Uz are already using lerus which show now be Explaine - Beet as The preding discussions are only a review of These previous to the invition of Qualizino, it has been thought desirable to give an EDplanalion of all The lerus to getter, as long as these already usue are guinally undustord -Quatinins forme a septim of Austylicae Leon stry, The name of which was give by Hamitten, an account of four qualities That sitter sorry there quaternion -A qualerin is The product of a busion and a men -A muser is The sum of a meter and a

### (56)

60.

particular direction in space. He showed that all directions can be represented by the imaginary, thus enabling mathematicians to represent any line or point equally well. We are already using terms which should now be explained. But as the preceding discussions are only a review of those previous to the invention of Quaternions, it has been thought desirable to give an explanation of all the terms together, as long as those already used are generally understood.

Quaternions form a System of Analytical Geometry, the name of which was given by Hamilton, on account of four quantities that enter every true quaternion. A quaternion is the product of a tensor and a versor A versor is the sum of a vector and a

(571 Scalar -Have in han The quantities Vector (V); Tensor (T); unsor (T); and scalar (S)-But un saw in ach. 58 That a quater-Mon can also be represented by C(cus. + ± sin. 01-1)\_ Hue There are two ways of representing any the quatorion : Q=TxT= C(cus. 0+ Lin. +1-1)\_ (15)\_ A neter is any line parallele to a give direction line, and if The line be a with in lugar, it is called a with retor live -A Tunor is a Multiplice, or literally That which stretches, an actor line Tile it coincides in lengthe with another oretor. A lessor is a quantity which turned me neter, about its orgin, tile it

(57)

scalar.

Hence we have the quantities vector (V); tensor (T); versor (U) and scalar(S). But we saw in Art. 58 that a quaternion can also be represented by

 $C(\cos \theta \pm \sin \theta \sqrt{-1}).$ Hence there are two ways of representing any true quaternion:

A vector is any line parallel to a given direction line, and if the line be a unit in length, it is called a unitvector line.

A tensor is a multiplier, or literally that which stretches, one vector line till it coincides in length with another vector.

A tensor is a quantity which turns one vector, about its origin, till it

 $Q = T \times U = C(\cos \theta \pm \sin \theta \sqrt{-1})$ 

(15).

(58) concides in direction with another weter, and if the augular space between The retors be goo, The our is called a quadraular uner-But in Equation (151 UZ Saw a qualemin ean be made up of a Mal, munerical part, and an imaginary - The Munerical part is called a scalar -61- Vector Equations: In Hig. II, let ABC les any Triangle. Denote The side AB by a: The side BC by Bi - B and the side the by r\_ Hig. II. Suppre The anons indicate The direction That The sides of The himyle are gue Ustur If a magnitude be transferred from A to C, there are two ways by which

### (58)

coincides in direction with another vector and if the angular space between the vectors be 90°, the vector is called a quadrantal versor But in equation (15) we saw a quaternion can be made up of a real, numerical part, and an imaginary. The numerical part is called a scalar.

- Vector Equations: 61.
  - [Figure]

In Fig. II, let ABC be any triangle. Denote the side AB by  $\alpha$ : the BC by  $\beta$ ; and the side AC by  $\gamma$ . Suppose the arrows indicate the direction that the sides of the triangle are generated. If a magnitude by transferred from A to C, there are two ways by which

it may be accomplished; one is by going directly along The line ac; The other, by going from & to B and There from Bto C. This may be represented by x+ 3= x, or  $\alpha + \beta - \gamma = 0,$ which is known as The actor Equation of The briangle. But the signs + and -, and =, here, do such have the same limited Aquification They do in Algebra -En quiral lauguage The above Equation May be read , a transformer typussed by meter &, following by a Transformer Expressed by actor B, is Equivalent to a transferrence Expussed by tetter r." 62 - The fundamental formulas of Trigmonutry care be Easily teduce by Means of a new Equation

### (59)

directly along the line AC; the other,

 $\alpha + \beta - \gamma = 0,$ 

signification they do in Algebra. transformer expressed by vector  $\gamma$ ."

62. The fundamental formulas of trigonometry can be easily deduced by means of a vector equation.

- it may be accomplished; one is by going by going from A to B and thence from B to C. This may be represented by  $\alpha + \beta = \gamma$ , or
- which is known as the vector equation of the triangle. But the signs + and -, and =, here, do not have the same limited In general language the above equation may be read, "a transformer expressed by vector  $\alpha$ , followed by a transformer expressed by vector  $\beta$ , is equivalent to a

(60) In dig. III, let All be the are of a circle whose radius is O.A. Let The augles Adl= a, ADD= B, and DCB= x; There, 7 A Fig. II. as The sinces of The sugles this her puper dicular to OA, which we will use for a direction line, They will all be accompanied by 1-1-Hunfon OB = OE + EB = co. (x+B) + Sin. (x+B)V-1\_ 0 D = 07 + 7 D = Cor. B + Sin. BF-1 -QL=09+9C = Lo. x + Zin. x F-1 -But un saw in ach. 54 that Osi Ocii Osion, or OD×013 = OC×00 -The live Of is the whit radius and is The direction line; here OA=1-The Equation because (16) OB= OC×OD

(60)

In Fig. III, let AM be the arc of a circle whose radius is 0A. Let the angles A0C= $\alpha$ , ACD= $\beta$ , and D0B= $\alpha$ ; then, as the series of the angles

[Figure]

will be perpendicular to 0A, which we will use for a direction line, they will all be accompanied by  $\sqrt{-1}$ . Therefore  $CB = 0E + EB = cos. (\alpha + \beta) + sin. (\alpha + \beta)\sqrt{-1}.$  $0D = 0F + FD = \cos \beta + \sin \beta \sqrt{-1}.$  $0C = 0G + GC = \cos \alpha + \sin \alpha \sqrt{-1}.$ But we saw in Art. 54 that 0A:0C::0D:0B, or  $0A \times 0B = 0C \times 0D.$ The line 0A is the unit radius and is the direction line; hence 0A = 1. The equation becomes

 $OB = OC \times OD$ .

(16)

(61) In This Equation replace The values of OB, Ol, and OD, by ingonomitrical functions as formed on page 60; The result is cos. (a+B) + lin. (a+B)V-1 = (cos. a + sin. aV-1)(cos. B + sin. BV-1)\_ (17). By Espanding and applying The principles of Undelemine Corfficients, in get Co. (x + 5) = Clo. x Co. 3 - zin. x Sin. B, and Rin. (x+B) = Rin. x Cor. B + Cor. X Sin. B-2/- (x+B) he replaced by (x-B), an get Cos. (x-B)= cos. x cos. B + zin. x sin. B, and Sin (x-B) = sin , x co, B - co, x sin . B -If a = 3, in Equation (17), the have ers. 2x + ziu. 2x1-7 = (crs. x + sin. x1-1)-If This operation be continue the hime, 42 Sch Con Max + sin, marin = (con x + sin, x V-1) ;

(61)

In this equation replace the values of 0B, 0C, and 0D, by trigonometrical functions as found on page 60;

the result is

 $\cos.(\alpha + \beta) + \sin.(\alpha + \beta)\sqrt{-1} = (\cos.\alpha + \sin.\alpha\sqrt{-1})(\cos.\beta + \sin.\beta\sqrt{-1}).$ By expanding and applying the principles of undetermined Coefficients, we get

 $cos. (\alpha + \beta) = cos. \alpha cos. \beta - sin. \alpha sin. \beta$ and

 $sin.(\alpha + \beta) = sin.\alpha cos.\beta + cos.\alpha sin.\beta.$ If  $(\alpha + \beta)$  be replaced by  $(\alpha - \beta)$ , we get  $\cos.(\alpha - \beta) = \cos.\alpha\cos.\beta + \sin.\alpha\sin.\beta.$ and

 $sin.(\alpha - \beta) = sin.\alpha cos.\beta - cos.\alpha sin.\beta.$ If  $\alpha = \beta$ , in equation (17), we have  $\cos 2\alpha + \sin 2\alpha \sqrt{-1} = (\cos \alpha + \sin \alpha \sqrt{-1}).$ If this operation be continued m times, we get  $\cos m\alpha + \sin m\alpha \sqrt{-1} = (\cos \alpha + \sin \alpha \sqrt{-1})^m$ ;

(17).

1621 which is another demonstration of Dellovino Fonula\_ 63\_ La Fig. II, let OI, OJ, and OK be Three untually perpendicular unil-Fig. IV rectoro - Prolongue These lives in The oppoint direction, a unit's dis. Tauce, and Thus draw ares of circles as indication in the figure -Let the live OJ revolve about OI as an asis until it coucies with the line OK. So for all the other line - The factors while ture there lives through the quadrant angle are the guadrante mors, muliment in act. 60\_ of 112 simply use The letters I, J, and K to durate the Metaugular meters

## (62)

which is another demonstration of De Moivre's Formula.

63. In Fig. IV, let 0I, 0J, and 0K be three mutually perpendicular unit-[Figure[ vectors. Prolongue these lines in the opposite direction, a unit's distance, and then draw arcs of circles as indicated in the figure. Let the line 0J revolve about 0I as an axis until it coincides with the line 0K. So for all the other lines. The factors which turn these lines through the quadrant angle are the quadrantal versors, mentioned in Art. 60.

If we simply use the letters, I, J, and K to denote the rectangular vectors

(63) OT, OJ, and OK, un have The following Malino :  $K = I, \quad \overline{K} = J, \quad aud \quad \overline{J} = K; \quad or$ K=IJ, I=JK, and J=KI. If we use The migation meters as asial lins, we have the following Matins: J = -I, K = -J, and J = -K; mJ=-IK, K=-JI, and T=-KJ. The minus Fign occurs here for two reasons; first, because The apis is consider-Ed regalin, and seeme, because The rolation is embrary to That give in The first case, where Mobilian was assund position -There two cases gin K = IJ, I = JK, and J = KI - (18)And

(63)

0I, 0J, and 0K, we have the following relations:

$$\frac{K}{J} = I, \frac{I}{K} = J, and \frac{J}{I} = K; \text{ or } K = IJ, I = JK, and J = KI.$$

If we use the negative vectors as axial lines, we have the following relations: J, and  $\frac{1}{I} = -K$ ; or -JI, and I = -KJ. J = -IK, K = The minus sign occurs here for two reasons; first, because the axis is considered negative, and second, because the rotation is contrary to that given in the first case, whose revolution was assumed

$$\frac{J}{K} = -I, \frac{K}{I} = -J$$
$$J = -IK, K = -J$$

positive.

These two cases give K = IJ, I = JK, and J = KIAnd

(18).

(64) K = -JI, I = -KJ, and J = -IK - (19)in both of The sets of Equations, we have usue The unit rectors I, J, and K as aves\_ But these are quadrante misors, and are querally denoted by inf, and k. Here a mit never may be suployed as a quadrantae men, having a plan perpendicular to the rectors; and the product or qualient of two perpendicular actions is a rector pupudicular to both-64- In algebraic Multiplication, In han learner that the product does not deput upon the order of the factors, and here is called the Consulation Principle -But ar saw in Equations (18) and (19) That qualimin Multiplication changes The sign of The product by drauging the order of The factors, and have is cauce the Um- Commulation Principle -

(64)

K = -JI, I = -KJ, and J = -IK. (19). In both of these sets of equations, we have used the axis vectors I, J, and K as axes. But these are quadrantal versors, and are generally denoted by i, j and k. Hence a unit vector may be employed as a quadrantal versor, having a plane perpendicular to the vectors; and the product or quotient of two perpendicular vectors is a vector perpendicular to both. In algebraic multiplication, we have learned that the product does not depend upon the order of the factors, and hence is called the Commutative Principle. But we saw in equations (18) and (19) that quaternion multiplication changes the sign of the product by changing the order of the factors, and hence is called the Non-Commutative Principle.

64.

(65) 65-The association Principle consists of Maintaining The cyclical order of The factors - If ijk be any know order of factor, There The association law is ijk= jki=kij= tc-But if This order he changed and the new cyclical order la asure the segus will be changed -66. In Equation (18), let J, K, and I he replace by J, K, and I of Equation (19). The result is I=-1, J=-1, and K=-1, or  $I^{2} = J^{2} = I^{2} = L^{2} = L^{2} = L^{2} = -1$ Jun i=j=k=±V=I, which is the relation believe The i, j. k. of Quaternions\_ 67. We much not consider a live and a vietor as synonymous. A viller disignature by x is Equivalent to a

(65)

65.

The Associative Principle consists of maintaining the cyclical order of the factors. If I j k be any known order of factors, then the associative law is ljk=jki=kji=&c.But if this order be changed and the new cyclical order be used, the signs will be changed.

- In equation (18), let J, K, and I be 66. The result is

Then

 $i = j = k = \pm \sqrt{-1}$ , which is the relation between the i, j, k, of Quaternions. We must not consider a line and a vector as synonymous. A vector designated by  $\alpha$  is equivalent to a

67.

replaced by J, K, and I of equation (19).

 $I^2 = -1$ ,  $J^2 = -1$ , and  $K^2 = -1$ , or  $I^{2} = J^{2} = K^{2} = i^{2} = j^{2} = k^{2} = -1.$ 

(66) live al, i being a with netor along The direction line; hence (ai)(ai) = 222 - as a live = a2 as a victor - But by art. 66, 6=-1-Hun an as a live = - at a sa order; or the square of a live is Equivalent to minus The square of The comepanding actor-68- The square of a with actor segardre as a quaturion is sometimes called an unur, and have it can be written x = - 1-Thue as x=1, the Tensor of a, Da=1, or Ta=1; here  $T_{\alpha}^{2} = -\alpha^{2}$ There any mit netor or quadrantal mor is a true representation of 17, and has are infinite number of values, buch They are all different fre the septebolie scales, V-1,

(66)

line  $\alpha i$ , *i* being a unit vector along the line =  $\alpha^2$  ------ as a vector. But by Art. 66,  $i^2 = -1.$ Then

68.

 $\alpha^2$  ----- as a line =  $-\alpha^2$  ----- as a vector; or the square of a line is equivalent to minus the square of the corresponding vector. The square of a unit vector regarded as a quaternion is sometimes called an inversor, and hence it can be written  $\alpha^2 = -1.$ And as  $\alpha = 1$ , the tensor of  $\alpha$ ,  $T\alpha = 1$ , or  $T^2\alpha = 1$ ; hence  $T^2 \alpha = -\alpha^2$ . Then any unit vector or quadrantaL versor is a true representative of  $\sqrt{-1}$ , and has an infinite number of values, but they are all different from the symbolic scaler,  $\sqrt{-1}$ ,

direction line; hence  $(\alpha i)(\alpha i) = \alpha^2 i^2$  ----- as a

(67) which occurs in This connection algebraic aualisis -But to discriminate This function in Quatunions proce The ordinary 1-1, Hamilton called qualinins, orctors, and scalars, con-Taining 1-1, biquaternions, biretors, and hiscalaro -Therefor a biquaterium becomes  $q = q_1 + q_1 - 1_1$ ie while q, and q, are Mal qualerning. It appears that these discriminations are 69. unucesary, for Fi much be involved in The production of These scalar punctions -69. If the have any quaternion, as, q= = = n TT, n being any ene inliger; Then a will be paralle to B and win have The sauce or opposite direction -Have a lecomes a prisition or negation Mal multer and is a scalar, but this

which occurs in this convection algebraic analysis.

But to discriminate this function in Quaternions from the ordinary  $\sqrt{-1}$ , Hamilton called quaternions, vectors, and scalars, containing  $\sqrt{-1}$ , biquaternions, bivectors, and biscalars. Therefore a biguaternion becomes  $q = q_{1} + q_{11}\sqrt{-1}$ in which q, and q, are real quaternions. It appears that these discriminations are unnecessary, for  $\sqrt{-1}$  must be involved in the production of these scalar functions. If we have any quaternions, as,  $q = \frac{\beta}{\alpha} = n\Pi$ , n being any even integer; then  $\alpha$  will be parallel to  $\beta$  and will have the same or opposite direction. Hence q becomes a positive or negative real number and is a scalar, but this

## (67)

(68) Acalar is me of Mobilion while The puciding is one of Matio -To\_ These sealars and tensors can be opplied to any line in space, and can occur in any qualimin -Frue the Mature of scalars, This product Must be a sealar ; and This conjugate, which considers the rolation in The opposite direction, is the identical Scalar itell-But as the imaginary, FI, is a scalar in the bignaterion q=q,+q,1-i: thee  $T(g, +g, )^{-1})^{=} \left\{ S(g, +g, V^{-1}) + V(g, +g, V^{-1}) \right\} \left\{ S(g, +g, V^{-1}) - V(g, +g, V^{-1}) \right\} =$ Tq = Tq = + 2 1-1 Sq Kq -If q=q, and Sq, Kq, =0, the whole lunor of The bignaterion reduces to o.

(68)

scalar is one of revolution while the preceding is one of ratio.

70. These scalars and tensors can be applied to any line in space, and can occur in any quaternion.

> From the nature of scalars, their product must be a scalar; and their conjugate, which considers the rotation in the opposite direction, is the identical scalar itself.

But as the imaginary,  $\sqrt{-1}$ , is a scalar in the biquaternion

$$q = q , + q , \sqrt{-1}: \text{ then}$$
  

$$T(q , + q , )\sqrt{-1})^{2} = [S(q , + q , \sqrt{-1}) + V(q + q , \sqrt{-1})] + V(q + q , \sqrt{-1}) + V(q + q , \sqrt{-1})]$$

If  $q_{1} = q_{11}$ , and  $Sq_{1}Kq_{11} = 0$ , the whole tensor of the biguaternion reduces to 0.

 $(1 + q_{11}\sqrt{-1})[S(q_{1} + q_{11}\sqrt{-1}) - V(q_{1} + q_{11}\sqrt{-1})] =$ 

(69) 11. The reciprocal of a quaternion is The quaternion is The quaternion of its saiplocal. Ihus q = q q = 1 If 3 = q ; There 3 = 1 = q - -Suppose - q= - a \* a = 1, or  $-\alpha = \frac{1}{\alpha}$ There 1= = - 1-1 If us take the leason of These Equations, us see That The Tusor of The recipiocae of a quaternion is the reciprocal fits lucor The veron has changed the direction. of its anyle, and to denote this nega-Tin monut, The conjugat quaternion is used. 72. The Pathajoreau Theonus can be Easily accuretatic by Quaternions-

(69)

The reciprocal of a quaternion is the 71. quaternion is the quaternion of its reciprocal. Thus  $q \frac{1}{a} = qq^{-1} = 1$ If  $\frac{\alpha}{\beta} = q$ ; then  $\frac{\beta}{\alpha} = \frac{1}{q} = q^{-1}$ . Suppose  $-\alpha^2 = -\alpha x \alpha = 1$ , or  $-\alpha = \frac{1}{\alpha}$ . Then  $\frac{1}{\sqrt{-1}} = -\sqrt{-1}$ 

> If we take the tensor of these equations, we see that the tensor of the reciprocal of a quaternion is the reciprocal of its tensor.

> The versor has changed the direction of its angle, and to denote this negative movement, the conjugate quaternion is used.

72.

The Pythagorean Theorem can be easily demonstrated by Quaternions.

(70) Let ABG, Fig. I. he a sight auglice triangle, right auglice at B. Stude The sides, AB by Al - 90: 1 B a, BE by s, and AC by M. Fig. I. Here by Ach. 61, V=x+B- as a VECTor-By squaring, r=x2+2x3+32\_ Dut by Un. 63, x 3 Equals another inter perfundicular to the place of rolation -Aun, in The above Equation,  $2\alpha\beta = 0$ , and  $r^{2} = \alpha^{2} + \beta^{2}$ . But in act, 67, us saw That x as a reter Equals - x 2 as a live -Thus The above Equation becomes -22 = - a 2 32 - a lines, or V= ~2+ (3-Have The Thomas-

(70)

Let ABC, Fig. V, be a rightangled triangle, right angled at B. Denote the sides, AB by [Figure]  $\alpha$ , BC by  $\beta$ , and AC by  $\gamma$ . Then by Art. 61,  $\gamma = \alpha + \beta$  ----- as a vector.

By squaring,  $\gamma^2 = \alpha^2 + 2\alpha\beta + \beta^2.$ 

perpendicular to the plane of rotation. Hence, in the above equation,

equals  $-\alpha^2$  as a line. Then the above equation becomes

$$-\gamma^2 = -\alpha^2$$
$$v^2 = \alpha^2 + \beta$$

Hence the theorem.

But by Art. 63,  $\alpha\beta$  equals another vector  $2\alpha\beta = 0$ , and  $\gamma^2 = \alpha^2 + \beta^2$ .

But in Art. 67, we saw that  $\alpha^2$  as a vector  $-\beta^2$  ----- a line, or

73\_ Leh AB6, Fig. II, les any triangle -B There by arh. 61, Fig. TI-AC= AB+BC, or  $d\overline{c}^2 = \mathcal{S}(dc \times dc) = \mathcal{S}(dc)(dB + Bc)_{-}$ Have 12=600. A. e+ 6 co. C. a, or b=cus. + + a cus. C, a une kunon plane trigonometrical relation -AB = AB,24-Ale= AB+BC, Auce AB+ac= AB(MB+BC)-Jake the retirs, and the purch is V(NB· NC) = U AB(AB+Be) = ch sin, A = ca sin. B, or a i le i ! sin. A ! sin. B.

73. Let ABC, Fig. VI, be any triangle. [Figure] Then by Art. 61, AC=AB+BC, or  $AC^2 = S(ACxAC) = S(AC)(AB+BC).$ Hence  $b^2 = b \cos A.C + b \cos C.A$ , or  $b = c \cos A + a \cos C$ , a well known plane trigonometrical relation. 74. AB = ABAC = AB + BC, Hence AB + AC = AB (AB+BC).Take the vectors, and the results is V(ABxAC)=VAB(AB+BC) =cb sin. A = ca sin. B, ora:b::sin. A: sin. B.

## (71)