01 Apr 1991

Nonlinear Dynamics and the Lorenz Water Wheel

Richard Margis

NONLINEAR DYNAMICS AND THE LORENZ WATER WHEEL

Rich Margis

Undoubtedly one of the most exciting frontiers in physics today is the study of nonlinear systems, more popularly called chaos. Chaos is the key to many time-dependent processes in as varied fields as meteorology (such as weather modeling), mechanics (the three-body problem in gravitational fields), and economics (prediction of trends in the stock market).

An easy way to get a grip on the basic ideas of chaos is with the so-called Butterfly Effect. The Butterfly Effect is the idea that a disturbance in an air mass in Beijing created by the wings of a butterfly is enough to make the weather in New York City due to that air mass unforecastable. The basic idea here is nonlinearity. A linear function, for those who have not studied differential equations or linear algebra, is any function y=f(x) such that f(a+b)=f(a)+f(b) and f(c*a)=c*f(a), where a and b are complex numbers or vectors. Simple linear functions include y=x and y=6: simple nonlinear functions include y=x^2 and y=sin(x).

Nonlinearity is evident in the following sequence, investigated by the researcher early in the course of the experiment:

$$x_0 = x \qquad x_{n+1} = q*x_n*(1-x_n) \qquad \text{eqn. 1}$$

where x is the first term of the sequence, restricted to be between zero and one, and q is the driving coefficient, restricted to be between zero and four. Different values of the driving coefficient produce the following behaviors:

value of q	behavior of sequence
less than one	approaches zero as n goes to infinity
between one and three	approaches some constant as n goes to infinity
about 3 to 3.6	oscillates between increasingly many pairs as q increases
about 3.6 to 3.85	chaotic behavior
about 3.85 to 3.90	oscillates between increasingly many triples as q increases
above 3.90	chaotic behavior

table 1

The sequence can be calculated easily with a hand calculator.

Naturally after investigating this intriguing sequence the researcher wanted to find out more about chaos. While reading the popular fiction book Chaos: Making a New Science by James Gleick the researcher discovered an interesting apparatus known as the Lorenz water wheel, which exhibits chaotic behavior. The apparatus in named for the meteorologist Edward Lorenz who invented it as a thought experiment in the early 1960's to explain the solution of a set of seven nonlinear heat convection differential equations derived by the physicist Barry Saltzman that reduce to three in certain ideal conditions. The form of these equations is:

$$dx/dt = -ax + ay \qquad \text{eqn. 2}$$

$$dy/dt = -xz + bx - y \qquad \text{eqn. 3}$$

$$dz/dt = xy - cz \qquad \text{eqn. 4}$$

The traditionally chosen values for the constants, a=10, b=28, and c=8/3, give the famous strange attractor plot when the solution of the differential equations is graphed in phase space.

The Lorenz water wheel itself is an ordinary-looking apparatus that consists basically of a water source and drain and a wheel mounted on a base. Cups with small holes punched in their bottoms are mounted to the outer radius of the wheel. This device works as follows. Water flows into the very top cup; the flow must be vertical such that it follows the diameter of the wheel down (unless the flow is intercepted by a cup) so that the flow itself exerts no torque on the wheel. Ideally, then, the wheel rotation will have to be started by hand unless the top cup is slightly off-center, but normally the flow exerts a small torque that overcomes the friction in the wheel and begins turning the wheel. Then the water in the cup slowly drains until the cup returns to the position at the top of the wheel. When the wheel is spinning slowly, the cups have a lot of time to fill at the top, as well as a lot of time to drain. This pulls the center of gravity of the wheel out close to the radius in the direction of the spin and puts a large torque on the wheel, increasing the angular acceleration. The wheel begins to spin faster and the cups get less and less time to fill and drain. Now the center of mass is pulled back toward center and over to the other side of the wheel, making the torque negative. The angular velocity begins to drop and the process repeats. After the initial awkwardness, the angular velocity seems to smooth out. But soon the fast-slow rhythm reappears, with barely noticeable but increasing amplitude. A little later the limps are highly pronounced, and the wheel almost stops once each rotation. Then it occurs that, on one rotation, the center of gravity of the wheel is far enough from the center that enough torque is exerted that the wheel stops and begins rotating backward! This is the beginning of the chaotic stage of the motion of the wheel and an example of the strange attractor in action. After this occurs, quantitatively predicting any long-term behavior of the wheel with any accuracy is impossible, because, to do so, it is necessary to know the EXACT state of the system at some time. Even the tiniest errors grow into hundred-percent uncertainties within fifty or so iterations; this is an example of the butterfly effect in action.

With all of this in mind, the researcher and the researcher's supervisor, Dr. Ralph Alexander of the Physics Department, planned a set of goals for the experiment; which included the following. First, a Lorenz water wheel would be constructed according to the basic description above in order to collect qualitative data. Second, library research would be performed to check if any suggestions concerning the experiment had been published and if any crash courses on the solution of differential equations, particularly nonlinear ones, existed (although the researcher was familiar only with mathematics up to the second semester of calculus when the experiment was begun, this was not considered a serious difficulty at the time!). Third, computer work would be done in learning how to operate MathCAD, Quattro, and similar programs so that iterated solutions could be computed on them. Finally, measurement devices would be attached to the wheel so that quantitative data could be taken, and, at least for a few iterations, fit with a theoretical curve.

The experiment was begun in the spring of 1990 in Dr. Alexander's Physics 27 (First Semester Honors Laboratory) class with completion intended by the end of the semester; the wheel, constructed of tape cases and small medicine cups (the sort found with Nyquil medicine), was finished by the Physics Shop, Dr. Alexander, and the researcher by mid-semester. Immediately it was found that the wheel was unbalanced; holes half-drilled in one side of the wheel partially corrected the problem. Then the job of playing with the wheel in order to find the parameters (number of cups, cup hole size, etc.) that make the wheel work correctly and easily was begun. In this task, there were no guides except somewhat vague ideas about the qualitative behavior of the wheel: it should rotate for awhile, uniformly after initial noise dies down and then with increasing limp, then the wheel should stop and begin rotating the opposite direction. To aid in further experimentation, more precise ideas of Lorenz water wheel behavior were needed.

To gain these ideas, the researcher proposed to try to model the water wheel's behavior by making successive ideal approximations of the wheel on the spreadsheet Lotus 1-2-3. The first step was to pick a set of variables that completely described the system. Variables to pick from included time, angle, angular velocity, angular acceleration, center of gravity, torque, flowrate of the water source, mass, and angular momentum. Angular momentum was immediately ruled out so that the problem would be defined in terms of easily visualized quantities, and time was ruled out soon after because it can be defined as the integral with respect to x of the reciprocal of the velocity and because time is implicit in many differential equations in mechanics and electrodynamics. Of the others, angle, angular velocity, and mass seemed the easiest to measure. Since center of gravity can be calculated from mass, position, and time, torque can be calculated from center of mass and mass, angular acceleration can be calculated from torque and mass, and mass can be calculated from flowrate, angle and time, all of the possible variables can be expressed in terms of the angle, the angular velocity, and the flowrate of the water source. If the flowrate is kept constant, the problem has only two degrees of freedom.

The researcher proceeded to compute the iterated solution of the center of mass and torque due to the water on an ideal wheel spinning at a constant velocity. The graphs of the center of mass and the torque on such a wheel are included; see pp. 10-11.

To supplement this computer activity, researching was done in the Curtis Laws Wilson library. The article by Lorenz concerning convection, "Deterministic Nonperiodic Flow", that the researcher learned of in Gleick's Chaos, was found in the March 1963 issue of the Journal of Atmospheric Sciences there. The article contained a numerical solution of the system of three differential equations presented earlier, as well as a graph of one of the variables versus time. This graph gave the researcher a much better feel for the expected behavior of the water wheel and explained the nature of phase space as well.

A final addition to the researcher's chaos understanding came near the end of the semester in the form of Chaos Demonstrations, a IBM Physics Academic Software program by Julien C. Sprott and George Rowlands. This easy-to-use program contains eighteen demonstrations on chaos, including such examples as the three-body problem, the magnetic quadrupole, the driven pendulum, as well as the strange attractor.

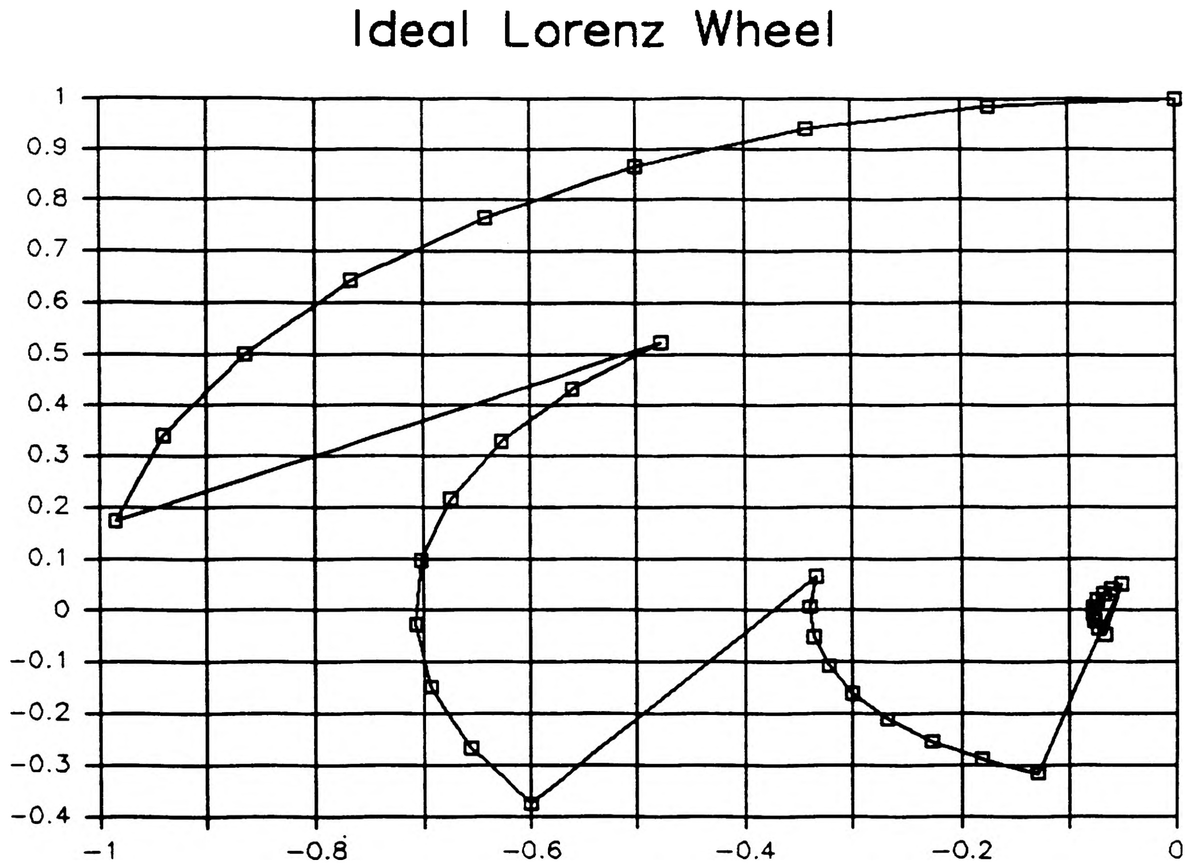
With this done at the end of the 1990 spring semester, the researcher was ready to attack the problem of getting data and decided to continue the unfinished experiment as a summer project for the OURE program.

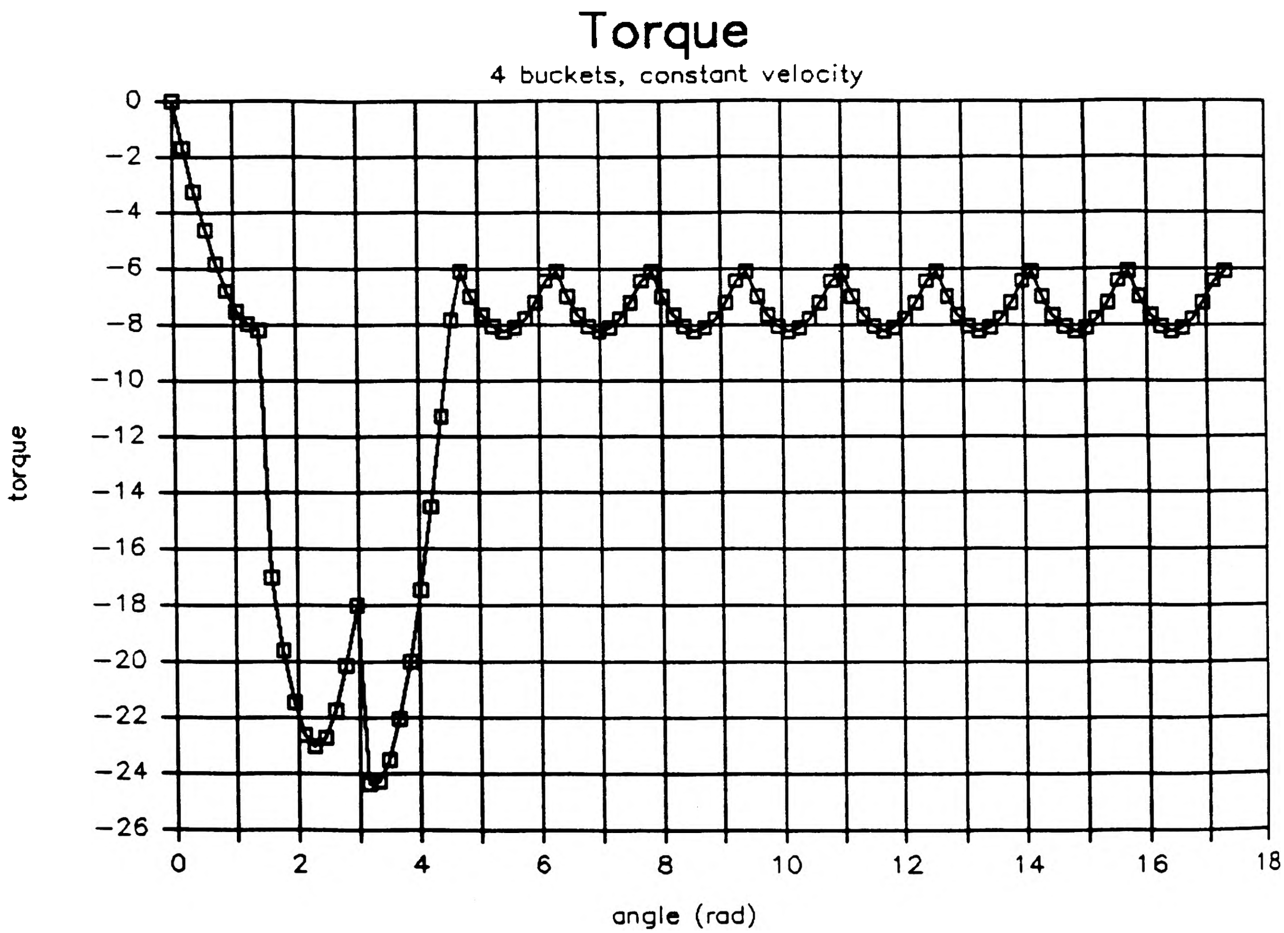
The summer portion of the experiment began optimistically with the nearly immediate confirmation that the behavior of the wheel is governed by a system of differential equations similar to equations 2, 3, and 4 above through the observation that a graph of the wheel's velocity over the fifteen or so minutes of each run would roughly agree with the graph mentioned above from Lorenz's article, as well as with the qualitative wheel predictions above.

Next it was necessary to refine the apparatus in order to make sure that the form of the data, a strange attractor in phase space, would not be thrown off by the conditions (of course in any case there would be enough uncertainty to throw the data points off from any expected values). Noted among the problems with the apparatus were: the wheel was still unbalanced after the holes were drilled in it to balance it; during operation the wheel got wet enough to affect its center of gravity; during operation when no cup was at the top of its cycle on the wheel, water got into lower cups; water did not flow out of the cups at a uniform rate, but at a rate proportional to some function of the amount of water in the cup; when the flowrate was high, as much water splashed out of the cups as remained inside, while when the flowrate was low, friction between the wheel and the base as well as between the cups and their mounts on the wheel prevented and chaotic motion from occurring; and the flowrate of the water source was unknown, while the direction of the flow was not always strictly vertical. The fact that the wheel remained unbalanced threw any ideal calculations off considerably; from time to time the researcher added small weights to the wheel to temporarily balance it, but this never did much good. A more practical idea, however, was discovered to keep the wheel from getting wet, namely, wax the wheel with car wax. Similar to the case of the unbalanced wheel, no good method to prevent water from falling in the lower cups on the wheel presented itself. The nonuniform flowrate out of the cups was more easily accounted for; one of the physicists Bernoulli investigated flow and his equation for flow can be found in the Physics 21 text by Pasachoff. The solution to the splash problem was to try to find an intermediate flowrate that avoided splashing as well as the nonchaotic behavior. Finally, the unknown flowrate could be held constant, ignored during datataking, and then solved for during analysis.

About midsummer came a period when Dr. Alexander was out of town and the researcher was occupied with other matters. When this ended, it was already quite late in the summer. The researcher began working on datataking apparatus, including some IBM data software, with the help of Dr. Alexander. Then, as the summer semester ended, it was decided that the project should not be prolonged into the succeeding semester for various reasons, so the project was terminated.

Although the project was never completed, the researcher still considers the experience a valuable one, for in doing the project the researcher gained experience in working with the nonidealities of the real world, as well as doing theory which tries to explain the phenomenon known as chaos. During the experiment, also, the researcher's appreciation of basic chaos ideas, such as the butterfly effect and nonlinearity, was enhanced. And finally, and perhaps most importantly, the researcher became aware of the beauty to be found in chaos.


Ideal Lorenz Wheel
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.8
−0.6
−0.4
−0.2
0


Torque
4 buckets, constant velocity
torque
angle (rad)
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
0
2
4
6
8
10
12
14
16
18