Solvability of differential systems near singular points

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NEAR SINGULAR POINTS

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TABLE OF CONTENTS

HOLOMORPHIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS NEAR SINGULAR POINTS

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ABSTRACT

Functional analysis techniques are used to prove a theorem, analogous to the Harris-Sibuya-Weinberg theorem for ordinary differential equations, which yields as corollaries a number of existence theorems for holomorphic solutions of linear functional differential systems of the form

$$z^D y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z)$$

in the neighborhood of the singularity at $z = 0$.

The existence of holomorphic solutions of ordinary differential systems near a singular point has been extensively studied. An elegant treatment of this question has been given by W.A. Harris, Jr., Y. Sibuya, and L. Weinberg [5] who used functional analysis techniques to establish a theorem which includes a number of classical results as corollaries.

Several authors [1,2,3,6] have studied functional differential equations with contracting arguments in the neighborhood of a singularity at the origin. In this note we extend the results of Harris, Sibuya , and Weinberg to a class of neutral differential systems. The principal result is the following theorem.

Theorem. Let $A(z)$, $B(z)$, and $C(z)$ be $n \times n$ matrices holomorphic at $z = 0$, let $D = \mathrm{diag}(d_1,\ldots,d_n)$ with nonnegative integers d_i, and let α, $|\alpha| < 1$, be a complex constant. Then for every positive integer N sufficiently large, and every polynomial $\phi(z)$ with $z^D\phi(z)$ of degree N, there exists a polynomial $f(z)$ (depending on A,B,C,α,ϕ, and N) of degree $N-1$ such that the linear neutral-differential system

(1) $z^D y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z) + f(z)$

has a solution $y(z)$ holomorphic at $z = 0$. Further, f and y are linear and homogeneous in ϕ and

$$z^D(y-\phi) = O(z^{N+1}) \text{ as } z \to 0.$$

Proof. The proof is an application of the Banach fixed point theorem as in [5]. Let $\delta > 0$ and let X be the set of all n-vector valued functions $f = f(z)$ whose components have absolutely convergent power series expansions in $|z| \le \delta$. For $f \in X$, $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_k = (f_k^1, \ldots, f_k^n)^T$, define $||f|| = \sum_{k=0}^{\infty} |f_k| \delta^k$, where $|f_k| = \sum_{j=1}^{n} |f_k^j|$. With this norm, X is a Banach space.

For a sufficiently large positive integer N, define the mapping L_N: $X \to X$ as follows: $L_N y = g$, where

$$y(z) = (y^1(z), \ldots, y^n(z))^T, \quad g(z) = (g^1(z), \ldots, g^n(z))^T,$$

with $y^j(z) = \sum_{k=0}^{\infty} y_k^j z^k$, $g^j(z) = \sum_{k=N}^{\infty} \frac{y_k^j}{k+1-d_j} z^{k+1-d_j}$. Hence

$$(2) \qquad ||L_N y|| \le \sum_{j=1}^{n} \frac{1-d_j}{N+1-d_j} ||y||.$$

Define $\hat{y}(z) = (y^1(\alpha z), \ldots, y^n(\alpha z))^T \equiv (\hat{y}^1(z), \ldots, \hat{y}^n(z))^T$ with $\hat{y}^j(z) = \sum_{k=0}^{\infty} y_k^j \alpha^k z^k$. Also define $y^*(z) = (y^{*1}(z), \ldots, y^{*n}(z))^T$ with $y^{*j}(z) = \sum_{k=0}^{\infty} (k+1)\alpha^k y_{k+1}^j z^k$.

Note that $\hat{y}$ and y^* have absolutely convergent power series expansions for $|z| < \delta$, and also that

$$(3) \qquad ||\hat{y}|| \le ||y||.$$

Furthermore, setting $\chi(z) = \sum_{k=0}^{\infty} (\sum_{j=1}^{n} |y_k^j|) z^k$, $|z| \le \delta$, we have

$$\chi'(|\alpha| z) = \sum_{k=1}^{\infty} k(\sum_{j=1}^{n} |y_k^j|) |\alpha|^{k-1} z^{k-1}, \quad |z| \le \delta.$$

By the Cauchy integral formula,

$$|\chi'(|\alpha| z)| \le \frac{\max_{|\zeta|=\delta} |\chi(\zeta)|}{\delta^2 (1-|\alpha|)^2} = \frac{||y||}{\delta^2 (1-|\alpha|)^2}, \quad |z| \le \delta.$$

Hence

(4) $$||y^*|| = |\chi'(|\alpha|\delta)| \leq \frac{||y||}{\delta^2(1-|\alpha|)^2} .$$

If M is an n × n matrix, $M = (m^{ij})$, with elements having absolutely convergent power series expansions for $|z| \leq \delta$, $m^{ij} = \sum_{k=0}^{\infty} m_k^{ij} z^k$, then for $f \in X$ we have $Mf \in X$ and $||Mf|| \leq ||M|| \cdot ||f||$, where $||M|| = \sum_{i,j=1}^{n} (\sum_{k=0}^{\infty} |m_k^{ij}| \delta^k)$.

Let $\phi = (\phi^1, \ldots, \phi^n)^T$ be a vector polynomial with $\phi^j(z) = \sum_{k=0}^{N-d_j} \phi_k^j z^k$, and consider the functional equation in X

(5) $$y = \phi + T_N[y] ,$$

where $T_N[y] = L_N(Ay + B\hat{y} + Cy^*)$. The estimates (2)-(4) imply that for N sufficiently large, $||T_N|| < 1$, and thus there exists a unique solution $y \in X$,

$$y(\cdot;\phi) = (I-T_N)^{-1}\phi .$$

From the definition of the mapping T_N it follows that the holomorphic solution of the functional equation (5) satisfies the linear differential system of the form (1), where

(6) $$f(z) \equiv \sum_{k=0}^{N-1} f_k z^k \equiv z^D \frac{d\phi}{dz} - \sum_{k=0}^{N-1} [Ay(\cdot;\phi)]_k z^k - \sum_{k=0}^{N-1} [B\hat{y}(\cdot;\phi)]_k z^k$$
$$- \sum_{k=0}^{N-1} [Cy^*(\cdot;\phi)]_k z^k .$$

Since the coefficients of $y(\cdot;\phi)$ (and thus also $\hat{y}$ and y^*) are linear in the coefficients of ϕ , this is also true for the f_k. The proof is complete.

The corollaries below follow from the theorem similarly to the proofs of corresponding results in [5].

Corollary 1. Let d = trace D and $n-d \geq 0$. Then the system

(7) $z^D y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z)$

has at least $n-d$ linearly independent solutions holomorphic at $z = 0$.

Corollary 2. Let $A(z) = \sum_{k=0}^{\infty} A_k z^k$, $B(z) = \sum_{k=0}^{\infty} B_k z^k$, and $C(z) = \sum_{k=1}^{\infty} C_k z^k$ be convergent for $|z| < a$ $(a > 0)$, and let $y(z) = \sum_{k=0}^{\infty} y_k z^k$ be a formal solution of

(8) $zy'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z)$.

Then $y(z)$ is convergent for $|z| < a$.

Corollary 3. Let A, B, and C be as in Corollary 2, let m be a fixed integer, let $\alpha \neq 0$, and let n_{m+k} be the number of linearly independent eigenvectors corresponding to the eigenvalue $m+k$ of the matrix

$$\Omega_{m+k} = [A_0 + \alpha^{m+k} B_0 + (m+k)\alpha^{m+k-1} C_1].$$

The number N_m (≥ 0) of linearly independent solutions of the differential system (8) of the form $y = \sum_{k=0}^{\infty} y_k z^{m+k}$ satisfies

$$N_m \leq n_m + n_{m+1} + \cdots .$$

If, in addition, $B_0 = C_1 = 0$, then

$$N_m \geq \max(n_m, n_{m+1}, \ldots) .$$

Remark 1. The results extend without change to systems with several deviating arguments of the same form.

Remark 2. If $A_0 = B_0 = C_0 = 0$, then $z = 0$ is an ordinary point for the system (8). Hence by Corollary 1, there exist at least n linearly independent solutions for this system. If in addition $C_1 = 0$, then the coefficients of each formal solution are determined recursively and there exist exactly n linearly independent solutions for the system.

Remark 3. Analogous results for nonlinear systems of the form

$$z^D y'(z) = h(z, y(z), y(\alpha z), y'(\alpha z)) + f(z)$$

can be obtained by considerations similar to those in the paper of Harris [4].

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HOLOMORPHIC SOLUTIONS OF SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS*

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ABSTRACT

By use of techniques developed by Harris, Sibuya, and Weinberg, various existence theorems are obtained for holomorphic solutions of functional differential systems near singular points, including generalizations of the theorems of Bass, Lettenmeyer, and Perron. Further results on Briot-Bouquet singularities are also given.

1. Introduction. In a classical paper, F. Lettenmeyer [8] showed that a linear ordinary differential system with an irregular singular point at $z = z_0$ may have several linearly independent solutions holomorphic at z_0, and estimated the number of such solutions. The Lettenmeyer theorem was extended to nonlinear systems by R.W. Bass [1], who effected a change of variable in order to apply Wintner's fixed-point theorem for analytic mappings in a separable Hilbert space.

Recent work of W.A. Harris, Jr., Y. Sibuya, and L. Weinberg [4], [6] has graetly simplified the proofs of the theorems of Lettenmeyer and Bass and has, in addition, yielded several theorems on systems of Briot-Bouquet type as corollaries.

In a recent note [2], we developed analogues of some of these results for neutral functional differential systems (NFDS) of the form

$$(1.1) \qquad z^D \frac{dy}{dz} = f(z, y(z), y(g(z)), y'(g(z)))$$

near $z = 0$, in the case where f is linear and homogeneous and $g(z) = \alpha z$, α constant, $|\alpha| < 1$. In this paper we obtain extensions to the nonlinear case and additional results for linear systems which yield an existence theorem for holomorphic solutions near $z = \infty$ of systems of the form

$$(1.2) \qquad y'(z) = A(z)y(z) + B(z)y(\beta z) ,$$

where A and B are $n \times n$ matrices holomorphic at ∞, and β is a complex constant, $|\beta| > 1$. This equation has been studied in the scalar constant-coefficient case by T. Kato and J.B. McLeod [7]. Some results for constant-coefficient systems have been given in another paper of McLeod [9] in the case $|\beta| < 1$. However, we obtain here only an existence theorem, while Kato and McLeod obtain asymptotic estimates. We also obtain simplified proofs and extensions of several results in the Russian literature, especially those of E.I. Grudo [3].

2. Two lemmata. In what follows, we shall make much use of power series representations for holomorphic functions. The following lemma, which can be proved by induction, gives such a representation for composite functions.

Lemma 2.1. Let $\rho > 0$, and let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $h(z) = \sum_{k=1}^{\infty} h_k z^k$ be holomorphic in $|z| \leq \rho$ with $|h(z)| \leq \rho$ for $|z| \leq \rho$. Set $P_n(z) = \sum_{k=1}^{n} f_k z^k$ and $Q_n(z) = \sum_{k=1}^{n} h_k z^k$, and write

$$P_n(Q_n(z)) = \sum_{k=1}^{n^2} \Big(\sum_{m=1}^{k} \gamma_{mk}\, f_m \Big) z^k .$$

Define $\gamma_{00} = 1$ and $\gamma_{0k} = 0$, $k \geq 1$. Then

$$f(h(z)) = \sum_{k=0}^{\infty} \Big(\sum_{m=0}^{k} \frac{\gamma_{mk}}{m!} f^{(m)}(0) \Big) z^k$$

where $f^{(m)}(0) = \dfrac{d^m f(0)}{dz^m}$, and the γ_{mk} are functions of the coefficients of $h(z)$ alone.

We shall also need to solve simultaneous systems of determining equations. For a linear differential system, the determining system is linear and the well-known theory of linear algebraic equations applies. In the nonlinear case, the determining system will be nonlinear, and the following lemma will be used:

Lemma 2.2. Consider the system of simultaneous equations

$$\begin{gathered} G_1(z_1,\ldots,z_n) = 0 \\ \vdots \\ G_m(z_1,\ldots,z_n) = 0, \end{gathered}$$

where G_k, $k = 1,\ldots,m$ are holomorphic functions all vanishing at the origin, but none vanishing identically. If $m<n$, then this system has at least an $(n-m)$-parameter algebroid family of solutions.

This is a special case of a theorem of Weierstrass; for a proof, see Osgood [10], pp. 132-133.

3. The nonlinear case. Let X denote the set of all n-vector valued functions $f = f(z)$ whose components have absolutely convergent power series expansions in $|z| \leq \delta$. For $f \in X$, $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_k = (f_k^1,\ldots,f_k^n)^T$, define a norm by

$$||f|| = \sum_{k=0}^{\infty} (\sum_{j=1}^{n} |f_k^j|)\delta^k = \sum_{j=1}^{n} (\sum_{k=0}^{\infty} |f_k^j|\delta^k).$$

It is easy to see that $(X,||\cdot||)$ is a Banach space. A central result is the following:

Theorem 3.1. Let $f(z,y,u,v)$ be an n-vector function of z, $y = (y^1,\ldots,y^n)^T$, $u = (u^1,\ldots,u^n)^T$, and $v = (v^1,\ldots,v^n)^T$ holomorphic in a neighborhood of the origin in $(3n+1)$-dimensional complex Euclidean space. Let D be an $n \times n$ matrix,

$D = \mathrm{diag}(d_1,\ldots,d_n)$ with nonnegative integers d_i. Let $h(z) = \sum_{k=1}^{\infty} h_k z^k$ be a scalar function holomorphic at $z = 0$ with $|h_1| < \alpha$, for some α, $0 < \alpha < 1$. Then for each positive integer N sufficiently large and each sufficiently small polynomial $\phi(z)$ with $z^D\phi(z)$ of degree N, there exists a polynomial $p(z;\phi)$ in z of degree N-1 with coefficients that depend on f, h, N, and ϕ, such that the nonlinear NFDS

$$z^D y'(z) = f(z,y(z),y(h(z)),y'(h(z))) + p(z;\phi) \tag{3.1}$$

has a solution y(z) holomorphic at z = 0. Further, y is homogeneous in ϕ and $z^D(y-\phi) = O(z^{N+1})$.

Proof. For N sufficiently large, define $L_N\colon X \to X$ by

$$L_N y = g, \quad y = (y^1(z),\ldots,y^n(z))^T,$$

$$g = (g^1(z),\ldots,g^n(z))^T, \text{ with}$$

$$y^j(z) = \sum_{k=0}^{\infty} y_k^j z^k, \quad g^j(z) = \sum_{k=N}^{\infty} \frac{y_k^j\, z^{k+1-d_j}}{k+1-d_j}.$$

Hence

$$||L_N y|| \le \left(\sum_{j=1}^{n} \left|\frac{1-d_j}{N+1-d_j}\right|\right)||y||. \tag{3.2}$$

Define $\tilde{y}(z) = y(h(z))$. Then by Lemma 2.1, $\tilde{y}^j(z) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} \gamma_{mk} y_m^j\right) z^k$. From this it follows that

$$||\tilde{y}|| \le ||y||. \tag{3.3}$$

Also define $\hat{y}(z) = (\hat{y}^1(z),\ldots,\hat{y}^n(z))^T$, with $\hat{y}^j(z) =$

$\sum_{k=0}^{\infty}(k+1)y_{k+1}^{j}(h(z))^{k}$. By Lemma 2.1,

$$\hat{y}^{j}(z) = \sum_{k=0}^{\infty}\Big(\sum_{m=0}^{k}\gamma_{mk}(m+1)y_{m+1}^{j}\Big)z^{k} .$$

Let $\delta > 0$ be chosen sufficiently small that $\sum_{k=1}^{\infty}|h_k|\delta^k < \alpha\delta$.
Set $\chi(z) = \sum_{j=1}^{n}\big(\sum_{k=0}^{\infty}|y_k^j|z^k\big)$ and $\eta(z) = \sum_{k=1}^{\infty}|h_k|z^k$ for $|z| \leq \delta$.
Then $||\hat{y}|| \leq |\chi'(\eta(\delta))|$ and by the Cauchy integral formula
$|\chi'(\eta(z))| \leq \dfrac{||y||}{\delta(1-\alpha)^2}$ for $|z| \leq \delta$, hence

$$(3.4) \qquad ||\hat{y}|| \leq \frac{1}{\delta(1-\alpha)^2}||y|| .$$

Let $\phi = (\phi^1,\dots,\phi^n)^T$ be the vector polynomial with
$\phi^j(z) = \sum_{k=0}^{N-d_j}\phi_k^j z^k$ and consider the functional equation in X

$$(3.5) \qquad y = \phi + L_N f(\cdot,y,\tilde{y},\hat{y}) \equiv \phi + \tilde{L}_N f(\cdot,y) .$$

The estimates (3.2)-(3.4) imply that, for N sufficiently large, $\tilde{L}_N f$ satisfies a Lipschitz condition in y with Lipschitz constant less than 1. Hence for sufficiently small ϕ there exists a unique solution of the functional equation (3.5), $y = y(z;\phi) \in X$.

It follows from the definition of L_N that the holomorphic solution y of the functional equation (3.5) satisfies the NFDS $z^D y'(z) = f(z,y,\tilde{y},\hat{y}) + p(z;\phi)$, where

$$p(z;\phi) = z^D\phi'(z) - \sum_{k=0}^{N-1}[f(z,y(z;\phi),y(h(z);\phi),y'(h(z);\phi)]_k\, z^k.$$

The proof is complete.

Remark 3.2. In case f(z,u,v,w) in Theorem 3.1 is linear and homogeneous in u,v,w, the solution y(z) will be linear and homogeneous in ϕ, see [2]. Further, in this case, the equation (3.5) has the form

(3.6) $$y = \phi + T_N[y], \text{ where } T_N[y] = L_N[Ay + B\tilde{y} + C\hat{y}],$$

and $[I-T_N]^{-1}$ will exist for N sufficiently large.

Theorem 3.3. Let f(z,y,u,v) and h(z) be as in Theorem 3.1, let $f(z,0,0,0) \equiv 0$, and let d = trace D and $n-d \geq 0$. Then the system

(3.7) $$z^D y'(z) = f(z,y(z),y(h(z)),y'(h(z)))$$

has at least an (n-d)-parameter algebroid family of solutions holomorphic at z = 0.

Proof. Consider the system of nN equations (determining equations) in the nN + n-d unknowns ϕ_k^j, represented by

(3.8) $$p(z;\phi) = 0.$$

Since $f(z,0,0,0) \equiv 0$, $p(z;0) = 0$, and hence by Lemma 2.2 there exists at least an (n-d)-parameter algebroid family of solutions of (3.8). For each ϕ thus obtained, Theorem 3.1 ensures the existence of a corresponding solution of (3.7).

Theorem 3.7. Let f and h be as in Theorem 3.1. Let $y(z) = \sum_{k=0}^{\infty} y_k z^k$ be a formal solution of the Briot-Bouquet equation

$$(3.9) \qquad zy'(z) = f(z,y(z),y(h(z)),zy'(h(z)))$$

in the sense of equality of formal power series. Then $y(z)$ is convergent in a neighborhood of $z = 0$.

Proof. Since $d_i = 1$, $i = 1,\ldots,n$, $\phi(z) = \sum_{k=0}^{N-1} \phi_k z^k$ and $y^* = \phi + O(z^N)$, where y^* is the holomorphic solution whose existence is guaranteed by Theorem 3.1. The determining equations can be represented as

$$(3.10) \qquad [f(z,y^*(\cdot;\phi),\tilde{y}^*(\cdot;\phi),z\hat{y}^*(\cdot;\phi))]_k - k\phi_k = 0,$$

$k = 0,1,\ldots,N-1$. However, the equations for the existence of a formal solution are

$$(3.11) \qquad [f(z,y(z),y(h(z)),zy'(h(z)))]_k - ky_k = 0,$$

$k = 0,1,\ldots,$ and hence the first N equations of (3.11) are the same as (3.10). The condition that $h(0) = 0$ implies that y_k is uniquely determined by the preceding coefficients, and thus $y^* = y(z)$, and the formal solution converges in a neighborhood of $z = 0$.

Corollary. Let $A(z) = \sum_{k=0}^{\infty} A_k z^k$, $B(z) = \sum_{k=0}^{\infty} B_k z^k$ and $C(z) = \sum_{k=1}^{\infty} C_k z^k$ be $n \times n$ matrices holomorphic at $z = 0$. Let $g(z) = \sum_{k=0}^{\infty} g_k z^k$ be an n-vector function holomorphic at $z = 0$, and let $h(z)$ be as in Theorem 3.4. Suppose that no eigenvalue of $A_0 + B_0$ is a nonnegative integer. Then the Briot-

Bouquet system

$$(3.12) \qquad zy'(z) = A(z)y(z) + B(z)y(h(z)) + C(z)y'(h(z)) + g(z)$$

has a solution holomorphic at $z = 0$.

Proof. The system (3.12) is of the form (3.9); the equations for the existence of a formal solution of the form $y(z) = \sum_{k=0}^{\infty} y_k z^k$ are

$$\begin{aligned} U_0 y_0 \qquad\qquad\qquad\qquad\qquad\qquad &= g_0 \\ U_1 y_0 + (U_0 - I)y_1 \qquad\qquad\qquad\quad &= g_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad & \\ U_k y_0 + \cdots + U_1 y_{k+1} + (U_0 - kI)y_k &= g_k \end{aligned}$$

where the $n \times n$ matrices U_k depend on the coefficients of $A(z)$, $B(z)$, $C(z)$ and $h(z)$, and in particular, $U_0 = A_0 + B_0$. The hypotheses of the corollary guarantee the existence of a formal solution which converges by Theorem 3.4.

4. The case trace $D > n$. The determining equation will now be used to prove existence of holomorphic solutions of systems with trace $D > n$. In particular, consider the system

$$(4.1) \qquad z^2 y'(z) = A(z)y(z)$$

where $A(z) = \sum_{k=0}^{\infty} A_k z^k$ is an $n \times n$ matrix holomorphic at $z = 0$. The polynomial $\phi(z)$ in Theorem 3.1 is of the form $\sum_{k=0}^{N-2} \phi_k z^k$

since $d_i = 2$, $i = 1,\ldots,n$, and the determining equation is $z^2\phi'(z) = \sum_{k=0}^{N-2} (A\phi)_k z^k + (Ay(z;\phi))_{N-1} z^{N-1}$. This equation is equivalent to the system

$$
(4.2)\qquad
\begin{aligned}
&A_0\phi_0 &&= 0\\
&A_1\phi_0 + A_0\phi_1 &&= 0\\
&A_2\phi_0 + A_1\phi_1 + A_0\phi_2 &&= \phi_1\\
&\quad\vdots\\
&A_{N-2}\phi_0 + A_{N-3}\phi_1 + \cdots + A_1\phi_{N-3} + A_0\phi_{N-2} &&= (N-3)\phi_{N-3}\\
&A_{N-1}\phi_0 + A_{N-2}\phi_1 + \cdots + A_1\phi_{N-2} + A_0 y_{N-1} &&= (N-2)\phi_{N-2}.
\end{aligned}
$$

From Remark 3.2, y_{N-1} in the last equation in (4.2) can be found using $y(\cdot;\phi) = (I-L_NA)^{-1}\phi$. Since for large enough N, $||L_NA|| < 1$, the mapping $(I-L_NA)^{-1}$ is given by the convergent series (see Taylor [11], p. 164) $I + L_NA + (L_NA)^2 + \ldots$. Define the matrices $R_{N,1}, R_{N,2},\ldots,R_{N,N-1}$ by

$$R_{N,k} = A_{N-k} + A_0\Big[\sum_{p=0}^{\infty} (L_NA)^p z^{k-1}\Big]_{N-1},\ k = 1,\ldots,(N-2);$$

$$R_{N,N-1} = A_1-(N-2)I + A_0\Big[\sum_{p=0}^{\infty} (L_NA)^p z^{N-2}\Big]_{N-1},$$

where $[\sum_{p=0}^{\infty} (L_NA)^p z^{k-1}]_{N-1}$ is the coefficient of $\phi_{k-1}z^{N-1}$ in the expansion $y(z) = (I-L_NA)^{-1}(\sum_{\ell=0}^{N-2} \phi_\ell z^\ell)$. Then the system (4.2) can be written as

$$(4.3)\qquad \begin{pmatrix} A_0 & 0 & & & 0 \\ A_1 & A_0 & 0 & & 0 \\ A_2 & A_1 - I & A_0 & & \\ & & & & \\ A_{N-2} & A_{N-3} & & A_1-(N-3)I & A_0 \\ R_{N,1} & R_{N,2} & & R_{N,N-2} & R_{N,N-1} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \\ \phi_{N-2} \end{pmatrix} = 0,$$

which represents nN linear equations in the n(N-1) unknowns ϕ_k^j. The matrix in (4.3) will be called the determining matrix.

Theorem 4.1. There exists at least one nontrivial solution y(z) holomorphic at z = 0 of the system

$$z^2 y'(z) = A(z)y(z)$$

if

$$\text{rank} \begin{pmatrix} A_0 & & & 0 \\ A_1 & A_0 & & \\ & & & \\ A_k & A_{k-1} & A_1-(k-1)I & A_0 \end{pmatrix} < kn$$

for at least one k, k = 1,2,... .

Proof. The determining equation (4.3) has nontrivial solutions, which guarantee nontrivial solutions of (4.1), if the rank of the determining matrix is less than n(N-1). The hypothesis of the theorem ensures this.

Example. Consider the system

$$z^2 \frac{dy}{dz} = A(z)y(z), \tag{4.4}$$

where $y = (y_1, y_2, y_3)^T$, and

$$A(z) = \begin{pmatrix} z \sin z & z & z \tan z \\ z^3 & \cos z & 0 \\ z^2 & 0 & z^2 \end{pmatrix}$$

Then the 6×6 block in the upper left-hand corner of the determining matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 2. Hence there exists a nontrivial solution of (4.4) holomorphic at $z = 0$ by Theorem 4.1.

Note that the equation (4.4) has an irregular singular point at $z = 0$, since matrix $A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not nilpotent, see [5].

The last n rows of the determining matrix, represented by the $R_{N,k}, k = 1,\ldots,(N-1)$, are observed to have a simpler form under additional hypotheses:

Theorem 4.2. If in the coefficients of the system

(4.1), the matrix A_0 is nilpotent of index $\ell \geq 2$ and if $A_0A_1 = A_0A_2 = \cdots = A_0A_{k-1} = 0$, then the matrices $R_{N,k}$ are given by

$$R_{N,k} = A_{N-k} + A_0([(I + L_NA)z^{k-1}]_{N-1}),$$

$k=1,2,\ldots,(N-2)$; and

$$R_{N,N-1} = A_1-(N-2)I + A_0([(I + L_NA)z^{N-2}]_{N-1}).$$

We omit the proof.

Theorem 4.1 can be generalized to the NFDS

$$z^2y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z) \tag{4.4}$$

where $A(z) = \sum_{k=0}^{\infty} A_kz^k$, $B(z) = \sum_{k=0}^{\infty} B_kz^k$, $C(z) = \sum_{k=0}^{\infty} C_kz^k$ are $n \times n$ matrices holomorphic at $z = 0$ and α is a complex constant, $|\alpha| < 1$. The determining equation is then

$$(A_0+B_0)\phi_0 + C_0\phi_1 = 0$$

$$(A_1+B_1)\phi_0 + (A_0+\alpha B_0+C_1)\phi_1 + 2\alpha C_0\phi_2 = 0$$

$$(A_2+B_2)\phi_0 + (A_1+\alpha B_1+C_2)\phi_1 + (A_0+\alpha^2B_0+2\alpha C_1)\phi_2 + 3\alpha^2C_0\phi_3 = \phi_1$$

$$\vdots$$

$$(A_{N-2}+B_{N-2})\phi_0 +\ldots+ (A_0+\alpha^{N-2}B_0+(N-2)\alpha^{N-3}C_1)\phi_{N-2} + (N-1)\alpha^{N-2}C_0y_{N-1} = (N-3)\phi_{N-3}$$

$$(A_{N-1}+B_{N-1})\phi_0 + \ldots + (A_0+\alpha^{N-1}B_0+(N-1)\alpha^{N-2}C_1)y_{N-1}$$
$$+ N\alpha^{N-1}C_0 y_N = (N-2)\phi_{N-2}$$

The coefficients y_{N-1} and y_N can be found using $y(\cdot;\phi) = (I-T_N)^{-1}\phi$ from (3.6). The last 2n rows of the determining matrix are very complicated; we shall not write the matrix explicitly.

The following result is analogous to Theorem 4.1 and is proved the same way:

Theorem 4.3. The NFDS (4.4) has a nontrivial solution holomorphic at $z = 0$ if

$$\text{rank}\begin{pmatrix} A_0+B_0 & C_0 & & & \\ A_1+B_1 & A_0+\alpha B_0+C_1 & 2\alpha C_0 & & \\ \vdots & & & & \\ A_k+B_k & A_{k-1}+\alpha B_{k-1}+C_k & \cdots & A_1+\alpha^{k-1}B_1+(k-1)\alpha^{k-2}C_2 & \\ & & & -(k-1)I \;\cdots & (k+1)^k\alpha C_0 \end{pmatrix} < kn$$

for at least one k, $k = 1,2,\ldots,(N-3)$.

Now consider the equation (4.4) in the case where $C(z) \equiv 0$. By making the change of variable $z = \frac{1}{\zeta}$, the equation (4.4) becomes

$$\frac{-dy}{d\zeta} = \sum_{k=0}^{\infty} A_k\left(\frac{1}{\zeta}\right)^k y(\zeta) + \sum_{k=0}^{\infty} B_k\left(\frac{1}{\zeta}\right)^k y\left(\frac{\zeta}{\alpha}\right)$$

or

$$\text{(4.5)} \qquad \frac{dy}{d\zeta} = \tilde{A}(\zeta)y(\zeta) + \tilde{B}(\zeta)y\left(\frac{1}{\alpha}\zeta\right)$$

Here $\tilde{A}$ and $\tilde{B}$ are holomorphic for $|\zeta|$ sufficiently large.

Since $|\alpha| < 1$, $|\frac{1}{\alpha}| > 1$ and so (4.5) is an equation of advanced type with a singularity at ∞. Hence the following theorem is an immediate corollary of Theorem 4.3.

Theorem 4.4. Let $A(z)$ and $B(z)$ be $n \times n$ matrices such that the series $A(z) = \sum_{k=0}^{\infty} A_k z^{-k}$ and $B(z) = \sum_{k=0}^{\infty} B_k z^{-k}$ converge. Let β be a complex constant with $|\beta| > 1$. Then the equation

$$(4.5) \qquad \frac{dy}{dz} = A(z)y(z) + B(z)y(\beta z)$$

has a nontrivial solution holomorphic at $z = \infty$ if

$$\operatorname{rank} \begin{pmatrix} A_0+B_0 & 0 & & & \\ A_1+B_1 & A_0+\frac{1}{\beta}B_0 & & & \\ \vdots & & & & \\ A_k+B_k & A_{k-1}+\frac{1}{\beta}B_{k-1} & \cdots & A_1+(\frac{1}{\beta})^{k-1}B_1+(k-1)I & \\ & & & & A_0+(\frac{1}{\beta})^{k}B_0 \end{pmatrix} < kn$$

for at least one $k = 1,2,\ldots,(N-2)$.

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VITA

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