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Abstract. A shooting method was developed to study eigenvalue problems derived from Schrödinger equation. The challenging problem, the two-dimensional hydrogen system with the logarithmic potential function, was successfully solved by the shooting method. But no complete proof was given for its rationale and correctness. This paper not only gives the complete proof for the shooting method, but also generalizes it to solve a large class of eigenvalue problems. In a certain sense, the shooting method proves more effective numerically and more powerful theoretically than the classical functional analysis approach.

I. INTRODUCTION

Recently, a shooting method was developed to study eigenvalue problems in quantum mechanics [1,2,3]. Originally, the method served to solve some specific eigenvalue problems associated with the Schrödinger operator, but as was pointed out [4], the shooting method ought to prove general in nature and be applicable for solving the Schrödinger equation with a wide variety of potential energy functions. In this paper, we will go far beyond even that: we will establish the validity of the shooting method for the following general problem.

Generalized Eigenvalue Problem. Let $Q(v, \lambda)$ be a continuous function of $v \in (-\infty, \infty)$ and $\lambda \in I$, where $I = [a, b]$ is a closed interval. Find all λ and corresponding nontrivial bounded solutions $R_\lambda(v)$ to

$$\frac{d^2R}{dv^2} = Q(v, \lambda)R. \tag{1}$$

It is not rigorous but convenient to call such numbers λ and corresponding solutions $R_\lambda(v)$ eigenvalues and eigenfunctions of (1) under the bounded condition, respectively.

Definitions. A function $R(v)$ defined on the real line is left-bounded if it is bounded when $v \in (-\infty, 0]$ and right-bounded if it is bounded when $v \in [0, +\infty)$.

Obviously, $R(v)$ is bounded on the whole real line if and only if it is both left and right- bounded. It leads to the basic idea of the shooting method. The idea is very simple in nature: first to find all the right-bounded solutions of (1) and then to select left-bounded ones among these solutions. To make it more specific, we need to impose some additional assumptions for $Q(v, \lambda)$.

A1. $Q(v, \lambda)$ has a continuous partial derivative with respect to λ and

$$\frac{\partial Q(v,\lambda)}{\partial \lambda} \le 0 \quad \left[\frac{\partial Q(v,\lambda)}{\partial \lambda} \ge 0\right] \quad \text{for all } v \in (-\infty, \infty) \text{ and } \lambda \in I.$$

A2. The set $\{\lambda \in I : \frac{\partial Q(v,\lambda)}{\partial \lambda} = 0 \text{ for all } v \in (-\infty, \infty)\}$ is of measure zero.

A3. $\int_{-\infty}^{+\infty} |v| Q_-(v, \lambda)dv < \infty$ for all $\lambda \in I$, where

$$Q_-(v, \lambda) = \begin{cases} -Q(v, \lambda) & Q(v, \lambda) < 0; \\ 0 & Q(v, \lambda) \ge 0. \end{cases}$$

As will be proved later, *A3* assures that for every λ, there exists a nontrivial right-bounded solution to (1), unique except for a nonzero factor. Let $Z(\lambda)$ be the number of zeros of such a right-bounded solution for every $\lambda \in I$. Several authors [1,2,3] notice that every discontinuity of $Z(\lambda)$ is an eigenvalue of (1) in some special cases. Actually, the discontinuities give all the eigenvalues of $Z(\lambda)$ provided $Q(v, \lambda)$ satisfies *A1*, *A2* and *A3*. We will establish the following four theorems.

Theorem A. If $Q(v,\lambda)$ satisfies *A3*, then for every λ, there exists a nontrivial right-bounded [left-bounded] solution to (1), unique except for a nonzero factor.

Theorem B. If $Q(v,\lambda)$ satisfies *A1* and *A3*, then every discontinuity of $Z(\lambda)$ is an eigenvalue of (1).

Theorem C. If $Q(v,\lambda)$ satisfies *A1*, *A2* and *A3*, every eigenvalue of (1) is a discontinuity of $Z(\lambda)$.

Theorem D. If $Q(v,\lambda)$ satisfies *A1* and *A3*, then $Z(\lambda)$ is a nondecreasing [nonincreasing] function in λ.

II. PROOF OF THEOREMS A-D

The following two lemmas are useful tools in our discussion later and can be easily proved in various ways.

Lemma E. Let ϕ and ψ be nontrivial solutions to $y'' = p(v)y$ and $y'' = q(v)y$ on $[v_0,\infty)$ $[(-\infty,v_0]]$ respectively, satisfying the same initial conditions: $\phi(v_0) = \psi(v_0)$ and $\phi'(v_0) = \psi'(v_0)$. If $p(v) \le q(v)$ and $\phi(v) > 0$ for all $v > v_0$ $[v < v_0]$, then

1° $\dfrac{\psi(v)}{\phi(v)} \ge \dfrac{\psi(v_1)}{\phi(v_1)}$ for all $v > v_1 > v_0$ $[v < v_1 < v_0]$.

2° $\psi(v) \ge \phi(v)$ for all $v > v_0$ $[v < v_0]$.

Lemma F. Let $q(v)$ be a non-positive continuous function defined on $[v_0,\infty)$ $[(-\infty,v_0]]$ which satisfies

$$\int_{v_0}^{\infty} |(v-v_0)q(v)|dv < 1 \quad \left[\int_{-\infty}^{v_0} |(v-v_0)q(v)|dv < 1\right].$$

Let $R(v)$ be the solution to

$$\frac{d^2R}{dv^2} = q(v)R$$

satisfying the initial conditions $R(v_0) = 0$ and $R'(v_0) = 1$ $[R'(v_0) = -1]$. Then

$$k|v-v_0| \le R(v) \le |v-v_0| \text{ for all } v > v_0 \ [v < v_0],$$

where $k = 1 - \displaystyle\int_{v_0}^{\infty} |(v-v_0)q(v)|dv$ $\left[k = 1 - \displaystyle\int_{-\infty}^{v_0} |(v-v_0)q(v)|dv\right]$.

Suppose $Q(v,\lambda)$ satisfies *A1* in our following arguments. For every $\lambda \in I$, there exists a positive number $v_0 = v_0(\lambda)$ such that

$$\int_{v_0}^{\infty} vQ_-(v,\lambda)dv < \frac{1}{2}.$$

Let F_λ and G_λ be the solutions to (1) satisfying the initial conditions $F_\lambda(v_0) = 1$, $F_\lambda'(v_0) = 0$ and $G_\lambda(v_0) = 0$, $G_\lambda'(v_0) = 1$, respectively. Compare G_λ with g_λ, the solution to

$$\frac{d^2R}{dv^2} = -Q_-(v,\lambda)R$$

satisfying the initial conditions $g_\lambda(v_0) = 0$, $g_\lambda(v_0) = 1$.

Since $\displaystyle\int_{v_0}^{\infty} (v-v_0)Q_-(v,\lambda)dv \le \int_{v_0}^{\infty} vQ_-(v,\lambda)dv \le \frac{1}{2}$, according to *Lemma F*,

$$\frac{v-v_0}{2} \le g_\lambda(v) \le v - v_0 \text{ for all } v > v_0,$$

and according to *Lemma E*,

$$\frac{v-v_0}{2} \le g_\lambda(v) \le G_\lambda(v) \text{ for all } v > v_0 \text{ and } \lambda \in I.$$

Thus $\int_v^\infty G_\lambda^{-2}(w)dw < \infty$ for all $v > v_0$. Define

$$R_\lambda(v) = G_\lambda(v)\int_v^\infty G_\lambda^{-2}(w)dw \text{ for } v \in (v_0, \infty).$$

According to *Lemma E*,

$$\frac{G_\lambda(v_2)}{g_\lambda(v_2)} \geq \frac{G_\lambda(v_1)}{g_\lambda(v_1)} \text{ for all } v_2 > v_1 > v_0.$$

Therefore

$$\frac{\int_v^\infty g_\lambda^{-2}(w)dw}{\int_v^\infty G_\lambda^{-2}(w)dw} \geq \frac{G_\lambda^2(v)}{g_\lambda^2(v)} \geq \frac{G_\lambda(v)}{g_\lambda(v)} \geq 1 \text{ for all } v > v_0.$$

Thus

$$R_\lambda(v) = G_\lambda(v)\int_v^\infty G_\lambda^{-2}(w)dw \leq g_\lambda(v)\int_v^\infty g_\lambda^{-2}(w)dw \leq (v - v_0)\int_v^\infty \frac{4}{(t-v_0)^2}dt = 4 \text{ for all } v > v_0,$$

namely, $R_\lambda(v)$ is right-bounded. It is not difficult to check that R_λ is a solution to (1) on (v_0, ∞). Extending it to a solution on the whole real line, we get a right-bounded solution to (1). Consequently, to every $\lambda \in I$, there corresponds a nontrivial right-bounded solution to (1). Obviously, for every λ, such a solution is unique except for a constant factor: in fact, it can be written as

$$c\left[F_\lambda(v) - G_\lambda(v)\lim_{w\to\infty}\frac{F_\lambda(w)}{G_\lambda(w)}\right].$$

Symmetrically, we can draw the same conclusions regarding left-bounded solutions of (1), and we have *Theorem A*.

The next lemma follows immediately.

Lemma G. If $Q(v,\lambda)$ satisfies *A3*, then a solution of (1) on $(-\infty,\infty)$, say $R_\lambda(v)$, is right-bounded [left-bounded] if and only if

$$\lim_{v\to+\infty}\frac{R_\lambda(v)}{G_\lambda(v)} = 0 \left[\lim_{v\to-\infty}\frac{R_\lambda(v)}{G_\lambda(v)} = 0\right].$$

Although we have introduced the function $Z(\lambda)$, we cannot take it for grant that it is a regular function which is finite everywhere. Namely, it may take ∞ as its value somewhere. But the following theorem eliminates such possibilities.

Theorem H. If $Q(v,\lambda)$ satisfies *A3*, then every nontrivial solution to (1) has only finitely many zeros.

Proof. Let R_λ be a nontrivial solution to (1). If it is not right-bounded, obviously, it must have finitely many zeros in $[0,\infty)$. Otherwise, if it is right-bounded, since

$$\frac{d[F_\lambda(v)/G_\lambda(v)]}{dv} = -\frac{1}{G_\lambda^2(v)} < 0 \text{ for } G_\lambda(v) \neq 0, \tag{2}$$

$$\frac{F_\lambda(v)}{G_\lambda(v)} > \lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)} \text{ for all } v > v_0.$$

Therefore,

$$R_\lambda(v) = c\left[F_\lambda(v) - G_\lambda(v)\lim_{w\to\infty}\frac{F_\lambda(w)}{G_\lambda(w)}\right] \neq 0 \text{ for all } v > v_0.$$

Thus, R_λ has finitely many zeros in $[0,\infty)$. Similarly, we can show R_λ has finitely many zeros in $(-\infty, 0)$. ∎

Let $T_\lambda(v)$ be the right-bounded solution to (1) satisfying the initial condition $T_\lambda(v_0) = 1$. It is easy to show

$$T_\lambda(v) = F_\lambda(v) - G_\lambda(v)\lim_{w\to\infty}\frac{F_\lambda(w)}{G_\lambda(w)}.$$

Notice the choice of the number v_0 depends on λ in our arguments. It is hard to obtain properties of F_λ, G_λ and T_λ as functions of λ. However, if $Q(v,\lambda)$ satisfies both *A1* and *A3*, we can choose a constant number $v_0 > 0$ such that

$$\int_{v_0}^{\infty} vQ_-(v,\lambda)dv < \frac{1}{2} \text{ for all } \lambda \in I = [a,b].$$

In fact, we can simply choose $v_0 > 0$ such that

$$\int_{v_0}^{\infty} vQ_-(v,b)dv < \frac{1}{2}.$$

Since $Q(v,\lambda)$ is nonincreasing in $\lambda \in I$, $Q_-(v,\lambda)$ is nondecreasing in λ. Consequently, for all $\lambda \in I$,

$$\int_{v_0}^{\infty} vQ_-(v,\lambda)dv \le \int_{v_0}^{\infty} vQ_-(v,b)dv < \frac{1}{2}.$$

Lemma I. If $Q(v,\lambda)$ satisfies *A1* and *A3*, then the function $\lim_{v\to+\infty} \frac{F_\lambda(v)}{G_\lambda(v)}$ is continuous with respect to $\lambda \in I$.

Proof. According to (2),

$$\left|\frac{F_\lambda(v_2)}{G_\lambda(v_2)} - \frac{F_\lambda(v_1)}{G_\lambda(v_1)}\right| \le \frac{1}{k^2}\left(\frac{1}{v_1 - v_0} - \frac{1}{v_2 - v_0}\right) \text{ for all } \lambda \in I \text{ and } v_2 > v_1 > v_0.$$

Thus $\frac{F_\lambda(v)}{G_\lambda(v)}$ converges uniformly with respect to λ. Since it is obvious that $\frac{F_\lambda(v)}{G_\lambda(v)}$ is continuous for every fixed $v > v_0$, it then follows that

$$\lim_{v\to+\infty} \frac{F_\lambda(v)}{G_\lambda(v)}$$

is a continuous function of λ. ∎

The following lemma shows a kind of stability for unbounded solutions of (1).

Lemma J. Let $R_\lambda(v,x,y)$ denote the solution to (1) satisfying the initial conditions $R_\lambda(a) = x$ and $R'_\lambda(a) = y$, where a is a constant, and suppose

$$\lim_{v\to+\infty} R_\lambda(v,x,y) = \infty \quad \left[\lim_{v\to-\infty} R_\lambda(v,x,y) = \infty\right]$$

at a certain point $(\lambda, x, y) = (\lambda_0, x_0, y_0)$. Then there is a neighborhood O of (λ_0, x_0, y_0) in which

1° $\lim_{v\to+\infty} R_\lambda(v,x,y) = \infty$ $\left[\lim_{v\to-\infty} R_\lambda(v,x,y) = \infty\right]$.

2° there exists a number v_1 such that $R_\lambda(v,x,y) \ne 0$ for all $v \ge v_1 [v \le v_1]$.

Proof. Without loss of generality, we suppose

$$\lim_{v\to+\infty} R_{\lambda_0}(v,x_0,y_0) = +\infty.$$

Choose $v_1 > v_0$ such that $R_{\lambda_0}(v,x_0,y_0) > 0$ for all $v \ge v_1$. Let $P_\lambda(v)$ and $S_\lambda(v)$ be the solutions to (1) with the initial conditions $P_\lambda(v_1) = 1$, $P'_\lambda(v_1) = 0$ and $S_\lambda(v_1) = 0$, $S'_\lambda(v_1) = 1$, respectively. Obviously, P_λ and S_λ have the same properties as we described before for F_λ and G_λ. Observe that

$$\frac{R_\lambda(v,x,y)}{S_\lambda(v)} = R_\lambda(v_1,x,y)\frac{P_\lambda(v)}{S_\lambda(v)} + R'_\lambda(v_1,x,y),$$

where $R_\lambda(v_1, x, y)$ and $R'_\lambda(v_1, x, y)$ are continuous functions of (λ, x, y). According to *Lemma I*, $\lim_{v \to +\infty} \frac{P_\lambda(v)}{S_\lambda(v)}$ is a continuous function of λ. So $\lim_{v \to +\infty} \frac{R_\lambda(v, x, y)}{S_\lambda(v)}$ is a continuous function of (λ, x, y). By *Lemma G*,

$$\lim_{v \to +\infty} \frac{R_{\lambda_0}(v, x_0, y_0)}{S_{\lambda_0}(v)} > 0.$$

Thus there exists a neighborhood O of (λ_0, x_0, y_0) in which

$$\lim_{v \to +\infty} \frac{R_\lambda(v, x, y)}{S_\lambda(v)} > 0 \text{ and } R_\lambda(v_1, x, y) > 0.$$

Conclusion 1° is valid for $(\lambda, x, y) \in O$ by *Lemma G*, and according to (2),

$$\frac{R_\lambda(v, x, y)}{S_\lambda(v)} \geq \lim_{v \to +\infty} \frac{R_\lambda(v, x, y)}{S_\lambda(v)} > 0 \text{ for all } v \geq v_1.$$

Thus 2° is also satisfied for $(\lambda, x, y) \in O$. ▌

Now we can prove *Theorem B*.

It suffices to prove that any λ for which T_λ is unbounded cannot belong to $D(Z)$. Let T_{λ_0} be unbounded. Since $T_\lambda(v_0) = 1$ and $T'_\lambda(v_0) = -\lim_{v \to +\infty} \frac{F_\lambda(v)}{G_\lambda(v)}$ where the latter is continuous with respect to λ by *Lemma I*, there must be a neighborhood O_1 of λ_0 and a number v_1 such that $T_\lambda(v) \neq 0$ for all $\lambda \in O_1$ and $v \leq v_1$ (see *Lemma E*). Thus all the zeros of such T_λ must lie in $[v_1, v_0]$. According to the continuity of $T_\lambda(v)$ on $I \times [v_1, v_0]$, there is a neighborhood O_2 of λ_0 in which all the T_λ have the same number of zeros on $[v_1, v_2]$ as $\lambda \in O_2$. Therefore, all the T_λ, $\lambda \in O_1 \cap O_2$, have the same number of zeros, so $Z(\lambda)$ is continuous at λ_0. ▌

The shooting method produces all the eigenvalues and eigenfunctions of (1) only if the converse of *Theorem B* holds. The converse, however, is not necessarily true if no additional conditions are imposed on $Q(v, \lambda)$ other than *A1* and *A3*. A counterexample can be simply constructed as

$$Q(v, \lambda) = 4v^2 - 2, \; v \in (-\infty, +\infty) \text{ and } \lambda \in [-1, 1].$$

Obviously, $Q(v, \lambda)$ satisfies *A1* and *A3*, and it is easy to check that $T_\lambda(v) = e^{-v^2}$ are bounded solutions of (1) for all $\lambda \in [-1, 1]$, but $Z(\lambda) = 0$ has no discontinuities. Nevertheless, we can prove the converse if *A2* is added. That is *Theorem C*.

The next lemma follows immediately from the definition of T_λ and *Theorem A*.

Lemma K. T_λ is bounded if and only if

$$\lim_{v \to -\infty} \frac{F_\lambda(v)}{G_\lambda(v)} = \lim_{v \to +\infty} \frac{F_\lambda(v)}{G_\lambda(v)}.$$

The following equality is extremely important, since it reveals some essential properties of the system (1) and (2).

Lemma L. If $Q(v, \lambda)$ has a continuous partial derivative with respect to λ on $(-\infty, \infty) \times I$, then for $G_\lambda(v) \neq 0$,

$$\frac{\partial [F_\lambda(v)/G_\lambda(v)]}{\partial \lambda}$$

exists and has the value

$$\int_{v_0}^{v} \frac{\partial Q(s, \lambda)}{\partial \lambda} \left[F_\lambda(s) - G_\lambda(s) \frac{F_\lambda(v)}{G_\lambda(v)} \right]^2 ds. \tag{3}$$

Proof. Since $Q(v,\lambda)$ is continuously partially differentiable with respect to λ, $z(v) = \partial F_\lambda(v)/\partial\lambda$ exists and is the solution to

$$\frac{d^2z}{dv^2} - Q(v,\lambda)z = \frac{\partial Q(v,\lambda)}{\lambda}F_\lambda(v)$$

satisfying the initial conditions

$$z(v_0) = z'(v_0) = 0;$$

thus

$$\frac{\partial F_\lambda(v)}{\partial\lambda} = \int_{v_0}^{v}\frac{\partial Q(s,\lambda)}{\partial\lambda}F_\lambda(s)[F_\lambda(s)G_\lambda(v) - F_\lambda(v)G_\lambda(s)]ds.$$

Similarly,

$$\frac{\partial G_\lambda(v)}{\partial\lambda} = \int_{v_0}^{v}\frac{\partial Q(s,\lambda)}{\partial\lambda}G_\lambda(s)[F_\lambda(s)G_\lambda(v) - F_\lambda(v)G_\lambda(s)]ds.$$

By a straightforward calculation, we get (3).

Lemma M. If $Q(v,\lambda)$ satisfy *A1* and *A2*, then, at a certain point λ_0,

1° there exists a neighborhood O of λ_0 in which

$$Z(\lambda) = Z(\lambda_0) \text{ if } \lambda < \lambda_0\ [\lambda > \lambda_0]$$

and

$$Z(\lambda) \geq Z(\lambda_0) \text{ if } \lambda > \lambda_0\ [\lambda < \lambda_0];$$

2° in addition, if $Q(v,\lambda)$ satisfies *A2*, O can be chosen such that

$$Z(\lambda) = Z(\lambda_0) + 1 \text{ if } \lambda > \lambda_0\ [\lambda < \lambda_0] \text{ and } \lambda \in O.$$

Proof. *Lemma K* implies that $|G_{\lambda_0}(v)| \to \infty$ as $v \to -\infty$. Without loss of generality, suppose $G_{\lambda_0}(v) \to +\infty$ as $v \to -\infty$. According to *Lemma E*, there exist a number v_1 and a neighborhood O_1 of λ_0 in which $G_\lambda(v) \to +\infty$ as $v \to -\infty$ and $G_\lambda(v) > 0$ for all $v \leq v_1$.

Formula (3) shows that as a function of λ, $F_\lambda(v)/G_\lambda(v)$ is nondecreasing for a fixed $v \leq v_1$ and nonincreasing for a fixed $v > v_0$, so

$$\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)} - \lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)} \leq 0 \text{ for all } \lambda \leq \lambda_0 \text{ and } \lambda \in O_1.$$

According to (2),

$$\begin{aligned}T_\lambda(v) &= G_\lambda(v)\left[\frac{F_\lambda(v)}{G_\lambda(v)} - \lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}\right] \\ &< G_\lambda(v)\left[\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)} - \lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}\right] \leq 0 \text{ for all } v \leq v_1,\ \lambda \in O_1 \text{ and } \lambda < \lambda_0.\end{aligned}$$

By the continuity of $T_\lambda(v)$ at every $(v,\ \lambda) \in [v_1, v_0] \times I$, there is a neighborhood $O \subset O_1$ of λ_0 in which all $T_\lambda(v)$ have the same number of zeros in $[v_1, v_0]$ and $T_\lambda(v_1) < 0$. Then 1° is obvious.

For $\lambda \in O$ and $\lambda > \lambda_0$, with (3) we have

$$\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)} - \lim_{v\to-\infty}\frac{F_{\lambda_0}(v)}{G_{\lambda_0}(v)} = \lim_{v\to-\infty}\int_{\lambda_0}^{\lambda}\int_{v_0}^{v}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s) - G_t(s)\frac{F_t(v)}{G_t(v)}\right]^2 ds dt.$$

Since

$$\frac{\partial Q(s,t)}{\partial t}\left[F_t(s) - G_t(s)\frac{F_t(v)}{G_t(v)}\right]^2 \leq 0 \text{ when } v \leq v_1,$$

according to Fatou's Lemma [6],

$$\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)}-\lim_{v\to-\infty}\frac{F_{\lambda_0}(v)}{G_{\lambda_0}(v)}\geq\int_{\lambda_0}^{\lambda}\left\{\lim_{v\to-\infty}\int_{v_0}^{v}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s)-G_t(s)\frac{F_t(v)}{G_t(v)}\right]^2ds\right\}dt$$

$$\geq\int_{\lambda_0}^{\lambda}\int_{v_0}^{-\infty}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s)-G_t(s)\lim_{v\to-\infty}\frac{F_t(v)}{G_t(v)}\right]^2dsdt$$

Similarly, we can prove

$$\lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}-\lim_{v\to+\infty}\frac{F_{\lambda_0}(v)}{G_{\lambda_0}(v)}\leq\int_{\lambda_0}^{\lambda}\int_{v_0}^{+\infty}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s)-G_t(s)\lim_{v\to+\infty}\frac{F_t(v)}{G_t(v)}\right]^2dsdt$$

when $\lambda\in O$ and $\lambda>\lambda_0$. Therefore,

$$\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)}-\lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}\geq\int_{\lambda_0}^{\lambda}\left\{\int_{v_0}^{-\infty}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s)-G_t(s)\lim_{v\to-\infty}\frac{F_t(v)}{G_t(v)}\right]^2ds\right.$$
$$\left.-\int_{v_0}^{+\infty}\frac{\partial Q(s,t)}{\partial t}\left[F_t(s)-G_t(s)\lim_{v\to+\infty}\frac{F_t(v)}{G_t(v)}\right]^2ds\right\}dt\geq0$$

if $\lambda\in O$ and $\lambda>\lambda_0$. If $Q(v,\lambda)$ satisfies *A2*, the equality can never hold, i.e.,

$$\lim_{v\to-\infty}\frac{F_\lambda(v)}{G_\lambda(v)}-\lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}>0.$$

Thus $T_\lambda(v)\to+\infty$ as $v\to-\infty$ if $\lambda\in O$ and $\lambda>\lambda_0$. Since $T_\lambda(v_1)<0$ and

$$\frac{F_\lambda(v)}{G_\lambda(v)}-\lim_{v\to+\infty}\frac{F_\lambda(v)}{G_\lambda(v)}$$

is decreasing in v for a fixed $\lambda\in O$. T_λ must have exactly one zero in (∞,v_1) when $\lambda\in O$ and $\lambda>\lambda_0$. 2° follows immediately. ▌

Now *Theorem C* is a direct inference of *Lemma M*, and it is also easy to show *Theorem D* by *Lemma M* and *Theorem H*.

If $Z(\lambda)$ is not a nondecreasing function, there exists λ_0 such that

$$\limsup_{\lambda\to\lambda_0-0}Z(\lambda)>\liminf_{\lambda\to\lambda_0+0}Z(\lambda).$$

Then λ_0 belongs to $D(Z)$, so T_{λ_0} is bounded (see *Theorem E*). This contradicts conclusion 1° of *Lemma J*. ▌

Theorem D reveals the structure of $D(Z)$, or equivalently, the eigenvalues of (1) under the bounded condition, when $Q(v,\lambda)$ satisfies *A1*, *A2*, and *A4*: $D(Z)$ must be composed of

$$\lambda_n=\sup\{\lambda:T_\lambda\text{ has }n-1\text{ zeros}\}\ [\lambda_n=\inf\{\lambda:T_\lambda\text{ has }n+1\text{ zeros}\}]$$

where $Z(a)<n\leq Z(b)$ $[Z(a)>n\geq Z(b)$. Therefore Z has only a finite number of discontinuities in $I=[a,b]$.

III. APPLICATIONS OF THE SHOOTING METHOD TO THE SCHRÖDINGER EQUATION

In this section, we apply the shooting method to the radial equation associated with the n-dimensional Schrödinger equation with a spherically symmetric potential [7]:

$$\frac{d^2R_0}{dr^2}+\left(\frac{n-1}{r}\right)\frac{dR_0}{dr}+\left(c[\lambda-V(r)]+\frac{k_n(l)}{r^2}\right)R_0=0 \tag{4}$$

subject to the condition

$$\int_0^\infty |R_0(r)|^2 r^{n-1} dr < \infty. \tag{5}$$

In these expressions, l is the angular number, $k_n(l)$ denotes the lth eigenvalue of the Laplace-Beltrami operator corresponding to the sphere in n-dimensional space, and c is a positive physical constant. Setting $r = e^{v/n}$ and $R_0(e^{v/n}) = e^{(1/n-1/2)v} R(v)$ in (4) and (5), yields

$$\frac{d^2 R}{dv^2} = \left(\frac{c}{n^2}[V(e^{v/n}) - \lambda] e^{2v/n} - \frac{k_n(l)}{n^2} + \left(\frac{1}{2} - \frac{1}{n}\right)^2 \right) R \tag{6}$$

and

$$\int_{-\infty}^\infty |R(v)|^2 e^{2v/n} dv < \infty. \tag{7}$$

If λ is real number for which there exists a nontrivial solution $R_\lambda(v)$ of (6) satisfying (7), then λ and $R_\lambda(v)$ are called an eigenvalue and an eigensolution, respectively, of (6)-(7).

We pause the general discussion to give three typical examples. (Positive physical coefficients in the potential functions are ignored).

Ex1. The two-dimensional hydrogen atom with the logarithmic potential function was discussed in [1, 2]; here, $n = 2$ and $V(r) = \log r$, so

$$\frac{d^2 R}{dv^2} = \left[\frac{l^2}{4} + c(\frac{v}{2} - \lambda) e^v \right] R \tag{8}$$

and

$$\int_{-\infty}^\infty |R(v)|^2 e^v dv < \infty. \tag{9}$$

Substituting $v = u + 2\lambda$ produces the simple form frequently used in standard works [1, 8]:

$$\frac{d^2 R}{du^2} - \left(\frac{l^2}{4} + \sigma u e^u \right) R = 0 \tag{10}$$

and

$$\int_{-\infty}^\infty |R(u)|^2 e^u du < \infty, \tag{11}$$

where $\sigma = ce^{2\lambda}/2$.

Ex2. The familiar three-dimensional hydrogen atom with the Coulomb potential, i.e. $n = 3$ and $V(r) = -1/r$, was discussed via the shooting method in [3].

$$\frac{d^2 R}{dv^2} = \left(\frac{c}{9}(-e^{-v/3} - \lambda) e^{2v/3} + \frac{l(l+1)}{9} + \frac{1}{36} \right) R \tag{12}$$

and

$$\int_{-\infty}^\infty |R(v)|^2 e^{2v/3} dv < \infty. \tag{13}$$

Ex3. The shooting method was used in [3] to study the three-dimensional Schrödinger equation with the isotropic harmonic oscillator potential. For this case, $n = 3$ and $V(r) = r^2$, so

$$\frac{d^2 R}{dv^2} = \left(\frac{c}{9}(e^{2v/3} - \lambda) e^{2v/3} + \frac{l(l+1)}{9} + \frac{1}{36} \right) R \tag{14}$$

and

$$\int_{-\infty}^\infty |R(v)|^2 e^{2v/3} dv < \infty. \tag{15}$$

Returning to the general case, it is clear that if $V(r)$ is continuous for $r \in (0, \infty)$, then

$$Q(v, \lambda) = \frac{c}{n^2}[V(e^{v/n}) - \lambda]e^{2v/n} - \frac{k_n(l)}{n^2} + \left(\frac{1}{2} - \frac{1}{n}\right)^2$$

satisfies both *A1* and *A2*. To ensure that *A3* holds, we will consider continuous potentials satisfying the following conditions.

A4. Let $\lim_{v \to +\infty} V(e^{v/n}) = \sup V(e^{v/n}) = M$, where M may be $+\infty$. For all $\lambda \geq M$, there exists v_0 such that $Q(v, \lambda)$ is negative and has a negative partial derivative with respect to v when $v > v_0$.

A5. $\int_{-\infty}^{0} |vQ_-(v, 0)| dv < \infty$.

Obviously, both *A4* and *A5* are satisfied in *Ex1*, *Ex2* and *Ex3*.

The next theorem implies the well-known fact that all eigenvalues in *Ex2* are negative [9].

Theorem N. If $V(r)$ is continuous for $r \in (0, \infty)$ and satisfies *A4*, then all eigenvalues of the system (6) and (7) lie in the interval $(-\infty, M)$.

Proof. If there is an eigenvalue $\lambda \geq M$ of (6) and (7), let $R(v)$ be a corresponding eigensolution. Since $-Q(v, \lambda)$ is positive and increasing when $v > v_0$, by Sturm's Separation Theorem [10], it is easy to see that $R(v)$ has a sequence of zeroes, say $\{v_n\}$, which approach $+\infty$.

If v is large enough, then

$$\frac{d\{-Q(v, \lambda)R^2(v) + [R'(v)]^2\}}{dv} = -\frac{\partial Q(v, \lambda)}{\partial v}R^2(v) \geq 0,$$

and this implies

$$\int_0^\infty [R'(v)]^2 dv - \int_0^\infty Q(v, \lambda)R^2(v) dv = \infty.$$

On the other hand,

$$\begin{aligned}\int_0^{v_n} Q(v, \lambda)R^2(v) dv &= \int_0^{v_n} R(v)R''(v) dv \\ &= -\int_0^{v_n} [R'(v)]^2 dv - R(0)R'(0).\end{aligned}$$

Since $v_n \to +\infty$ as $n \to \infty$, it is easy to deduce

$$\int_0^\infty Q(v, \lambda)R^2(v) dv = \infty.$$

Clearly, there exists a constant $k > 0$ such that $Q(v, \lambda) < ke^{2v/n}$ when v is large enough, and therefore

$$\int_0^\infty R^2(v)e^{2v/n} dv = \infty,$$

a contradiction. ∎

It is not difficult to obtain the following theorem.

Theorem O. If $V(r)$ is continuous and satisfies both *A4* and *A5*, then the corresponding $Q(v, \lambda)$ satisfies *A3* on any closed interval $I \subset (-\infty, M)$.

If $V(r)$ satisfies both *A4* and *A5*, then the shooting method can be applied to find all the eigenvalues and eigensolutions of the system consisting of (6) and the bounded condition

$$R_\lambda(v) \text{ is bounded for } v \in (-\infty, +\infty). \tag{16}$$

The next result is obvious but needs to be claimed to show the relation between this eigenvalue problem and the one consisting of (6) and (7).

Theorem P. Let $V(r)$ be continuous for $r \in (0, \infty)$ and satisfy both *A4* and *A5*, then every bounded solution of (6) must satisfy (7).

The shooting method can only be used to find the bounded solutions of (6), and although we have seen in *Theorem P* that bounded solutions must be eigensolutions of (6) and (7), the converse is not always true. In the ground state, that is, when $l = 0$, *Ex1* and *Ex3* have all the real numbers as their eigenvalues of (6) and (7), while *Ex2* has all the negative numbers as its eigenvalues of (6) and (7) [2]. In these cases, the shooting method only produces a part of all the eigenvalues and eigensolutions. In fact, the eigensolutions to (6) largely depend on the domain of Schrödinger operator. It is not surprising at all that we get different eigenvalues and eigensolutions if we change condition (7). But it is also easy to see in our three examples, when $l \neq 0$, the eigensolutions of (6) and (7) must be bounded. Even in the ground state, the bounded solutions are still the most interesting ones in a physical sense [11]. The shooting method is sufficiently general to solve a wide class of Schrödinger equations.

By the shooting method, we can easily get a generalization of a classical theorem regarding *Ex1* [12].

Theorem Q. The eigenvalues of (6) and (16) are isolated and only have finite multiplicities, that is, for every λ, there are only a finite number l's such that λ is an eigenvalue of (6) and (16).

Proof. The isolation follows directly from our previous arguments.

It is known that $k_n(l)$ is a polynomial and non-negative for $l \in Z$. Let σ denote $-k_n(l)/n^2$. For a fixed λ, $P(v, \sigma) = Q(v, \lambda)$ satisfies both *A1*, *A2* and *A3*, so we can apply the shooting method for (6) and (16) with respect to σ. Let T_λ have N zeros when $\sigma = 0$. Since the number of zeros of T_λ is nonincreasing in σ (see *Theorem D*) and $\sigma \geq 0$, the system (6) and (16) has at most N eigenvalues for σ. Thus there are only finitely many l's such that λ is an eigenvalue.

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