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SMALL SAMPLE INFERENCE FOR EXPONENTIAL SURVIVAL TIMES WITH HEAVY RIGHT-CENSORING

by

NOROHARIVELO VOLANIAINA RANDRIANAMPY

A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

2012

Approved by:

Dr. Robert Paige, Advisor
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Dr. Xuerong Wen
Dr. Gayla Olbricht
Dr. Xiaoping Du

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DEDICATION

This dissertation is dedicated to my family

Sahondra, Mamy, Hanta, Oly, Emmanuella, Christiano and Timothy.

This dissertation is also in loving memory to

my dad

Rowlands Randrianampy,

and my grandparents

Rasamison and Berthe Razaiarinoro.

ABSTRACT

We develop a saddlepoint-based method and several generalized Bartholomew methods for generating confidence intervals about the rate parameter of an exponential distribution in the presence of heavy random right-censoring. Butler's conditional moment generating function formula is used to derive the relevant moment generating function for the rate parameter score function which provides access to a saddlepoint-based bootstrap method. Moment generating functions also play a key role in the generalized Bartholomew methods we develop. Since heavy censoring is assumed, the possible non-existence of the rate parameter maximum likelihood estimate (MLE) is nonignorable. The overwhelming majority of existing methods condition upon the event that the number of observed failures is non-zero (rate parameter MLE exists). With heavy censoring, these methods may not be able to produce confidence interval an appreciable percentage of times. Our proposed methods are unconditional in the sense that they can produce confidence intervals even when the rate parameter MLE does not exist. The unconditional saddlepoint method in particular defaults in a natural way to a proposed generalized Bartholomew method when the rate parameter MLE fails to exist. We find that the proposed saddlepoint method outperforms competing Bartholomew methods in the presence of heavy censoring and small sample sizes.

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1. INTRODUCTION

This dissertation has been prepared in the style utilized by Missouri University of Science and Technology. Pages 1-79 will be submitted to the Scandinavian Journal of Statistics.

We develop methods for making small sample inference about the rate parameter of an exponential distribution in the presence of heavy and random right-censoring. We let $T_1, T_2, \ldots, T_n$ denote the independent and identically distributed (IID) exponential survival times, with hazard rate λ, for n subjects or items. These survival times are censored at the right by IID random variables $C_1, C_2, \ldots, C_n$ which are independent of the survival times. It is assumed that the censoring time distribution admits a moment generating function (MGF) and is indexed by parameter θ which may be of infinite dimension, e.g. if the censoring time distribution does not have an assumed parametric form. The right-censored data are denoted by $(Z_1, \Delta_1), (Z_2, \Delta_2), \ldots, (Z_n, \Delta_n)$ where the time on study is $Z_i = \min\{T_i, C_i\}$ and the survival indicator function is $\Delta_i = 1\{T_i \leq C_i\}$. The observed right-censored data are denoted as (z_1, δ_1), $(z_2, \delta_2), \ldots, (z_n, \delta_n)$.

The joint likelihood function for λ and θ is

$$\begin{aligned}\mathcal{L}(\lambda, \theta) &= \prod_{i=1}^{n} \left\{ f(z_i; \lambda)^{\delta_i} S_f(z_i; \lambda)^{1-\delta_i} \right\} \prod_{i=1}^{n} \left\{ g(z_i; \theta)^{1-\delta_i} S_g(z_i; \theta)^{\delta_i} \right\} \\ &= \mathcal{L}(\lambda) \mathcal{L}(\theta)\end{aligned} \tag{1.1}$$

where $f(\cdot; \lambda)$ and $S_f(\cdot; \lambda)$ are the density and survival functions of the exponential survival times and, $g(\cdot; \theta)$ and $S_g(\cdot; \theta)$ are the density and survival functions of the censoring times (Kalbfleish & Prentice, 2002; Lawless, 2003). The joint likelihood for λ and θ is said to be separable since it factorizes into a product of a likelihood for λ and a likelihood for θ. Furthermore, one could say that parameters λ and θ are orthogonal (Severini, 2000, sec. 3.6.4) since their MLEs are asymptotically uncorrelated.

One consequence of likelihood separability is that, from a purely likelihood point of view, inference about λ can be based upon $\mathcal{L}(\lambda)$ while ignoring $\mathcal{L}(\theta)$. In particular, the maximum likelihood estimate (MLE) for λ_0, the true value of λ, is given as the unique root of the score estimating equation from $\mathcal{L}(\lambda)$;

$$U(\lambda) = \sum \{\delta_i/\lambda - z_i\} \tag{1.2}$$

so that $\hat{\lambda} = \{\sum z_i\}^{-1} \sum \delta_i$ when $\sum \delta_i > 0$. Another consequence of likelihood separability is that $\hat{\theta}(\lambda)$, the constrained or conditional MLE for true parameter value θ_0, coincides with $\hat{\theta}$, the ordinary MLE.

Note however that, even though MLEs $\hat{\lambda}$ and $\hat{\theta}$ may be determined separately from one another, the distribution of $\hat{\lambda}$ is quite complicated and will in general depend upon θ_0. Furthermore, $\hat{\lambda}$ is no longer sufficient for making inference about λ and therefore large-sample likelihood methods are often considered (Kalbfleish and Prentice 2002, sec. 3.3) since MLEs are generally asymptotically sufficient (Cox and Hinkley 1974, sec. 9.2). However, in the presence of small samples or heavy censoring there is a need to consider exact distribution theory (or nearly exact results gotten from saddlepoint approximations; Butler, 2007) since asymptotic methods may provide poor distributional approximations (Crowder, Kimber, Smith and Sweeting 1991, sec. 3.5).

Furthermore, factorization of the likelihood suggests that asymptotically information about the censoring time distribution is not useful in making inference about λ. This does not, however, imply that this information is not useful in small samples. Lawless (2003, sec. 2.2.1.2) notes that while it may be desirable to make inferences conditional upon the observed censoring times it may actually be of interest to average over the censoring time distribution as, for instance, one would do when planning a study. We consider a form of censoring time averaging in the confidence interval methods we develop.

Lawless (2003, sec. 4.1) provides an overview of existing confidence interval methods for λ. The methods discussed therein are said to be *conditional* since their construction is conditional upon event $\sum \delta_i > 0$. In the presence of heavy censoring this condition becomes restrictive and this is the motivation for *unconditional* methods which can be applied even when $\sum \delta_i = 0$.

Abe and Iwasaki (2005) provide an unconditional confidence interval method for exponential survival times in the presence of Type 1 censoring. Their method is based on the unconditional cumulative distribution function (CDF) for $\hat{\lambda}$ and is a generalization of the conditional CDF for $\hat{\lambda}$ given in Bartholomew (1963). In this same work, Bartholomew (1963) proposes an unconditional confidence interval method based on the discrete distribution of $\sum \Delta_i$. He notes that this method, while exact, may be inefficient in the presence of light or moderate censoring.

The scope of this dissertation is to develop small-sample methods of unconditional confidence interval construction. These methods make use information about the censoring time distribution since such information is useful in small samples. Our methods are developed for fixed and random censoring times and are generalizations of two methods for Type 1 censoring; (i) Abe and Iwasaki's unconditional CDF method and (ii) Bartholomew's exact method.

The proposed saddlepoint-based method, which generalizes Abe and Iwasaki's method, is based in part upon the automatic percentile method bootstrap confidence interval method of DiCiccio & Romano (1995). Suppose that one desires a $(1-\alpha)100\%$ conditional confidence interval for λ_0, say (λ_L, λ_U). This interval may be determined as the solution of the following set of equations:

$$\begin{aligned} P\left(\hat{\lambda} \leq \hat{\lambda}_{obs} | \lambda_L, \hat{\theta}\,(\lambda_L) \equiv \hat{\theta}, \textstyle\sum \Delta_i > 0\right) &= 1 - \alpha/2 \\ P\left(\hat{\lambda} \leq \hat{\lambda}_{obs} | \lambda_U, \hat{\theta}\,(\lambda_U) \equiv \hat{\theta}, \textstyle\sum \Delta_i > 0\right) &= \alpha/2 \end{aligned} \tag{1.3}$$

where $\hat{\lambda}_{obs}$ denotes the observed value of $\hat{\lambda}$. DiCiccio & Romano (1995) show that this method provides second-order accuracy, in the sense of Hall (1988), without a bias correction or an acceleration constant. Furthermore, one needs only to compute one conditional MLE value for θ_0, since $\hat{\theta}\,(\lambda) \equiv \hat{\theta}$ for all λ, due to the separability of the likelihood function in (1.1).

One drawback to the automatic percentile method bootstrap method is that one rarely has a closed-form expression for $F_{\hat{\lambda}}(\cdot|\lambda,\theta)$. Furthermore, Monte Carlo estimation of this function can be computationally expensive. Note however that since the random score function

$$U(\lambda) = \sum \{\Delta_i/\lambda - Z_i\}$$

is monotonically decreasing in λ for all realizations of $(\sum \Delta_i, \sum Z_i)$, when $\sum \Delta_i > 0$, then

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda,\theta,\sum \Delta_i > 0\right) = P\left(U\left(\hat{\lambda}_{obs}\right) \leq 0|\lambda,\theta,\sum \Delta_i > 0\right).$$

This observation is useful since the MGF function of $U(\lambda)$ is derived in closed-form and used to determine a highly accurate saddlepoint approximation to $P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda,\theta,\sum \Delta_i > 0\right)$.

The conditional procedure in (1.3), which makes use of the saddlepoint CDF approximation is an example of a *saddlepoint-based bootstrap (SPBB)* confidence interval method. Paige, Trindade and Fernando (2009) developed the SPBB methodology for estimating equations, in the presence of no censoring, which are quadratic forms in normal random variables and coined the term *saddlepoint-based bootstrap*. Furthermore, it was shown in this work that replacing the CDF of an estimating equation root by its saddlepoint approximation preserves the inherent second-order accuracy of the automatic percentile method. Finally, Paige and Trindade (2010) and Paige and Trindade (preprint) make further use of the SPBB methodology for quadratic forms.

More generally, saddlepoint approximations have replaced Monte Carlo estimation in a number of other bootstrapping applications including those described in Davison and Hinkley (1988, 1997, sec. 9.5), Daniels and Young (1991), Butler and Bronson (2002), and Butler (2007, chap. 14).

The important contributions of this dissertation are as follows: (i) we consider censoring whereas SBPP methods thus far have not; (ii) our method may be applied in truly semiparametric settings where the censoring time distribution is unspecified and is estimated via the Kaplan-Meier survival curve; (iii) we derive a number of novel closed-form expressions for various moment generating functions which can be used to study probabilistic properties of exponential survival times with right-censoring in even greater detail and (iv) we develop a number of unconditional confidence interval methods which are applicable even when MLE $\hat{\lambda}$ does not. This last contribution is particularly important in the presence of heavy censoring.

Few procedures for handling non-existent MLEs have been considered in the statistics literature and notable examples include the aforementioned unconditional CDF method of Abe and Iwasaki (2005) for Type 1 censoring which accounts for the non-existence of MLE $\hat{\lambda}$ in a natural way. On the other hand, the saddlepoint method considered in Paige, Chapman and Butler (2011) defaulted to a likelihood ratio method when the MLEs for the underlying logistic regression model did not exist. In this work, the saddlepoint method did not converge in any sense to the likelihood ratio method as the MLE approached infinite values and it was not possible to easily characterize when MLEs failed to exist or even to compute the probability of this happening without simulation.

Heavy censoring is known to create special problems for the analysis of survival data, besides a substantial loss of information and the resulting poor performance of asymptotic methods. Examples in the literature include; (i) Prentice and Marek (1979) who find very different results for the Mantel-Cox and Breslow test statistics in the presence of heavy censoring; (ii) Lee, Häkkinen and Rosenqvist (2007) who note that in the presence of heavy censoring it is difficult to determine the best treatment, among several competing treatments, because sample location measures, such as the mean and median, may not be identifiable, and (iii) the unreliability, in the presence of heavy censoring, of the Kaplan-Meier in the tails of a distribution which often necessitates that one considers some form of weighted Kaplan-Meier estimate (Susan, 2001).

Note that in our work we consider Kaplan-Meier estimates of the censoring time distribution and as a result heavy censoring is actually beneficial.

Furthermore, our methodology should be robust against Weibull departures from exponentiality, for certain levels of heavy censoring. The results of Emerson (1981) and Harder (1990) suggest that for a percentage of censored observations in the neighborhood of 50% or at least 90%, the classical exponential confidence intervals for the median survival time, $\ln(2)\,\lambda_0$, will have coverage probabilities that are very close to the expected coverage probabilities (assuming exponential of the survival times), when the survival times are in fact Weibull. Note that when this is the case, the quantity being estimated is the median survival time of the underlying Weibull model.

The remainder of this dissertation is organized as follows. In sections 2 and 3, we discuss the Abe and Iwasaki (2005) unconditional method for Type 1 censoring in detail and describe how it handles non-existent MLEs. In section 4, we derive the conditional ($\sum\Delta_i > 0$) and unconditional ($\sum\Delta_i \geq 0$) MGFs of estimating equation $U(\lambda)$. In section 5 , we explain our choice of estimating equation. In section 6, an unconditional SPBB method, which accounts for the possible non-existence of MLE $\hat{\lambda}$ in a natural way, is developed. Section 7 considers the case of completely observed data and how we handle that case. In section 8, we develop a class of generalized Bartholomew confidence intervals for λ_0. In section 9, we discuss approximate MGFs for censoring time distributions. In section 10, tail-completed Kaplan-Meier MGF estimates are discussed. In section 11, we prove large sample properties of our proposed methods. In section 12, we present an application and in section 13, a Monte Carlo study. Finally, in section 14 we present concluding remarks.

2. ABE-IWASAKI CONFIDENCE INTERVALS

The Abe-Iwasaki confidence intervals are obtained, in part, from (1.3) but where $\hat{\theta}(\lambda) = \theta_0$, the fixed and known Type 1 censoring time. Abe and Iwasaki (2005) provide a formula for the unconditional survival function of $\hat{\lambda}^{-1}$, or equivalently the unconditional CDF of $\hat{\lambda}$, under Type 1 censoring at θ_0.

Using the well-known relationship between the survival function of a gamma random variable with integral shape parameter value and the CDF of the appropriate Poisson random variable, Abe and Iwasaki's CDF formula may be expressed as follows:

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0\right) = \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{i-1}\binom{n}{i}\binom{i}{j}(-1)^j \exp\left\{-\lambda_0\theta_0(n-i+j)\right\} \times a(i,j)^k \exp\left\{-a(i,j)\right\}/k! \tag{2.1}$$

where the summand in k is taken to be 1 when $i = 0$, λ_0 denotes the (assumed) true value of λ,

$$a(i,j) = \lambda_0\left\langle ic^{-1} - \theta_0(n-i+j)\right\rangle,$$

and $\langle\bullet\rangle$ denotes $\max(0, \bullet)$.

To derive formula (2.1), Abe and Iwasaki (2005) wrote the unconditional CDF of $\hat{\lambda}$ as

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0\right) = P\left(\hat{\lambda} \leq c|\lambda_0, , \theta_0, \textstyle\sum\Delta_i = 0\right) P\left(\textstyle\sum\Delta_i = 0|\lambda_0, \theta_0\right) + P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \textstyle\sum\Delta_i > 0\right) P\left(\textstyle\sum\Delta_i > 0|\lambda_0, \theta_0\right)$$

and simplified this to their final form

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0\right) = \exp\left\{-n\theta\lambda_0\right\} + P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \textstyle\sum\Delta_i > 0\right)\left[1 - \exp\left\{-n\theta_0\lambda_0\right\}\right] \tag{2.2}$$

where

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum \Delta_i > 0\right) = \frac{1}{1-\exp(-n\theta_0\lambda_0)} \sum_{i=1}^{n} \sum_{j=0}^{i} \sum_{k=0}^{i-1} \binom{n}{i} \binom{i}{j} (-1)^j \times \exp\left\{-\lambda_0\theta_0 (n-i+j)\right\} a(i,j)^k \exp\left\{-a(i,j)\right\}/k! \tag{2.3}$$

is Bartholomew's (1963) conditional CDF for $\hat{\lambda}$.

Bartholomew derives this CDF by first deriving the conditional MGF of $\hat{\lambda}$ (conditional upon $\sum \Delta_i > 0$) and then inverting this MGF in closed-form to obtain the conditional distribution of $\hat{\lambda}$. We take a somewhat similar approach to approximate the conditional CDF of $\hat{\lambda}$ under random censoring; in section 4 we derive the conditional MGF of $U(\lambda)$, the estimating equation for $\hat{\lambda}$, and then approximately invert this with a saddlepoint approximation in section 6.

3. ACCOUNTING FOR A NON-EXISTENT MLE

In the derivation of the final form of their CDF (2.2), Abe and Iwasaki (2005) prove three results, which handle the possibility of an indefinite MLE, that we will make use of, in section 6, where we develop our unconditional SPBB confidence interval. First, they showed that

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum\Delta_i = 0\right) = 1$$

for any value of $c > 0$. The idea here is that when $\sum\delta_i = 0$ then MLE $\hat{\lambda}$ may be characterized as the smallest real number greater than zero. Note that this MLE technically does not exist since it is the maximum of a monotone increasing function on an open interval. Second, they showed that

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0, \sum\Delta_i > 0\right) = 0$$

when $\sum\delta_i = 0$. This result makes sense given the aforementioned characterization of MLE $\hat{\lambda}$ when $\sum\delta_i = 0$. Also, from Bartholomew's conditional CDF formula in (2.3), it is easily verified that

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \to 0$$

as $\hat{\lambda}_{obs} \to 0$. Lastly, Abe and Iwasaki (2005) show that when $\sum\delta_i = 0$ the unconditional CDF defaults to

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) = \exp\left(-n\theta_0\lambda_0\right)$$

which easily follows from the first two results they prove.

4. DERIVATION OF $M_{U(\lambda)}\left(S|\sum\Delta_I > 0\right)$

Exact distributional results for $\hat{\lambda}$ are hard to obtain for many reasons including requisite condition $\sum\Delta_i > 0$ and the fact that in general $\sum Z_i$ and $\sum\Delta_i$ are not independent except in a proportional hazards setting; Allen (1963). It is however possible to determine a closed-form expression for the joint MGF of $(\sum\Delta_i, \sum Z_i)$ conditional upon $\sum\Delta_i > 0$. We first derive the joint MGF for (Δ, Z), the censored data for a single subject or item, as

$$\begin{aligned}
M(s,t) &= E\left[e^{\Delta s + Zt}\right] \qquad (4.1)\\
&= \sum_{\delta=0}^{1}\int_0^\infty e^{s\delta+tz}\left[f\left(z;\lambda_0\right)S_g\left(z;\lambda_0\right)\right]^{\delta}\left[g\left(z;\theta_0\right)S_f\left(z;\theta_0\right)\right]^{1-\delta}dz\\
&= \int_0^\infty e^{(t-\lambda_0)z}g(z;\theta_0)dz + e^s\int_0^\infty \lambda_0 e^{(t-\lambda_0)z}S_g(z;\theta_0)dz\\
&= M_C(t-\lambda_0) + \frac{\lambda_0 e^s}{\lambda_0 - t}\left\{1 - M_C(t-\lambda_0)\right\}
\end{aligned}$$

where $M_C(\cdot)$ is the MGF of the censoring time distribution and, for notational convenience, we suppress the dependence of this function on θ_0. Note that this joint MGF exists for unrestricted values of s and for any value of t where $M_C(t)$ exists.

It now easily follows that the joint MGF of $(\sum\Delta_i, \sum Z_i)$ is $[M(s,t)]^n$. Butler's conditional MGF formula (Butler 2007, sec. 4.4.4) provides the means for deriving the joint MGF of $(\sum\Delta_i, \sum Z_i)$ conditional upon $\sum\Delta_i > 0$. From this formula, we first obtain the conditional MGF of $\sum Z_i$ given that $\sum\Delta_i = k$ as

$$\begin{aligned}
M\left(t|\sum\Delta_i = k\right) &= \frac{\left.[M_p(s,t)]^{n-k}\left[\frac{\lambda_0}{\lambda_0-t}\left\{1-M_C(t-\lambda_0)\right\}\right]^k\right\}_{s=0}}{\left.[M_p(s,t)]^{n-k}\left[\frac{\lambda_0}{\lambda_0-t}\left\{1-M_C(t-\lambda_0)\right\}\right]^k\right\}_{s=0,t=0}} \qquad (4.2)\\
&= \frac{[M_C(t-\lambda_0)]^{n-k}\left[\frac{\lambda_0}{\lambda_0-t}\left\{1-M_C(t-\lambda_0)\right\}\right]^k}{[M_C(-\lambda_0)]^{n-k}\left[1-M_C(-\lambda_0)\right]^k}
\end{aligned}$$

where $M_p(s,t) := M(\ln s, t)$ is the *probability-moment generating function* of Δ and Z. From conditional MFG $M\left(t|\sum\Delta_i = k\right)$, we may obtain the joint MGF of $\left(\sum\Delta_i, \sum Z_i\right)$ conditional upon $\sum\Delta_i > 0$ as

$$
\begin{aligned}
M\left(s,t|\sum\Delta_i > 0\right) &= \sum_{k=1}^{n} e^{sk} M\left(t|\sum\Delta_i = k\right) P\left(\sum\Delta_i = k|\sum\Delta_i > 0\right) \qquad (4.3)\\
&= \frac{1}{1-[M_C(-\lambda_0)]^n} \sum_{k=1}^{n} \binom{n}{k} [M_C(t-\lambda_0)]^{n-k} \\
&\times \left[\frac{\lambda_0 e^s}{\lambda_0 - t}\left\{1 - M_C(t-\lambda_0)\right\}\right]^k \\
&= \frac{[M(s,t)]^n - [M_C(t-\lambda_0)]^n}{1-[M_C(-\lambda_0)]^n} \\
&= \frac{[M(s,t)]^n - M\left(t|\sum\Delta_i = 0\right) P\left(\sum\Delta_i = 0\right)}{P\left(\sum\Delta_i > 0\right)}.
\end{aligned}
$$

Note that as $P\left(\sum\Delta_i = 0\right) \to 0$, conditional MGF $M\left(s,t|\sum\Delta_i > 0\right) \to [M(s,t)]^n$, as one would expect. Note also that to obtain the final form of this MGF in (4.3) we made use of the fact that $P\left(T_i < C_i\right) = P\left(\Delta_i = 1\right) = 1 - M_C\left(-\lambda_0\right)$. This follows since the probability of censoring $P\left(\Delta_i = 0\right)$ may be written in terms of a Riemann–Stieltjes integral with respect to the censoring time CDF $F_g\left(\cdot;\theta\right)$;

$$
\begin{aligned}
P\left(\Delta_i = 0\right) &= P\left(T_i > C_i|\lambda_0, \theta_0\right) \qquad (4.4)\\
&= \int_0^\infty P\left(T_i > c_i|\lambda_0, \theta_0\right) dF_g\left(c_i;\theta_0\right) \\
&= \int_0^\infty e^{-\lambda_0 c_i} dF_g\left(c_i;\theta_0\right) = M_C\left(-\lambda_0\right).
\end{aligned}
$$

It now follows that the MGF of random score function

$$
U\left(\lambda\right) = \sum\left\{\Delta_i/\lambda - Z_i\right\}
$$

conditional upon $\sum\Delta_i > 0$ is

$$
M_{U(\lambda)}\left(s|\sum\Delta_i > 0\right) = M\left(s/\lambda, -s|\sum\Delta_i > 0\right). \qquad (4.5)
$$

5. CHOICE OF ESTIMATING EQUATION

Estimating equation $U(\lambda)$ is unbiased when one does not assume $\sum\Delta_i > 0$ but is biased otherwise. From conditional MGF $M_{U(\lambda)}(s|\sum\Delta_i > 0)$ it can be shown, see Appendix B, that the expected value of estimating equation $U(\lambda)$ at $\lambda = \lambda_0$, conditional upon $\sum\Delta_i > 0$, is non-zero;

$$E\left[U(\lambda_0)\,|\sum\Delta_i > 0\right] = \frac{n\left[M_C(-\lambda_0)\right]^{n-1} M_C'(-\lambda_0)}{1-\left[M_C(-\lambda_0)\right]^n}.$$

It is easily shown that this bias term is positive for any value of λ_0 meaning that $U(\lambda)$ is expected to consistently yield estimates for λ_0 which are too large.

Given this fact, we initially considered a bias adjusted estimating equation, $\tilde{U}(\lambda)$ which is defined as

$$\tilde{U}(\lambda) := U(\lambda) - \frac{n\left[M_C(-\lambda)\right]^{n-1} M_C'(-\lambda)}{1-\left[M_C(-\lambda)\right]^n},$$

in place of classical estimating equation $U(\lambda)$. Nonetheless, we opted to use $U(\lambda)$, as the basis for our saddlepoint method in section 6, for reasons which we describe next.

First, in preliminary computations we found that estimating equation $\tilde{U}(\lambda)$ can yield negative estimates for λ_0 when $\Sigma\delta_i$ is small but positive. Secondly, estimating equation $\tilde{U}(\lambda)$ corresponds to the following pseudo log-likelihood:

$$\ln\tilde{\mathcal{L}}(\lambda) = \sum\left\{\Delta_i \ln(\lambda) - \lambda Z_i\right\} - \ln\left\{1-\left[M_C(-\lambda)\right]^n\right\}$$

and likelihood function

$$\tilde{\mathcal{L}}(\lambda) = \mathcal{L}(\lambda)\left\{1-\left[M_C(-\lambda)\right]^n\right\}^{-1} \tag{5.1}$$

which involves a weighting term which is the probability that $\sum\Delta_i > 0$ as a function of λ but which also depends upon unknown parameter θ_0.

For example, consider IID exponential censoring times so that

$$\begin{aligned} M\left(t|\sum\Delta_i = k\right) &= \frac{\left.[M_p(s,t)]^{n-k}\left[\frac{\lambda}{\lambda-t}\left\{1-M_C(t-\lambda)\right\}\right]^k\right\}_{s=0}}{\left.[M_p(s,t)]^{n-k}\left[\frac{\lambda}{\lambda-t}\left\{1-M_C(t-\lambda)\right\}\right]^k\right\}_{s=0,t=0}} \\ &= \frac{[M_C(t-\lambda)]^{n-k}\left[\frac{\lambda}{\lambda-t}\left\{1-M_C(t-\lambda)\right\}\right]^k}{[M_C(-\lambda)]^{n-k}\left[1-M_C(-\lambda)\right]^k} \\ &= \left(\frac{\theta+\lambda}{\theta+\lambda-t}\right)^n = \left[M_Z\left(t\right)\right]^n \end{aligned}$$

as one would expect based upon the results of Allen (1963). As a result

$$M\left(s,t|\sum\Delta_i > 0\right) = \left[M_Z\left(t\right)\right]^n \left\{\frac{\left[\left\{1-M_C(-\lambda)\right\}e^s + M_C(-\lambda)\right]^n - [M_C(-\lambda)]^n}{1-[M_C(-\lambda)]^n}\right\}$$

so that $\sum Z_i$ has an exponential distribution with rate $\theta+\lambda$, $\sum\Delta_i$ has a zero-truncated binomial distribution and summary statistics $\sum Z_i$ and $\sum\Delta_i$ remain independent of one another. The likelihood for (λ,θ) based upon $\left(\sum\Delta_i, \sum Z_i\right)|\sum\Delta_i > 0$ is given as

$$\breve{\mathcal{L}}\left(\lambda,\theta\right) = \mathcal{L}\left(\lambda\right)\mathcal{L}\left(\theta\right)\left\{1-\left[\theta/\left(\theta+\lambda\right)\right]^n\right\}^{-1}$$

which is equivalent to the penalized likelihood $\tilde{\mathcal{L}}\left(\lambda\right)$ in (5.1). In general, one could use the joint saddlepoint approximation for the joint density of $\left(\sum\Delta_i, \sum Z_i\right)|\sum\Delta_i > 0$ to provide an approximate likelihood for (λ,θ) from (4.3), see for instance Butler (2007, chap. 15) for an application of this idea in Bayesian computations. However, given the poor performance of the resulting unbiased likelihood-based estimating equation, $\tilde{U}\left(\lambda\right)$, for λ we did not consider this methodology and simply worked with the unconditional likelihood function in (1.1).

6. UNCONDITIONAL SPBB CONFIDENCE INTERVALS

We extend the Abe-Iwasaki confidence interval to random censoring settings. The unconditional CDF for this method, in (2.2), may be written as follows:

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0\right) = \left[M_C\left(-\lambda_0\right)\right]^n + P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum\Delta_i > 0\right)\left\{1 - \left[M_C\left(-\lambda_0\right)\right]^n\right\}. \quad (6.1)$$

Here we make use of the fact that the MGF for Type 1 censoring is $M_C(s) = e^{s\theta_0}$. Were it not for the possibility of a non-existent MLE, $\hat{\lambda}$ would have a continuous distribution and

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) \sim Unif\,(0, 1)$$

since this is simply the probability integral transform for a continuous random variable. Furthermore, if the true value of θ_0 were known then a $(1-\alpha)100\%$ confidence interval for λ_0 would be determined as

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_L, \theta_0\right) = 1 - \alpha/2 \quad \text{and} \quad P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_U, \theta_0\right) = \alpha/2.$$

The distribution of $\hat{\lambda}$, in reality, is mixed with a point mass essentially at 0 of size $\left[M_C\left(-\lambda_0\right)\right]^n$, and is continuous otherwise. This follows from the fact that

$$\left[M_C\left(-\lambda_0\right)\right]^n = \lim_{\hat{\lambda}_{obs} \to 0} P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) \leq P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right).$$

Therefore, $P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) = \left[M_C\left(-\lambda_0\right)\right]^n$ with probability $\left[M_C\left(-\lambda_0\right)\right]^n$ and

$$\begin{aligned} P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) &\sim \left[M_C\left(-\lambda_0\right)\right]^n + Unif\,(0, 1)\left\{1 - \left[M_C\left(-\lambda_0\right)\right]^n\right\} \quad (6.2) \\ &= Unif\left(\left[M_C\left(-\lambda_0\right)\right]^n, 1\right) \end{aligned}$$

with probability $1 - \left[M_C\left(-\lambda_0\right)\right]^n$.

The unconditional CDF for $\hat{\lambda}$ is simply a weighted average of CDFs for the continuous and discrete portions of the mixed distribution for $\hat{\lambda}$. This observation and the fact that Abe and Iwasaki's three results (which account for the possibility of an indefinite MLE), from section 3, hold for arbitrary censoring time distributions, shows that the unconditional CDF expression in (6.1) holds more generally for any censoring time distribution admitting a MGF.

An important component of CDF $P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0\right)$ is the conditional CDF for $\hat{\lambda}$. This CDF is available in closed-form for Type 1 censoring times (Bartholomew's conditional CDF in equation (2.3)) and for exponential censoring times, see Appendix A, but appears to be intractable otherwise. Note, however, that in the exponential censoring case

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum \Delta_i > 0\right) \to P\left(\sum Z_i \geq c^{-1}|\lambda_0, \theta_0\right) \neq 1,$$

where $\sum Z_i$ has a gamma distribution with rate parameter θ_0, as $\lambda_0 \to 0$, as shown in Appendix A. As a result, a conditional approach to confidence interval construction, where one uses the conditional (instead of the unconditional) CDF for $\hat{\lambda}$, is not guaranteed to yield a solution to equations (1.3). In contrast, the unconditional CDF in (6.1) approaches 1 as λ_0 approaches zero.

For general censoring time distributions, it is advantageous to express this CDF in terms of the CDF for random score function

$$U(\lambda) = \sum \left\{\Delta_i/\lambda - Z_i\right\}.$$

This estimating equation is monotonically decreasing in λ for all realizations of $(\sum \Delta_i, \sum Z_i)$ when $\sum \Delta_i > 0$, and as such

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum \Delta_i > 0\right) = P\left(U(c) \leq 0|\lambda_0, \theta_0, \sum \Delta_i > 0\right)$$

for $0 < c < \infty$.

The idea of relating the CDF of an estimator to the CDF of its monotonic estimating equation is often referred to as the *Device of Daniels* (see for instance Paige, Trindade and Fernando 2009) and was introduced in Daniels (1987).

There do not seem to be any censoring time distributions, besides the Type 1 and exponential cases, for which the CDF of estimating equation $U(\lambda)$ can be derived in closed-form. The closed-form expression for conditional MGF $M_{U(\lambda)}(s|\sum\Delta_i > 0)$, which was derived in section 4, does however provide easy access to the Lugannani and Rice (1980) saddlepoint CDF approximation to

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \equiv P\left(U(c) \leq 0|\lambda_0, \theta_0, \sum\Delta_i > 0\right)$$

which is given as

$$\hat{P}\left(U(c) \leq 0|\lambda_0, \theta_0, \sum\Delta_i > 0\right) = \begin{cases} \Phi(\hat{w}) + \phi(\hat{w})\left[\hat{w}^{-1} - \hat{u}^{-1}\right], & \text{if } E\left[U(c)\right] \neq 0, \\ \frac{1}{2} + K^{(3)}_{U(\lambda)}(0)\left[72\pi K^{(2)}_{U(\lambda)}(0)^3\right]^{-1/2}, & \text{if } E\left[U(c)\right] = 0, \end{cases}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal CDF and probability density (PDF) functions respectively, $K_{U(\lambda)}(s) := M_{U(\lambda)}(s|\sum\Delta_i > 0)$ is the conditional cumulant generating function (CGF) of random score function $U(\lambda)$, $K^{(i)}_{U(\lambda)}(s)$ is the ith derivative of this CGF (with respect to s) for $i = 1, 2, 3$, $\hat{w} = sgn(\hat{s})\sqrt{-2K_{U(\lambda)}(s)}$, $\hat{u} = \hat{s}\sqrt{K^{(2)}_{U(\lambda)}(\hat{s})}$ and $\hat{s}$ is the solution to saddlepoint equation $K^{(1)}_{U(\lambda)}(\hat{s}) = 0$.

Except for Type 1 censoring, nuisance parameter θ_0 will be unknown. When the censoring time distribution has an assumed parametric form, this means we compute the MLE for θ_0, call it $\hat{\theta}$, and use it in place of θ_0 in our confidence interval computations. Note however that since every term in the unconditional CDF for $\hat{\lambda}$ may be determined from MGF $M_C(s)$ we can shift our focus from estimating θ_0 to estimating this transform. In the parametric case, the MLE for $M_C(s)$, call it $\hat{M}_C(s)$, is obtained by replacing θ_0 with MLE $\hat{\theta}$ in this function. In the nonparametric case, the MGF estimate, which we also denote as, $\hat{M}_C(s)$ is one of the tail-completed KM estimators we consider in section 10.

Without loss of generality, our approximation to the unconditional CDF of $\hat{\lambda}$ is given as

$$\hat{P}\left(\hat{\lambda} \leq c|\lambda_0\right) = \left[\hat{M}_C\left(-\lambda_0\right)\right]^n + \hat{P}\left(\hat{\lambda} \leq c|\lambda_0, \sum\Delta_i > 0\right)\left\{1 - \left[\hat{M}_C\left(-\lambda_0\right)\right]^n\right\} \quad (6.3)$$

which becomes

$$\hat{P}\left(\hat{\lambda} \leq c|\lambda_0\right) = \left[\hat{M}_C\left(-\lambda_0\right)\right]^n$$

when $\sum\delta_i = 0$. As a result, in the presence of total censoring, the unconditional SPBB confidence interval defaults to a generalized Bartholomew (Clopper-Pearson) confidence interval which is described in section 8.

7. THE $\sum \Delta_I = N$ CASE

When $\sum \delta = n$, the unconditional SPBB method equates to an exact confidence interval for λ_0 which is exact in terms of its coverage probability as well as the distribution theory used in its derivation.

Note that when $\sum \delta_i = n$ special care must be taken with the conditional CDF of $\hat{\lambda}$ since $\hat{\theta}$ does not exist. We circumvent this difficulty by noting that the MLE for the distribution of Δ satisfies $\hat{\Delta} = 1$ with probability 1. In such case, Δ is independent of Z and the MLE of the joint MGF of these random variables (equation 4.1) reduces to

$$\hat{M}(s,t) = \hat{M}_{\Delta}(s)\,\hat{M}_{Z_i}(t) = e^s \frac{\hat{\lambda}}{\hat{\lambda} - t}$$

and as a result the MLE of the joint MGF of $(\sum \Delta_i, \sum Z_i)$ is simply

$$M(s,t) = \left(\frac{\hat{\lambda} e^s}{\hat{\lambda} - t}\right)^n$$

and the conditional SPBB method is easily shown to yield the exact confidence interval for λ_0 which is based upon the gamma distribution and which is described in section 3.3 of Kalbfleish and Prentice (2002).

8. GENERALIZED BARTHOLOMEW METHODS

Bartholomew (1963) develops an exact unconditional confidence interval for λ_0 under Type 1 censoring at θ_0. Here, he makes use of the fact $\sum\Delta_i$ has a binomial distribution with probability of success

$$p = P\left(T_i < C_i\right) = 1 - \exp\left\{-\theta_0\lambda_0\right\},$$

which is estimated by $n^{-1}\sum\delta_i$. This MLE is then used to generate the exact Clopper-Pearson confidence interval for parameter p, call it $(\hat{p}_L, \hat{p}_U)$. This interval is inverted to obtain the following exact confidence interval for λ_0:

$$\left(\hat{\lambda}_L, \hat{\lambda}_L\right) = \left[-\ln\left(1-\hat{p}_L\right)/\theta_0, -\ln\left(1-\hat{p}_U\right)/\theta_0\right]. \tag{8.1}$$

It is noted in Sundberg (2001) that this interval may perform well under heavy censoring but will be inefficient when censoring is of light to moderate in frequency.

Generalizations of Bartholomew's confidence interval under Type 1 censoring can be obtained by considering other confidence interval methods for proportions whose end-points are then inverted using equation (8.1) to generate a confidence interval for λ_0. Popular methods for proportions, besides the Clopper-Pearson interval, include the Wilson confidence interval (Wilson, 1927), the Agresti–Coull confidence interval (Agresti and Coull, 1998), and the Jeffreys (Bayesian) confidence interval (Berger, 1985). Brown, Cai and DasGupta (2001) compare these three confidence intervals with the Clopper-Pearson confidence interval and the classical Wald confidence interval and generally recommend the Wilson or Jeffreys confidence intervals for small sample sizes.

Bartholomew's method can also be extended to generate confidence intervals for λ_0 in the presence of random censoring where, by equation (4.4),

$$p = P\left(T_i < C_i\right) = 1 - M_C\left(-\lambda_0\right).$$

We can then invert estimate $1-\hat{M}_C\,(-\lambda_0)$, obtained either by maximum likelihood (ML) or Kaplan-Meier methods, to obtain an approximate Bartholomew confidence interval for λ_0 from any of the four confidence intervals for a proportion, that we consider. In section 13 we compare the performance of the unconditional SPBB method with our four proposed generalized Bartholomew methods (Clopper-Pearson, Wilson, Agresti–Coull and Jeffreys).

9. CENSORING TIME MGF

The formula for conditional MGF $M_{U(\lambda)}\left(\cdot|\sum\Delta_i > 0\right)$ and Bartholomew confidence intervals described in the previous section depend upon the MGF of the censoring time distribution $M_C(\cdot)$. We assume that censoring times $C_1, \ldots, C_n$ are independent random variables with either a specified or unspecified distribution. The parametric censoring time distributions we consider are the degenerate (Type 1 censoring), the exponential, gamma and the Weibull. For the last three distributions we use ML estimation of MGF $M_C(\cdot)$. When the form of the distribution for the censoring times is not assumed to be known, tail-completed Kaplan-Meier estimators (Moeschberger and Klein, 1985) are used as the basis for Kaplan-Meier integrals which approximate $M_C(\cdot)$.

For Type 1 censoring, we assume that $C_1, \ldots, C_n$ are degenerate with unit point masses at $c_1, \ldots, c_n$, (which are fixed before the sample is collected) and $c_1 = \cdots = c_n = \theta$ so that $M_C(s) = e^{\theta s}$. We could also consider progressive Type 1 censoring where the unit point masses at $c_1, \ldots, c_n$ satisfy $c_1 \neq \cdots \neq c_n$, after straightforward changes to the MGF formulas in section 4, but do not do so here. For exponential censoring times, we assume rate parameter of θ and MGF $M_C(s) = (1 - s/\theta)^{-1}$ and more generally gamma censoring times with rate parameter θ_1, shape parameter θ_2, and MGF $M_C(s) = (1 - s/\theta_1)^{-\theta_2}$. Finally, we consider censoring times which follow a two-parameter Weibull distribution with PDF

$$g\left(c\right) = \theta_1\theta_2\left(\theta_1 c\right)^{\theta_2 - 1}\exp\left\{-\left(\theta_1 c\right)^{\theta_2}\right\} \text{ for } c > 0$$

where the MGF is not given in a simple form but is given as a power series;

$$M_C(s) = \sum_{k=0}^{\infty}\frac{s^k}{\theta_1^k k!}\Gamma\left(1 + \frac{k}{\theta_2}\right)$$

when $\theta_2 \geq 1$.

The MLE for θ in the exponential PDF is given in closed-form as $\hat{\theta} = \left\{\sum z_i\right\}^{-1}\left(n - \sum\delta_i\right)$. For the Gamma PDF, the MLEs for θ_1 and θ_2 are determined numerically in Fortran 77 using results from Wilk et al. (1962). Finally, the MLEs for θ_1 and θ_2 in the Weibull are also determined numerically in Fortran 77 from the profile likelihood function for θ_2.

We considered approximating the Weibull's MGF with finite sums with convergence acceleration techniques (Small 2010, sec. 8.6). For negative values of s, van Wijngaarden's technique (Press et al. 1992, sec. 5.1) for alternating series was used. We also used the percentage relative error bound from Butler and Paige (2011, equation 23) where our stopping criterion was that this bound is less than $\varepsilon = 10^{-12}$. For positive values of s, Aitken's δ^2 process (Aitken 1926) was used with the approximate relative error (Butler and Paige, 2011, sec. 4.1) and a stopping criterion requiring that this error be less than the $\varepsilon = 10^{-12}$. These convergence acceleration procedures worked well in practice. Unfortunately for the simulation studies we considered, where the true value of θ_2 was set at 3, MLE $\hat{\theta}_2$ was less than unity often enough that we opted to always use IMSL numerical integration routine DQDAGI, instead of convergence acceleration techniques, to approximate the MLE of the Weibull MGF.

10. TAIL-COMPLETED KAPLAN-MEIER MGF ESTIMATORS

When the form of the censoring time distribution is unspecified the Kaplan-Meier (KM) estimator of censoring time survival function $S_g(t)$ is used to determine a discrete or mixed approximation to it. As a result, observed survival times play the role of censored censoring times in these computations. One issue here is that the KM estimator of the failure time survival distribution is not defined for $t > t_{\max}$, where $t_{\max}$ denotes the largest time recorded, if $t_{\max}$ is a censoring time. Note that since we use the KM estimator to estimate the censoring time distribution then the KM estimator in our setting is undefined when $t_{\max}$ is a failure time (a censored censoring time). The (right-continuous) KM estimator is defined as

$$\hat{S}_g(t) = \prod_{t_i \le t} \frac{n_i - d_i}{n_i}$$

where $t_1 < t_2 < \cdots < t_k$ are the distinct times at which censoring occurs, d_i is the number of censoring events at time t_i and n_i is the number of people at risk for censoring at time t_i.

For $c \le t_{\max}$, the KM estimator defines a probability mass function approximation to the censoring time distribution as follows:

$$\hat{g}(c) = \begin{cases} \frac{d_1}{n_1} & c = t_1 \\ \left\{ \prod_{i=1}^{j-1} \left(1 - \frac{d_i}{n_i}\right) \right\} \frac{d_j}{n_j} & c = t_j \text{ for } j = 2, 3, \ldots, k \\ 0 & \text{elsewhere for } c \le t_{\max}. \end{cases} \tag{10.1}$$

For c larger than $t_{\max}$, when $t_{\max}$ is in fact a failure time, we consider a number of "tail completion" methods.

We considered the tail completion method introduced in Efron (1967), which we refer to as "KM Efron". Here the KM estimator is set equal to zero for $c > t_{\max}$. This results in a discrete approximation to $g(\cdot)$.

We also considered tail completion methods described in Moeschberger and Klein (1985). These consist of the "Estimated Order Statistic" method (KM EOS) and the "Restricted Weibull Maximum Likelihood" method (KM Weibull) which they developed and the Brown-Hollander-Korwar method (KM Exponential) described in Brown, Hollander & Korwar (1974). In our setting, the KM EOS method generates estimates for the censoring times for the observed failure times that exceed the largest censoring time. The KM Weibull method on the other hand fits a Weibull distribution which dovetails with $\hat{S}_g(t_{\max})$ and is a more general case of the KM Exponential method which fits an exponential distribution which agrees with $\hat{S}_g(t)$ at $t = t_{\max}$. The KM Weibull and KM Exponential methods yield mixed approximations to $g(\cdot)$. For any of the aforementioned tail-completion methods, an approximate censoring time MGF $\hat{M}_C(\cdot)$ is determined as the Riemann–Stieltjes integral with respect to the approximate censoring time CDF $\hat{F}_g(\cdot)$;

$$\hat{M}_C(s) = \int_0^{\infty} e^{sc} d\hat{F}_g(c) \,. \tag{10.2}$$

When this integral could not be evaluated in closed-form, IMSL numerical integration routine DQDAGI was used to approximate it.

As a final note, Satten and Somnath (2001) show that the KM estimator at time t is an inverse-probability-of-censoring weighted average. Since we know that our survival (censored censoring) times are exponential it stands to reason that it may be possible to improve upon the KM estimator. Suzukawa (2004, sec. 4) develops improved KM estimators which are unbiased in small samples and which are appropriate when censoring times are known to be exponential. We found however that MGF estimators based on these methods yielded confidence intervals that performed much worse that confidence interval based on the classical KM estimator, in the small sample settings we consider. The reason for this is probably the increase in variance that comes with the decreased bias of Suzukawa's estimators.

11. LARGE SAMPLE RESULTS

In this section, we present a number of large sample results including the generalization of the uniform consistency and weak convergence results for the empirical saddlepoint approximation, as derived in Feuerverger (1989).

11.1. CENSORING TIME SADDLEPOINT APPROXIMATION

Here we consider convergence results for the saddlepoint approximation based upon $\hat{M}_C(s)$ in (10.2). These results are a generalization of the large sample results from Feuerverger (1989) since in the presence of no censoring the KM distribution in (10.1) coincides with the empirical distribution. The approximate integral in (10.2), for fixed value of s, is an example of a Kaplan-Meier (KM) integral. Furthermore, the j-th derivative of this approximation is itself a Kaplan-Meier integral which approximates the j-th derivative of $M_C(s)$, which we denote as $M_C^{(j)}(s)$. Large sample properties of KM integrals, with Efron tail-completion, which is also known as Efron's self-consistent estimator (Efron, 1967), have been considered in a number of papers including Breslow and Crowley (1974), Gill (1983), Schick, Susarla and Koul (1988), Yang (1994) and Stute (1995). The self-consistent KM estimator introduces no bias when the largest time on study is censored; denote this event as $\left\{\Delta_{(n)} = 0\right\}$ where $\Delta_{(n)}$ is taken to be the survival indicator function for largest time on study $Z_{(n)}$. Maller and Zhou (1993) show that

$$\lim_{n\to\infty} P\left(\Delta_{(n)} = 0\right) = \lim_{t\to\tau_H} \frac{h_g\left(t;\theta_0\right)/h_f\left(t;\lambda_0\right)}{h_g\left(t;\theta_0\right)/h_f\left(t;\lambda_0\right)+1}$$

where $h_g\left(\cdot;\theta_0\right)$ is the hazard rate function for the censoring time distribution, $h_f\left(\cdot;\lambda_0\right)$ is the hazard rate for the survival times, $\tau_G = \min\left\{z : S_g(z) = 0\right\}$, $\tau_F = \min\left\{z : S_f(z) = 0\right\} = \infty$ and $\tau_H = \min\left(\tau_G, \tau_F\right)$.

In our setting, this reduces to

$$\lim_{n\to\infty} P\left(\Delta_{(n)}=0\right)=\lim_{t\to\tau_G}\frac{h_g\left(t;\theta_0\right)}{h_g\left(t;\theta_0\right)+\lambda_0}.$$

As a result, we see that for $\tau_G=\infty$ and censoring distributions with increasing failure rates (IFR)

$$\lim_{n\to\infty} P\left(\Delta_{(n)}=0\right)=1$$

and for censoring distributions with decreasing failure rates (DFR) that this probability approaches 0.

It appears that Yang (1994) provides the most efficient way to prove the asymptotic normality of the j-th derivative of $\hat{M}_C(s)$, denote this by $\hat{M}_C^{(j)}(s)$, where j is any non-negative integer, i.e. $j\in\mathbb{Z}^*:=\{0\}\cup\mathbb{Z}^+$. From Corollary 1 of Theorem 2 we obtain that if $F_g(\cdot)$ is continuous (which is what we shall assume throughout this section) and

$$\int_0^\tau \left(c^j e^{sc}\right)^2 e^{\lambda_0 c} dF_g\left(c\right)=M_C^{(2j)}\left(2s+\lambda_0\right)<\infty$$

then, for any fixed value of s, pointwise weak convergence follows;

$$\sqrt{n}\left(\hat{M}_C^{(j)}(s)-M_C^{(j)}\left(s\right)\right)\overset{D}{\to} N\left(0,\sigma_{j,s}^2\right) \tag{11.1}$$

for $j=0,1,\ldots$ where

$$\sigma_{j,s}^2=M_C^{(2j)}\left(2s+\lambda_0\right)-\left[M_C^{(j)}\left(s\right)\right]^2-E_F\left[\left(M_C^{(j)}(s;z)\right)^2\exp(2\lambda_0 z)S_g(z)\right],$$

$M_C^{(j)}\left(s;z\right)$ is the j-th derivative of the MGF for the truncated censoring time distribution on the interval (z,∞) and where $E_F\left(\cdot\right)$ denotes expectation with respect to the exponential survival time distribution with rate λ_0.

In the particular case of exponential censoring times, one obtains

$$\sigma_{j,s}^2 = \frac{(2j)!\theta_0}{(\theta_0 - 2s - \lambda_0)^{2j+1}} - \frac{\theta_0^2}{(\theta_0 - s)^{2j+2}} \times E_F \left\{ (j!)^2 + [\Gamma(j+1, (\theta_0 - s)z)]^2 \exp\left(-\left(\theta_0 - 2\lambda_0\right) z\right) \right\}$$

where $\Gamma(\cdot,\cdot)$ denotes the incomplete Gamma function (Abramowitz and Stegun, 1972, sec. 6.5).

Note however that in the proposed confidence interval methods one requires an approximation for $M_C^{(j)}(-\lambda)$ where $\lambda > 0$ and $j = 0, 1, 2$. As a result, for any j, Yang's condition is satisfied for $\lambda_0 - 2\lambda$ in the common convergence strip of the $M_C^{(j)}(\cdot)$'s which we denote as $(-\infty, \zeta)$ for $\zeta > 0$. Therefore, convergence in distribution for MGF derivatives of all orders is guaranteed for $\{\lambda : \max\left[\left(\lambda_0 - \zeta\right)/2, 0\right] \leq \lambda\}$.

The pointwise convergence in distribution in (11.1) may be extended to uniform consistency and distributional convergence over closed intervals contained in the convergence strip of censoring time MGF $M_C(\cdot)$. Good references for the empirical process techniques we use include van der Vaart (1998, Chapt. 19) and Kosorok (2008).

Consider the following classes of Borel measurable and integrable functions

$$\mathcal{F}_r^{(j)} = \left\{ f_s^{(j)}(c) = c^j \exp(sc) : s \in [a/r, b/r] \right\}$$

for $j \in \mathbb{Z}^*$ and $1 \leq r < \infty$. The $L_r(F_g)$ ε-bracketing number for $\mathcal{F}_r^{(j)}$, for a fixed value j, which we denote as $N_{[\,]}(\varepsilon, \mathcal{F}_r^{(j)}, L_r(F_g))$, is the smallest number of ε-brackets needed to ensure that every $f_s^{(j)} \in \mathcal{F}_r^{(j)}$ lies in at least one ε-bracket. A bracket in $\mathcal{F}_r^{(j)}$ is defined in the context of two $L_r(F_g)$ functions $b_{1,j}$ an $b_{2,j}$ which, by definition, satisfy

$$\left\{ \int \left[b_{i,j}(c)\right]^r dF_g(c) \right\}^{1/r} < \infty$$

for $i = 1, 2$.

The associated bracket $[b_{1,j}, b_{2,j}]$ consists of all $f_s^{(j)}$ functions in $\mathcal{F}_r^{(j)}$ satisfying

$$P_{F_g}\left[b_{1,j} \leq f_s^{(j)} \leq b_{2,j}\right] := \int I\left[b_{1,j}(c) \leq f_s^{(j)}(c) \leq b_{2,j}(c)\right] dF_g(c) = 1$$

where $I[A]$ denotes the indicator function for event A. Furthermore, a bracket $[b_{1,j}, b_{2,j}]$ is an ε-bracket if

$$\left\{\int \left[b_{1,j}(c) - b_{2,j}(c)\right]^r dF_g(c)\right\}^{1/r} \leq \varepsilon.$$

To show that class $\mathcal{F}_1^{(j)}$ is F_g-Glivenko-Cantelli for any $j \in \mathbb{Z}^*$, i.e. establish uniform consistency for said class;

$$\sup_{s\in[a,b]}\left|\hat{M}_C^{(j)}(s) - M_C^{(j)}(s)\right| \to 0,$$

we need to prove that the $L_1(F_g)$ ε-bracketing number, $N_{[\,]}(\varepsilon, \mathcal{F}_1^{(j)}, L_1(F_g))$, is finite for every $\varepsilon > 0$.

Theorem 11.1. *$L_1(F_g)$ ε-bracketing number $N_{[\,]}(\varepsilon, \mathcal{F}_r^{(j)}, L_r(F_g))$ is finite for every $\varepsilon >$ 0 and for every $j \in \mathbb{Z}^*$.*

Proof: Let $j \in \mathbb{Z}^*$. From the Mean Value Theorem, for $1 \leq r < \infty$, we obtain with $s_1, s_2 \in [a/r, b/r]$, after taking absolute values, the following inequality

$$\left|f_{s_2}^{(j)}(c) - f_{s_1}^{(j)}(c)\right| \leq c^{j+1}\exp(s_0 c)\left|s_2 - s_1\right| \leq c^{j+1}\exp(bc/r)\left|s_2 - s_1\right|, \tag{11.2}$$

where s_0 is between s_1 and s_2.

Then

$$\int \left[c^{j+1}\exp(bc/r)\right]^r dF_g(c) = \int c^{r(j+1)}\exp(bc)dF_g(c) < \infty \tag{11.3}$$

since by definition

$$\int f_b^{(0)}(c) dF_g(c) = \int \exp(bc) dF_g(c) < \infty$$

and the exponential in the former integrand, $\exp(bc)$, dominates power term $c^{r(j+1)}$. The fact that

$$\mathcal{F}_r^{(j)} = \left\{ f_s^{(j)}(c) = c^j \exp(sc) : s \in [a/r, b/r] \right\}$$

satisfies properties (11.2) and (11.3) establishes that the $\mathcal{F}_r^{(j)}$ are parametric classes in the sense of van der Vaart (1989, Chapt. 19). From this it follows that there exists a constant $k_{1,r,j}$ with

$$N_{[\,]}(\varepsilon, \mathcal{F}_r^{(j)}, L_r(F_g)) \leq \frac{k_{1,r,j}\,|b-a|}{\varepsilon r k_{2,r,j}}$$

for every $0 < \varepsilon < (b-a)/r$ where

$$k_{2,r} = \left\{ \int c^{r(j+1)} \exp(bc) dF_g(c) \right\}^{1/r}.$$

Note that by definition if $\varepsilon_1 < \varepsilon_2$ then

$$N_{[\,]}(\varepsilon_2, \mathcal{F}_r^{(j)}, L_r(F_g)) \leq N_{[\,]}(\varepsilon_1, \mathcal{F}_r^{(j)}, L_r(F_g))$$

so that, for every $\varepsilon > 0$,

$$N_{[\,]}(\varepsilon, \mathcal{F}_r^{(j)}, L_r(F_g)) < \infty.$$

Letting $r = 1$ establishes that class $\mathcal{F}_1^{(j)}$ is F_g-Glivenko-Cantelli.

□

To show that $\mathcal{F}_2^{(j)}$ is F_g-Donsker, meaning that

$$\sqrt{n}\left(\hat{M}_C^{(j)}(s) - M_C^{(j)}(s) \right)$$

converges weakly to a mean zero Gaussian process for $s \in [a/2, b/2]$, we need to show that the *unit bracketing integral* for class $\mathcal{F}_2^{(j)}$;

$$J_{[\,]}(\mathcal{F}_2^{(j)}, L_2(F_g)) = \int_0^1 \sqrt{\ln N_{[\,]}(\varepsilon, \mathcal{F}_2^{(j)}, L_2(F_g))} d\varepsilon$$

is finite.

Theorem 11.2. *The unit bracketing integral for class $\mathcal{F}_2^{(j)}$ is finite for every $j \in \mathbb{Z}^*$.*

Proof: From the proof of Theorem 11.1

$$\begin{aligned} J_{[\,]}(\mathcal{F}_2^{(j)}, L_2(F_g)) &\leq \int_0^1 \sqrt{\ln\left(\frac{k_{1,2,j}\,|b-a|}{2\varepsilon k_{2,2,j}}\right)} d\varepsilon \\ &= \sqrt{\ln\left(\frac{k_{1,2,j}\,|b-a|}{2k_{2,2,j}}\right)} \int_0^1 \sqrt{\ln\left(\frac{1}{\varepsilon}\right)} d\varepsilon \\ &= \frac{1}{2}\sqrt{\pi \ln\left(\frac{k_{1,2,j}\,|b-a|}{2k_{2,2,j}}\right)} \end{aligned}$$

where the last integral in ε is evaluated in closed-form after a basic substitution which yields $\Gamma\left(\frac{3}{2}\right)$.

□

Next we consider the covariance function of the limiting Gaussian process of $\sqrt{n}\left(\hat{M}_C^{(j)}(s) - M_C^{(j)}(s)\right)$ for $s \in [a/2, b/2]$ where $j \in \mathbb{Z}^*$. To do so we first need to consider functions used in the von Mises expansion of the KM integral, which is given in Bae and Kim (2003). Good references for von Mises expansions include van der Vaart (1998, Chapt. 20) and Fernholz (1983).

Theorem 11.3. *The covariance function of the limiting Gaussian process for $\sqrt{n}\left(\hat{M}_C^{(j)}(s) - M_C^{(j)}(s)\right)$ with $s_1, s_2 \in [a/2, b/2]$, is given as*

$$Cov\left(\Psi\left(s_1\right), \Psi\left(s_2\right)\right)$$

where

$$\Psi(s) = f_s^{(j)}(Z)\kappa(Z)(1-\Delta) + \tilde{f}_s^{(j)}(Z)\Delta - \breve{f}_s^{(j)}(Z),$$

$$\kappa(z) = e^{\lambda_0 z},$$

$$\tilde{f}_s^{(j)}(z) = e^{\lambda_0 z} M_C^{(j)}(s; z)$$

and

$$\breve{f}_s^{(j)}(z) = \int_0^z \lambda_0 e^{\lambda_0 t_1} M_C^{(j)}(s; t_1)\, dt_1.$$

Proof: From (4.1) we have the following subdistribution functions for time on study random variable Z:

$$\begin{aligned} F_0(z; \lambda_0, \theta_0) &= P(Z \leq z, \Delta = 0) = \int_0^z e^{-\lambda_0 t} g(t; \theta_0) dt, \\ F_1(z; \lambda_0, \theta_0) &= P(Z \leq z, \Delta = 1) = \int_0^z \lambda_0 e^{-\lambda_0 t} S_g(t; \theta_0) dt. \end{aligned}$$

Now define

$$\begin{aligned} \kappa(z) &= \exp\left[\int_0^z \frac{dF_1(t; \lambda_0, \theta_0)}{S_g(t; \theta)\, e^{-\lambda_0 t}}\right] = \exp\left[\int_0^z \frac{\lambda_0 e^{-\lambda_0 t} S_g(t; \theta_0)}{S_g(t; \theta_0)\, e^{-\lambda_0 t}} dt\right] \\ &= e^{\lambda_0 z} \end{aligned}$$

and

$$\begin{aligned} \tilde{f}_s^{(j)}(z) &= \frac{1}{S_g(z; \theta_0)\, e^{-\lambda_0 z}} \int_0^\infty I[z < t]\, f_s^{(j)}(t)\kappa(t)\, dF_0(t; \lambda_0, \theta_0) \\ &= e^{\lambda_0 z} \int_z^\infty \frac{t^j e^{ts} g(t; \theta_0)}{S_g(z; \theta_0)} dt \\ &= e^{\lambda_0 z} M_C^{(j)}(s; z). \end{aligned}$$

In addition, define

$$\begin{aligned}
\breve{f}_s^{(j)}(z) &= \int_0^\infty \int_0^\infty \frac{I\left[t_1 < z, t_1 < t_0\right] f_s^{(j)}(t_0)\kappa\left(t_0\right)}{S_g^2\left(t_1;\theta_0\right)e^{-2\lambda_0 t_1}} dF_0\left(t_0;\lambda_0,\theta_0\right) dF_1\left(t_1;\lambda_0,\theta_0\right) \\
&= \int_0^z \int_{t_1}^\infty \frac{t_0^j e^{t_0 s} e^{\lambda_0 t_0}}{S_g^2\left(t_1;\theta_0\right)e^{-2\lambda_0 t_1}} e^{-\lambda_0 t_0} g(t_0;\theta_0)\lambda_0 e^{-\lambda_0 t_1} S_g(t_1;\theta_0) dt_0 dt_1 \\
&= \int_0^z \int_{t_1}^\infty \frac{t_0^j e^{t_0 s}}{S_g\left(t_1;\theta_0\right)} \lambda_0 e^{\lambda_0 t_1} g(t_0;\theta_0) dt_0 dt_1 \\
&= \int_0^z \lambda_0 e^{\lambda_0 t_1} \int_{t_1}^\infty \frac{t_0^j e^{t_0 s} g(t_0;\theta_0)}{S_g\left(t_1;\theta_0\right)} dt_0 dt_1 \\
&= \int_0^z \lambda_0 e^{\lambda_0 t_1} M_C^{(j)}\left(s;t_1\right) dt_1.
\end{aligned}$$

From Bae and Kim (2003, Thm. 2.1) the result now follows.

□

Pointwise convergence of the CGF process

$$\sqrt{n}\left(\hat{K}_C(s) - K_C\left(s\right)\right)$$

easily follows from the continuity of the logarithm function. More generally, since we have established uniform consistency and distributional convergence for the MGF process these properties automatically hold for the CGF process by the Continuous Mapping Theorem. Furthermore, since the j-th derivative of the CGF, $\hat{K}_C^{(j)}(s)$ is the ratio of a polynomial function in the $\hat{M}_C^{(k)}(s)$, for $k = 0, 1, \ldots, j$, and $\left[\hat{M}_C(s)\right]^{2j}$ it follows from the Multivariate Delta Method that uniform consistency holds for $\hat{K}_C^{(j)}(s)$;

$$\sup_{s\in[a,b]} \left|\hat{K}_C^{(j)}(s) - K_C^{(j)}(s)\right| \to 0$$

where $j \in \mathbb{Z}^*$. The $\hat{K}_C^{(j)}(s)$ can be shown to be F_g-Donsker by first using multivariate versions of the Central Limit Theorem and Delta Method, which establish weak convergence over a finite grid of s-values, and then by using the method of proof for Theorem 2.4 in Feuerverger (1989). From this it follows that the saddlepoint approximation for survival time density $f(z;\lambda_0)$, based upon $\hat{M}_C(s)$, converges weakly to the saddlepoint approximation for this density based upon true MGF $M_C(s)$, for $s \in [a/2, b/2]$.

11.2. CONVERGENCE OF THE BARTHOLOMEW METHODS

With regards to the solution of equations

$$1 - \hat{M}_C\left(-\breve{\lambda}_L\right) = \hat{p}_L$$
$$1 - \hat{M}_C\left(-\breve{\lambda}_U\right) = \hat{p}_U$$

for determining the approximate generalized Bartholomew confidence intervals, Theorem 1.1 of Stute and Wang (1993) guarantees that $\breve{\lambda}_L$ and $\breve{\lambda}_U$ converge to the true confidence bounds, $\hat{\lambda}_L$ and $\hat{\lambda}_U$, which solve the following set of equations:

$$1 - M_C\left(-\hat{\lambda}_L\right) = \hat{p}_L$$
$$1 - M_C\left(-\hat{\lambda}_U\right) = \hat{p}_U.$$

12. EXAMPLE

In this section, we discuss the application of our proposed methods to a real data set. The data appears in Der and Everitt (2006, sec. 14.2.1) and concerns the survival times in weeks for 20 patients with late stage (stage 3 or 4) melanoma. Here z_i represents the time on study for the ith subject and δ_i is the associated survival indicator function. The data is shown in Table 12.1.

Table 12.1. Survival Times and Indicator Function Values for 20 Subjects with Stage 3 or Stage 4 Melanoma

Time z_i (weeks)	**Survival Indicator δ_i**	**Time z_i (weeks)**	**Survival Indicator δ_i**
12.8	1	77.2	1
15.6	1	82.4	1
24	0	87.2	0
26.4	1	94.4	0
29.2	1	97.2	0
30.8	0	106	0
39.2	1	114.8	0
42	1	117.2	0
58.4	0	140	0
72	0	168	0

PROC LIFETEST in SAS was used to generate a plot of the negative natural logarithm of the KM estimator for the survival time distribution. Since this plot, (c.f. Figure 12.1), displayed no serious departures from linearity, we assume that the survival times are exponentially distributed. Furthermore, the censoring fraction for this data set is 0.60 since 12 of the original 20 subjects were still alive at the end of the study.

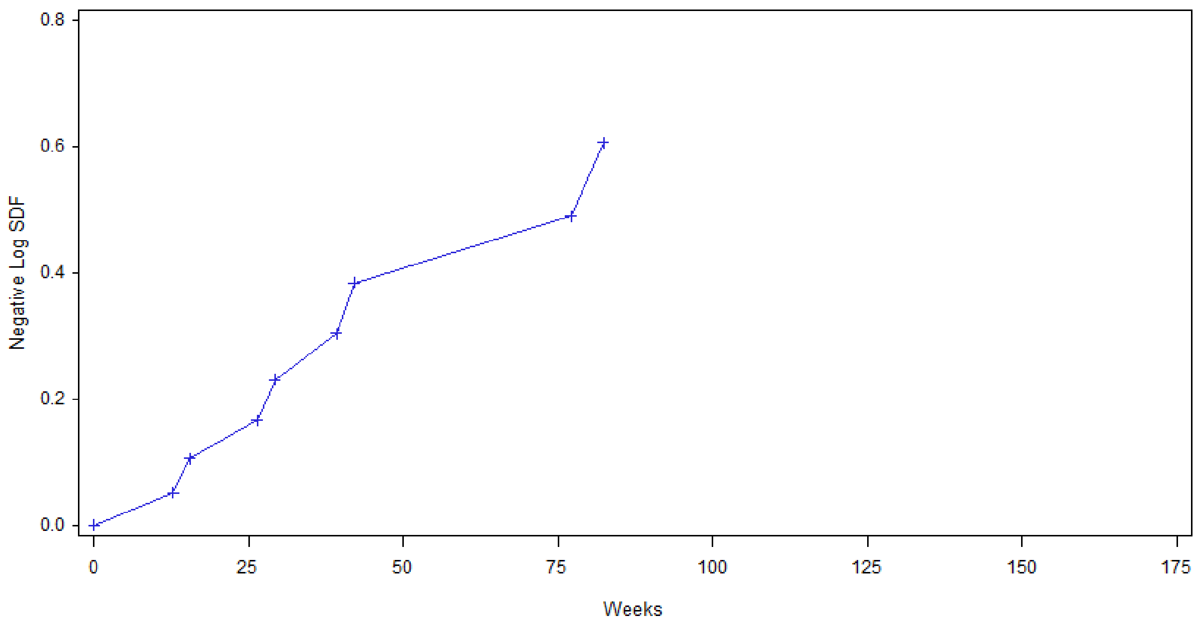

Figure 12.1. Negative Logarithm of Survival Function for Melanoma Data

Table 12.2 provides the lower 95% confidence bounds for the survival rate of patients with late stage melanoma, for each of the censoring models we consider. Column heading "SPA" represents the SPBB lower bounds for the parameter λ.

Table 12.2. Coverage Probabilities of Lower Bounds for Melanoma Data

	Methods				
Distribution	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**
Exponential	0.0056707	0.0023188	0.0026070	0.0026704	0.0026758
Gamma	0.0056706	0.0025598	0.0028469	0.0029092	0.0029145
Weibull	0.0056741	0.1403105	0.1559195	0.1593057	0.1595921
KM Efron	0.0056705	0.0025514	0.0028349	0.0028965	0.0029016

Note that for this data set, since the largest time on study is censored then the KM estimator for the censoring distribution has a finite tail. As a result, the KM Efron, KM Weibull, KM EOS and KM Exponential censoring models are all the same. From Table 12.2, we see that the SPBB method is robust to choice of censoring model and there is little difference between the lower 95% confidence bounds gotten from the different censoring models. This is interesting since the negative natural logarithm of the KM estimator for the censoring times is strongly non-linear which suggests that the Exponential censoring model is not appropriate. In contrast, the Bartholomew methods; the Clopper-Pearson interval (Clopper), Jeffreys confidence interval (Jeffreys), the Agresti-Coull confidence interval (Agresti), and the Wilson confidence interval (Wilson), appear to be less stable than the saddlepoint method and in particular the Bartholomew lower bounds for the Weibull censoring model differ significantly from the lower bounds for the other censoring models.This is due to Weibull MLEs which are close to the support boundary.

Table 12.3 presents one-sided 95% upper confidence bounds for the survival rate of patients with late stage melanoma.

Table 12.3. Coverage Probabilities of Upper Bounds for Melanoma Data

	Methods				
Distribution	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**
Exponential	0.0056823	0.0239717	0.0116663	0.0116360	0.0116184
Gamma	0.0056823	0.0160748	0.0098170	0.0097983	0.0097874
Weibull	0.0056763	0.8573972	0.5288966	0.5279053	0.5273294
KM Efron	0.0056827	0.0156697	0.0096247	0.0096067	0.0095960

Table 12.3 provides results which are similar to those in Table 12.2. The saddlepoint approximation is stable while the other methods are not and the Weibull censoring model again yields large upper bounds due to unusually small MLE values.

Table 12.4 provides 95% confidence intervals for the survival rate of patients with stage 3 or 4 melanoma.

Table 12.4. Coverage Probabilities of Lower and Upper Bounds for Melanoma Data

	Methods					
Distribution	**Bound**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**
Exponential	Lower	0.0056704	0.0019770	0.0022317	0.0023350	0.0023426
	Upper	0.0056827	0.0279085	0.0134210	0.0133019	0.0132711
Gamma	Lower	0.0056703	0.0022115	0.0024718	0.0025761	0.0025837
	Upper	0.0056827	0.0176785	0.0108663	0.0107972	0.0107792
Weibull	Lower	0.0056749	0.1213402	0.1355226	0.1411982	0.1416106
	Upper	0.0056763	0.9410986	0.5842754	0.5806337	0.5796860
KM Efron	Lower	0.0056703	0.0022063	0.0024643	0.0025674	0.0025750
	Upper	0.0056827	0.0172287	0.0106374	0.0105709	0.0105537

Here again we see that the SPBB method is quite robust with respect to assumed censoring distribution and the Bartholomew methods with a Weibull censoring model, that again has usually small MLEs, yields results which differ substantially from those for the other censoring models. Furthermore, the SPBB method yields the shortest intervals of all the unconditional methods.

13. MONTE CARLO STUDIES

In this section, we perform simulation studies to compare the performance of the unconditional SPBB confidence intervals with confidence intervals from the four generalized Bartholomew methods we propose. When designing our simulation studies, we took into account how the coverage of the unconditional SPBB method depends upon $[M_C(-\lambda_0)]^n$, the probability that $\hat{\lambda}$ is indefinite.

Recall from equation (6.2) that $P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) = [M_C(-\lambda_0)]^n$ with probability $[M_C(-\lambda_0)]^n$ and with probability $1 - [M_C(-\lambda_0)]^n$:

$$P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) \sim Unif\left([M_C(-\lambda_0)]^n, 1\right).$$

The $(1-\alpha)100\%$ unconditional confidence interval computed from $P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda, \theta_0\right)$ will contain the true survival rate λ_0 if and only if

$$\alpha/2 \leq P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) \leq 1 - \alpha/2.$$

If $[M_C(-\lambda_0)]^n > \alpha/2$ then $P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right)$ can never lie below $\alpha/2$ and $\hat{\lambda}_U$, the upper confidence bound of the interval, will never lie below λ_0. As a result, the coverage probability for an interval with nominal $(1-\alpha)100\%$ coverage will be $1 - \alpha/2$.

Note, however, that

$$\begin{aligned} P\left\{P\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right) \leq 1 - \alpha/2\right\} &= [M_C(-\lambda_0)]^n + \left(1 - [M_C(-\lambda_0)]^n\right) \\ &\times \left(\frac{1 - \alpha/2 - [M_C(-\lambda_0)]^n}{1 - [M_C(-\lambda_0)]^n}\right) \\ &= 1 - \alpha/2 \end{aligned}$$

so that $(\alpha/2)\,100\%$ percent of the time $\hat{\lambda}_L$, the lower confidence bound of the interval, will lie above λ_0. Therefore, one can always construct a valid lower confidence bound for λ_0, regardless of the $[M_C(-\lambda_0)]^n$ value.

In practice, $[M_C(-\lambda_0)]^n$ will exceed $\alpha/2$ for small samples and heavy censoring only. These are, however, the types of situations we consider in our simulation study. Thus, for ease of both presentation and comparison we only present results for the 95% lower confidence bound for λ_0 (which incidentally equates to a 95% upper confidence bound for the mean survival time).

Another issue we encountered was numerical instability of the conditional MGF in (4.5) when computing the Luganani and Rice CDF approximation. The numerator of this transform involves the difference of two MGFs. In a small number of simulation runs, when the value of $|\hat{s}|$ was large, the values of these two MGFs were so close to each other that their difference was evaluated as zero in double precision Fortran 77. In the numerical analysis literature, this phenomenon is known as catastrophic cancelation (Datta, 2010, sec. 3.8).

Saddlepoint approximations for mixed distributions were considered in Lund, Butler and Paige (1999). Here it is shown that the Luganani and Rice CDF approximation naturally detects and adjusts for point masses in the support of a mixed random variable. This means that the Luganani and Rice CDF approximation should be accurate when one does not condition upon the event $\sum \Delta_i > 0$. This unconditional approximation, which is similar to the conditional approximation in equation (6.3), is of the form $\hat{P}\left(\hat{\lambda} \leq \hat{\lambda}_{obs}|\lambda_0, \theta_0\right)$ where the unconditional MGF of random estimating equation $U(\lambda)$;

$$M_{U(\lambda)}(s) = [M(s/\lambda, -s)]^n$$

with $M(\cdot, \cdot)$ given in equation (4.1) is used. In the overwhelming majority of simulated data sets, both types of Luganani and Rice approximations (conditional and unconditional) were computable and we found that the coverage probabilities of the resulting confidence intervals were nearly identical to three significant digits. As a result, in our simulations we used the unconditional Luganani and Rice approximation.

We took the survival rate λ_0 to be 1 and assumed four types of censoring distributions: Type 1, Exponential, Gamma and Weibull. The assumed parameter values for these distributions depended in part on the *censoring fraction* (cf) which is the probability that the survival time for either a single subject or item is censored, or equivalently the expected fraction of censored observations in the sample. Recall that this probability is given in terms of $M_C(\cdot)$ as

$$cf = \int_0^\infty e^{-\lambda c} dFg(c)\, dc = M_C(-\lambda_0)\,. \tag{13.1}$$

For each censoring distribution, we defined a grid of censoring fraction values;

$$cf = (0.5, 0.6, 0.7, 0.8, 0.9)$$

and a grid of sample size values;

$$n = (10, 15, 20, 30)\,.$$

At each configuration, where the censoring distribution, censoring model, cf and n values are set, we performed $10,000$ simulations to estimate the coverage of the nominal 95% lower confidence bounds.

From equation (13.1), $\theta = -\ln(cf)/\lambda_0$ for Type 1 censoring times and for Exponential censoring times this means that $\theta = \lambda_0 cf/(1 - cf)$. For Gamma censoring times we took rate parameter θ_1 to be 1 and let shape parameter θ_2 vary so that $\theta_2 = -\ln(cf)/\ln(2)$. With this choice of parameters, the Gamma censoring time distribution becomes increasing thin tailed (in comparison with the Exponential) as the censoring fraction increases. For the Weibull censoring times we set θ_2 at 3 and determined θ_1 numerically. With this choice of parameters, the Weibull censoring time distribution goes from being strongly right skewed to being very diffuse and symmetric as censoring fraction cf approaches 1.

In the following sections we discuss our simulation results first for Type 1 censoring and then for random censoring.

13.1. TYPE 1 CENSORING

Table 13.1 displays results for Type 1 censoring. In this table all numerical entries are percentages, and column heading "$\sum \delta_i = 0$" represents the percentage of times there was total censoring. Column heading "SPA" represents the saddlepoint based-bootstrap coverage. In addition, column heading "Exact" represents the coverage probability for Abe-Iwasaki confidence intervals which make use of the exact unconditional CDF formula in (2.2). Next to each estimated coverage probability is either a "b" or a "g". When the 95% large-sample confidence interval for the true coverage probability, $(\hat{p}_L^c, \hat{p}_U^c,)$, does not contain 0.95 then we assign a "b" which is short for a "bad" coverage probability that differs significantly from the nominal value. In a similar fashion, when confidence interval $(\hat{p}_L^c, \hat{p}_U^c,)$ contains 0.95 we assign a "g" for a "good" coverage probability that does not differ significantly from the nominal value. Here we see that the SPBB method (SPA) generally outperforms the generalized Bartholomew methods; the Clopper-Pearson interval (Clopper), Jeffreys confidence interval (Jeffreys), the Agresti-Coull confidence interval (Agresti), and the Wilson confidence interval (Wilson) in terms of coverage. Furthermore, this performance does not seem to depend upon the percentage of times there was total censoring. The poorer performance of the SPBB method, in comparison to Abe and Iwasaki's exact method, for the $cf = 0.9$ cases seems to be due to the following reasons: (i) numerical instabilities which occur when one approaches the edge of the support for estimator $\hat{\lambda}$ when $\sum \delta_i$ has a value of 1 or 2 and (ii) the inaccuracies that result from replacing the conditional saddlepoint CDF for $\hat{\lambda}$ with the unconditional saddlepoint CDF. The Luganani and Rice CDF approximation does detect and adjust for point mass at 0, but as seen in Lund, Butler and Paige(1999) this adjustment can exhibit varying degrees of accuracy. Nonetheless, the SPBB method generally outperforms the generalized Bartholomew methods. Note also that the generalized Jeffreys, Agresti and Wilson Bartholomew methods yield the same coverage probabilities for each configuration. This is due to the fact that the underlying confidence interval methods, for proportions, are generally comparable in terms of performance. Furthermore, with Type 1 censoring we have no censoring time parameter to estimate so, as a result, there is no added variation when one inverts censoring time MGF $M_C(s)$, as in (8.1), to obtain the confidence interval for λ_0 from the confidence interval for $p = M_C(-\lambda_0)$.

Table 13.1. Coverage Probabilities for the Type 1 Censoring.

Type 1 Censoring Times and Model								
cf	**n**	**SPA**	**Exact**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$
0.5	10	94.90,g	94.96,g	99.05,b	94.52,b	94.52,b	94.52,b	0.13
0.5	15	94.75,g	94.75,g	97.93,b	94.05,b	94.05,b	94.05,b	0.01
0.5	20	95.00,g	95.00,g	98.11,b	94.30,b	94.30,b	94.30,b	0.00
0.5	30	95.20,g	95.20,g	95.24,g	95.24,g	95.24,g	95.24,g	0.00
0.6	10	95.06,g	95.02,g	98.68,b	94.59,g	94.59,g	94.59,g	0.70
0.6	15	94.55,b	94.57,g	96.47,b	96.47,b	96.47,b	96.47,b	0.03
0.6	20	94.97,g	94.97,g	97.89,b	94.09,b	94.09,b	94.09,b	0.00
0.6	30	95.14,g	95.13,g	95.30,g	95.30,g	95.30,g	95.30,g	0.00
0.7	10	95.24,g	95.07,g	95.22,g	95.22,g	95.22,g	95.22,g	2.66
0.7	15	95.08,g	95.02,g	98.34,b	94.98,g	94.98,g	94.98,g	0.36
0.7	20	95.17,g	95.15,g	95.28,g	95.28,g	95.28,g	95.28,g	0.05
0.7	30	95.02,g	95.06,g	96.12,b	96.12,b	96.12,b	96.12,b	0.00
0.8	10	95.20,g	94.60,g	96.51,b	96.51,b	96.51,b	96.51,b	10.63
0.8	15	94.55,b	94.83,g	98.12,b	93.83,b	93.83,b	93.83,b	3.31
0.8	20	95.23,g	95.13,g	97.07,b	97.07,b	91.13,b	91.13,b	1.25
0.8	30	94.63,g	95.01,g	97.46,b	93.79,b	93.79,b	93.79,b	0.11
0.9	10	93.29,b	95.00,g	98.62,b	92.83,b	92.83,b	92.83,b	34.53
0.9	15	94.23,b	94.83,g	98.59,b	94.23,b	94.23,b	94.23,b	20.67
0.9	20	95.67,b	95.03,g	95.67,b	95.67,b	95.67,b	95.67,b	12.27
0.9	30	93.96,b	94.18,b	97.33,b	92.66,b	92.66,b	92.66,b	4.58

13.2. RANDOM CENSORING TIMES

13.2.1. Exponential Censoring Times. Table 13.2 displays results for simulations where the censoring distribution is Exponential and the assumed censoring model is also Exponential. Here column heading “Exact” refers to coverage probabilities for 95% lower confidence bounds obtained with the exact unconditional distribution of $\hat{\lambda}$ given in Appendix A. Here we see that the SPBB method generally outperforms the Bartholomew methods and basically performs as well as the exact procedure with the exception of the $cf = 0.9$ and $n = 20$ case (and here the difference in simulated coverage probabilities is 0.02%). Furthermore, its’ performance is good as long as “$\sum \delta_i = 0$” is not too large. Tables 13.3 and 13.4 display results for when the assumed censoring model is Gamma and Weibull, respectively. Here exact procedures are not available but one sees that the SPBB method continues to outperform the Bartholomew methods for nearly every configuration; one exception is the Wilson method which performs exceedingly well when $cf = 0.9$ and $n = 30$ and the Agresti method which often performs well for $cf = 0.9$ cases. In addition, the Jeffreys method on occasion performs well. Tables 13.5, 13.6, 13.7 and 13.8 display results for the KM Exponential, KM Efron, KM Weibull and KM EOS censoring models, respectively. The column heading “Adjust” refers to the percentage of times that the largest time on study was a survival time and therefore the KM estimator required tail completion. This tail adjustment percentage is particularly high for small cf values meaning that the choice of KM tail completion method is particularly important when the level of censoring is relatively modest. Here we see that the KM Efron method yields coverage probabilities which seem to be different from those obtained from the other tail completion methods and, as a result, the KM Efron model may not be the best choice in such settings. Overall, for KM tail completion models we have the same basic conclusions as those for the Exponential, Gamma and Weibull censoring models: (i) the SPBB method generally outperforms the Bartholomew methods (regardless of the values of “Adjust” and “$\sum \delta_i = 0$”) and (ii) and the Wilson and Agresti methods can be competitors to the SPBB method when

$cf = 0.9$. For small censoring fraction values, the occasional good performance of these methods is probably due to a "Lucky n, Lucky p" phenomenon like that discussed in Brown, Cai and DasGupta (2001, sec. 2.1) wherein, for certain values of n and p, the oscillatory coverage probability of binomial confidence interval happens to be very close to the nominal confidence level. Overall, the four tail-completed KM models yield very similar results. In terms of computation complexity, one would prefer the KM Exponential or perhaps KM Efron methods since, unlike the KM Weibull and KM EOS methods, one does not need to fit a Weibull model to the data.

Table 13.2. Coverage Probabilities for the Exponential Censoring Distribution and the Exponential Censoring Model.

Exponential Censoring Model								
cf	**n**	**SPA**	**Exact**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.98,g	94.96,g	99.76,b	99.13,b	99.12,b	99.03,b	0.01
0.5	15	95.06,g	95.05,g	99.73,b	99.30,b	99.28,b	99.26,b	0.01
0.5	20	95.19,g	95.19,g	99.71,b	99.17,b	99.22,b	99.20,b	0.00
0.5	30	95.17,g	95.16,g	99.73,b	99.21,b	99.25,b	99.23,b	0.00
0.6	10	95.13,g	95.12,g	99.59,b	98.57,b	98.46,b	98.44,b	0.72
0.6	15	95.13,g	95.12,g	99.50,b	98.66,b	98.54,b	98.49,b	0.04
0.6	20	95.19,g	95.19,g	99.27,b	98.43,b	98.36,b	98.36,b	0.00
0.6	30	95.28,g	95.28,g	99.32,b	98.56,b	98.53,b	98.53,b	0.00
0.7	10	94.87,g	94.85,g	99.26,b	97.74,b	97.42,b	97.38,b	3.10
0.7	15	95.07,g	95.06,g	99.12,b	97.64,b	97.33,b	97.30,b	0.42
0.7	20	94.63,g	94.63,g	98.74,b	97.60,b	97.28,b	97.25,b	0.04
0.7	30	95.21,g	95.21,g	98.82,b	97.78,b	97.55,b	97.50,b	0.00
0.8	10	95.32,g	95.31,g	98.66,b	96.71,b	96.12,b	95.95,b	11.16
0.8	15	95.37,g	95.36,g	98.39,b	96.52,b	96.05,b	95.96,b	3.24
0.8	20	94.97,g	94.96,g	98.21,b	96.57,b	96.19,b	96.04,b	1.18
0.8	30	95.25,g	95.25,g	98.12,b	96.70,b	96.27,b	96.20,b	0.08
0.9	10	95.49,b	95.48,b	98.14,b	95.56,b	95.10,g	94.00,b	35.33
0.9	15	95.27,g	95.25,g	98.12,b	95.57,b	95.14,g	94.31,b	20.10
0.9	20	95.38,b	95.36,g	97.87,b	95.86,b	95.39,g	94.65,g	12.03
0.9	30	95.39,g	95.38,g	97.70,b	95.90,b	95.56,b	95.05,g	4.46

Table 13.3. Coverage Probabilities for the Exponential Censoring Distribution and the Gamma Censoring Model.

Gamma Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.84,g	99.18,b	98.06,b	98.09,b	98.02,b	0.01
0.5	15	95.06,g	99.39,b	98.59,b	98.62,b	98.60,b	0.01
0.5	20	95.16,g	99.41,b	98.78,b	98.78,b	98.75,b	0.00
0.5	30	95.16,g	99.62,b	99.09,b	99.10,b	99.08,b	0.00
0.6	10	95.07,g	99.09,b	97.73,b	97.55,b	97.51,b	0.72
0.6	15	95.04,g	99.22,b	98.22,b	98.11,b	98.08,b	0.04
0.6	20	95.18,g	99.14,b	98.11,b	98.06,b	98.05,b	0.00
0.6	30	95.26,g	99.17,b	98.42,b	98.35,b	98.33,b	0.00
0.7	10	94.84,g	98.87,b	97.05,b	96.77,b	96.69,b	3.10
0.7	15	94.97,g	98.94,b	97.26,b	96.88,b	96.84,b	0.42
0.7	20	94.65,g	98.61,b	97.50,b	97.11,b	97.09,b	0.04
0.7	30	95.18,g	98.77,b	97.66,b	97.43,b	97.40,b	0.00
0.8	10	95.25,g	98.50,b	96.36,b	95.72,b	95.55,b	11.16
0.8	15	95.31,g	98.32,b	96.45,b	95.91,b	95.80,b	3.24
0.8	20	94.99,g	98.09,b	96.51,b	95.94,b	95.84,b	1.18
0.8	30	95.25,g	98.14,b	96.63,b	96.28,b	96.20,b	0.08
0.9	10	95.42,b	98.06,b	95.48,b	94.98,g	93.99,b	35.33
0.9	15	95.27,g	98.09,b	95.55,b	95.03,g	94.31,b	20.10
0.9	20	95.33,g	97.88,b	95.64,b	95.26,g	94.55,b	12.03
0.9	30	95.34,g	97.70,b	95.82,b	95.49,b	94.99,g	4.46

Table 13.4. Coverage Probabilities for the Exponential Censoring Distribution and the Weibull Censoring Model.

Weibull Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$
0.5	10	94.87,g	99.12,b	97.95,b	98.00,b	97.94,b	0.01
0.5	15	95.06,g	99.38,b	98.51,b	98.57,b	98.51,b	0.01
0.5	20	95.18,g	99.35,b	98.61,b	98.65,b	98.62,b	0.00
0.5	30	95.15,g	99.57,b	99.01,b	99.03,b	99.01,b	0.00
0.6	10	95.09,g	99.07,b	97.66,b	97.49,b	97.43,b	0.72
0.6	15	95.08,g	99.21,b	98.15,b	98.04,b	98.00,b	0.04
0.6	20	95.17,g	98.99,b	98.12,b	98.04,b	98.04,b	0.00
0.6	30	95.27,g	99.17,b	98.44,b	98.41,b	98.38,b	0.00
0.7	10	94.82,g	98.84,b	97.12,b	96.76,b	96.68,b	3.10
0.7	15	94.95,g	98.92,b	97.32,b	96.89,b	96.88,b	0.42
0.7	20	94.64,g	98.61,b	97.52,b	97.09,b	97.04,b	0.04
0.7	30	95.20,g	98.72,b	97.77,b	97.54,b	97.51,b	0.00
0.8	10	95.31,g	98.57,b	96.50,b	95.83,b	95.64,b	11.16
0.8	15	95.28,g	98.32,b	96.56,b	96.02,b	95.87,b	3.24
0.8	20	94.98,g	98.08,b	96.60,b	96.03,b	95.88,b	1.18
0.8	30	95.23,g	98.13,b	96.69,b	96.34,b	96.25,b	0.08
0.9	10	95.44,b	98.14,b	95.62,b	95.22,g	94.18,b	35.33
0.9	15	95.26,g	98.11,b	95.71,b	95.22,g	94.45,b	20.10
0.9	20	95.34,g	97.92,b	95.71,b	95.39,g	94.69,g	12.03
0.9	30	95.35,g	97.75,b	95.82,b	95.55,b	94.98,g	4.46

Table 13.5. Coverage Probabilities for the Exponential Censoring Distribution and the Kaplan-Meier Exponential Censoring Model.

KM Exponential Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	99.34,b	98.30,b	98.29,b	98.19,b	0.01	49.76
0.5	15	95.07,g	99.48,b	98.78,b	98.81,b	98.74,b	0.01	49.97
0.5	20	95.12,g	99.26,b	98.59,b	98.64,b	98.60,b	0.00	49.70
0.5	30	95.17,g	99.48,b	98.93,b	98.92,b	98.92,b	0.00	49.87
0.6	10	95.05,g	99.06,b	97.83,b	97.55,b	97.53,b	0.72	39.76
0.6	15	95.01,g	99.24,b	98.15,b	97.95,b	97.93,b	0.04	39.76
0.6	20	95.11,g	98.93,b	97.92,b	97.75,b	97.74,b	0.00	40.05
0.6	30	95.25,g	99.02,b	98.31,b	98.23,b	98.23,b	0.00	39.95
0.7	10	94.88,g	98.73,b	96.98,b	96.68,b	96.59,b	3.10	29.68
0.7	15	94.90,g	98.83,b	97.13,b	96.74,b	96.72,b	0.42	30.63
0.7	20	94.67,g	98.42,b	97.19,b	96.90,b	96.87,b	0.04	30.32
0.7	30	95.14,g	98.70,b	97.63,b	97.32,b	97.31,b	0.00	30.02
0.8	10	95.17,g	98.54,b	96.24,b	95.60,b	95.48,b	11.16	20.64
0.8	15	95.31,g	98.24,b	96.38,b	95.91,b	95.76,b	3.24	20.19
0.8	20	94.97,g	98.02,b	96.39,b	95.89,b	95.72,b	1.18	19.99
0.8	30	95.20,g	98.10,b	96.71,b	96.43,b	96.34,b	0.08	19.89
0.9	10	95.41,b	98.02,b	95.29,g	94.87,g	93.73,b	35.33	9.31
0.9	15	95.22,g	97.97,b	95.40,g	95.03,g	94.24,b	20.10	9.31
0.9	20	95.36,g	97.82,b	95.59,b	95.16,g	94.43,b	12.03	9.31
0.9	30	95.34,g	97.68,b	95.78,b	95.49,b	94.88,g	4.46	9.62

Table 13.6. Coverage Probabilities for the Exponential Censoring Distribution and the Kaplan-Meier Efron Censoring Model.

KM Efron Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	95.06,g	97.59,b	95.03,g	95.24,g	94.95,g	0.01	49.76
0.5	15	95.20,g	98.09,b	96.08,b	96.21,b	96.09,b	0.01	49.97
0.5	20	95.20,g	98.10,b	96.68,b	96.77,b	96.68,b	0.00	49.70
0.5	30	95.28,g	98.70,b	97.71,b	97.80,b	97.74,b	0.00	49.87
0.6	10	95.14,g	97.67,b	95.18,g	94.88,g	94.79,g	0.72	39.76
0.6	15	95.09,g	98.13,b	96.13,b	95.79,b	95.78,b	0.04	39.76
0.6	20	95.16,g	97.99,b	96.61,b	96.33,b	96.32,b	0.00	40.05
0.6	30	95.36,g	98.29,b	97.19,b	97.03,b	97.02,b	0.00	39.95
0.7	10	95.04,g	97.59,b	94.86,g	94.44,b	94.35,b	3.10	29.68
0.7	15	95.00,g	97.84,b	95.69,b	95.26,g	95.21,g	0.42	30.63
0.7	20	94.79,g	97.86,b	96.06,b	95.58,b	95.54,b	0.04	30.32
0.7	30	95.18,g	98.03,b	96.85,b	96.47,b	96.45,b	0.00	30.02
0.8	10	95.25,g	97.72,b	94.91,g	94.12,b	93.96,b	11.16	20.64
0.8	15	95.38,g	97.63,b	95.59,b	95.04,g	94.88,g	3.24	20.19
0.8	20	95.03,g	97.58,b	95.77,b	95.20,g	95.04,g	1.18	19.99
0.8	30	95.24,g	97.75,b	96.17,b	95.90,b	95.75,b	0.08	19.89
0.9	10	95.56,b	97.72,b	94.78,g	94.39,b	93.08,b	35.33	9.31
0.9	15	95.24,g	97.76,b	95.09,g	94.73,g	93.80,b	20.10	9.31
0.9	20	95.39,g	97.63,b	95.33,g	94.92,g	94.17,b	12.03	9.31
0.9	30	95.35,g	97.54,b	95.62,b	95.33,g	94.69,g	4.46	9.62

Table 13.7. Coverage Probabilities for the Exponential Censoring Distribution and the Kaplan-Meier Weibull Censoring Model.

KM Weibull Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	99.00,b	97.95,b	98.03,b	97.87,b	0.01	49.76
0.5	15	95.07,g	99.35,b	98.63,b	98.63,b	98.60,b	0.01	49.97
0.5	20	95.12,g	99.20,b	98.52,b	98.51,b	98.48,b	0.00	49.70
0.5	30	95.17,g	99.48,b	98.91,b	98.91,b	98.90,b	0.00	49.87
0.6	10	95.08,g	98.94,b	97.60,b	97.32,b	97.30,b	0.72	39.76
0.6	15	95.01,g	99.20,b	98.07,b	97.92,b	97.90,b	0.04	39.76
0.6	20	95.12,g	98.89,b	97.94,b	97.80,b	97.80,b	0.00	40.05
0.6	30	95.24,g	99.02,b	98.33,b	98.26,b	98.25,b	0.00	39.95
0.7	10	94.92,g	98.72,b	96.85,b	96.54,b	96.45,b	3.10	29.68
0.7	15	94.91,g	98.76,b	97.08,b	96.81,b	96.78,b	0.42	30.63
0.7	20	94.67,g	98.47,b	97.15,b	96.85,b	96.81,b	0.04	30.32
0.7	30	95.13,g	98.73,b	97.69,b	97.37,b	97.36,b	0.00	30.02
0.8	10	95.16,g	98.56,b	96.33,b	95.68,b	95.57,b	11.16	20.64
0.8	15	95.32,g	98.21,b	96.46,b	95.98,b	95.87,b	3.24	20.19
0.8	20	94.97,g	98.04,b	96.49,b	95.94,b	95.80,b	1.18	19.99
0.8	30	95.20,g	98.15,b	96.73,b	96.49,b	96.38,b	0.08	19.89
0.9	10	95.42,b	98.07,b	95.36,g	95.02,g	93.92,b	35.33	9.31
0.9	15	95.22,g	97.99,b	95.46,b	95.11,g	94.34,b	20.10	9.31
0.9	20	95.35,g	97.86,b	95.66,b	95.27,g	94.49,b	12.03	9.31
0.9	30	95.34,g	97.71,b	95.78,b	95.51,b	94.91,g	4.46	9.62

Table 13.8. Coverage Probabilities for the Exponential Censoring Distribution and the Kaplan-Meier with Expected Order Statistics Censoring Model.

KM EOS Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.91,g	99.04,b	97.91,b	97.90,b	97.78,b	0.01	49.76
0.5	15	95.07,g	99.34,b	98.63,b	98.64,b	98.59,b	0.01	49.97
0.5	20	95.12,g	99.24,b	98.53,b	98.57,b	98.54,b	0.00	49.70
0.5	30	95.16,g	99.51,b	98.93,b	98.97,b	98.96,b	0.00	49.87
0.6	10	95.03,g	98.92,b	97.53,b	97.34,b	97.31,b	0.72	39.76
0.6	15	95.02,g	99.23,b	97.99,b	97.84,b	97.81,b	0.04	39.76
0.6	20	95.09,g	98.95,b	97.98,b	97.86,b	97.85,b	0.00	40.05
0.6	30	95.22,g	99.05,b	98.43,b	98.35,b	98.35,b	0.00	39.95
0.7	10	94.88,g	98.67,b	96.90,b	96.59,b	96.48,b	3.10	29.68
0.7	15	94.89,g	98.78,b	97.14,b	96.80,b	96.76,b	0.42	30.63
0.7	20	94.63,g	98.44,b	97.21,b	96.90,b	96.88,b	0.04	30.32
0.7	30	95.13,g	98.77,b	97.71,b	97.38,b	97.37,b	0.00	30.02
0.8	10	95.13,g	98.57,b	96.39,b	95.59,b	95.49,b	11.16	20.64
0.8	15	95.32,g	98.18,b	96.48,b	95.99,b	95.85,b	3.24	20.19
0.8	20	94.96,g	98.06,b	96.42,b	95.89,b	95.77,b	1.18	19.99
0.8	30	95.20,g	98.15,b	96.74,b	96.49,b	96.39,b	0.08	19.89
0.9	10	95.38,g	98.04,b	95.41,b	94.99,g	93.87,b	35.33	9.31
0.9	15	95.21,g	98.01,b	95.46,b	95.10,g	94.33,b	20.10	9.31
0.9	20	95.35,g	97.84,b	95.64,b	95.23,g	94.48,b	12.03	9.31
0.9	30	95.34,g	97.71,b	95.78,b	95.51,b	94.89,g	4.46	9.62

13.2.2. Gamma Censoring Times. Tables 13.9, 13.10, 13.11, 13.12, 13.13, 13.14 and 13.15 present results for simulations where the censoring distribution is Gamma. The SPBB method again generally outperforms the Bartholomew methods and the Wilson, Agresti and Jeffreys methods are sometimes competitive with the SPBB method especially when $cf = 0.9$. For smaller values of the censoring fraction, these methods on occasion yield very good coverage probabilities most likely due to some form of "Lucky n, Lucky p" phenomenon, as discussed in the previous section. In comparison, with the Exponential censoring results, the SPBB method and the Bartholomew methods do not perform as well overall, except for when the assumed censoring model is correctly specified as Gamma. This suggests that when the censoring distribution is thin tailed the proper choice of censoring model is important and especially so when one wants to make inference about λ_0 at higher censoring fraction values. Furthermore, among the KM tail-completed models one should probably choose the KM Exponential or KM Efron methods since they appear to perform as well as the KM Weibull and KM EOS but with much less programming and computation. Furthermore, for lower cf values one should probably choose the KM Exponential method.

Table 13.9. Coverage Probabilities for the Gamma Censoring Distribution and the Exponential Censoring Model.

Exponential Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i$= **0**
0.5	10	94.98,g	99.76,b	99.13,b	99.12,b	99.03,b	0.01
0.5	15	95.06,g	99.73,b	99.30,b	99.28,b	99.26,b	0.01
0.5	20	95.19,g	99.71,b	99.17,b	99.22,b	99.20,b	0.00
0.5	30	95.16,g	99.73,b	99.21,b	99.25,b	99.23,b	0.00
0.6	10	95.50,b	99.58,b	98.64,b	98.50,b	98.46,b	0.64
0.6	15	95.22,g	99.34,b	98.32,b	98.18,b	98.15,b	0.05
0.6	20	95.48,b	99.33,b	98.39,b	98.32,b	98.30,b	0.01
0.6	30	95.41,g	99.19,b	98.58,b	98.52,b	98.51,b	0.00
0.7	10	95.15,g	98.89,b	97.18,b	96.70,b	96.54,b	2.73
0.7	15	94.74,g	98.68,b	97.06,b	96.64,b	96.58,b	0.42
0.7	20	94.77,g	98.64,b	97.18,b	96.82,b	96.76,b	0.04
0.7	30	94.97,g	98.52,b	97.37,b	97.12,b	97.04,b	0.00
0.8	10	94.79,g	98.36,b	95.73,b	95.12,g	94.66,g	10.75
0.8	15	94.88,g	98.12,b	95.94,b	95.33,g	95.03,g	3.44
0.8	20	95.18,g	98.27,b	96.41,b	95.95,b	95.70,b	1.18
0.8	30	95.03,g	97.94,b	96.46,b	96.02,b	95.83,b	0.12
0.9	10	93.18,b	97.72,b	93.22,b	94.57,g	91.28,b	34.55
0.9	15	93.84,b	97.48,b	93.66,b	94.04,b	91.78,b	20.53
0.9	20	94.34,b	97.44,b	94.47,b	94.59,g	92.93,b	12.28
0.9	30	94.49,b	97.09,b	94.78,g	94.66,g	93.45,b	4.14

Table 13.10. Coverage Probabilities for the Gamma Censoring Distribution and the Gamma Censoring Model.

Gamma Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.86,g	99.17,b	97.90,b	98.03,b	97.93,b	0.01
0.5	15	95.09,g	99.36,b	98.47,b	98.55,b	98.51,b	0.01
0.5	20	95.19,g	99.39,b	98.66,b	98.72,b	98.68,b	0.00
0.5	30	95.16,g	99.61,b	99.07,b	99.09,b	99.07,b	0.00
0.6	10	95.66,b	99.29,b	98.17,b	98.10,b	98.03,b	0.64
0.6	15	95.30,g	99.37,b	98.42,b	98.40,b	98.33,b	0.05
0.6	20	95.63,b	99.40,b	98.77,b	98.69,b	98.68,b	0.01
0.6	30	95.48,b	99.49,b	98.97,b	98.91,b	98.89,b	0.00
0.7	10	95.42,b	99.37,b	98.03,b	97.63,b	97.54,b	2.73
0.7	15	95.05,g	99.27,b	98.11,b	97.87,b	97.84,b	0.42
0.7	20	95.02,g	99.36,b	98.52,b	98.29,b	98.25,b	0.04
0.7	30	95.17,g	99.44,b	98.81,b	98.59,b	98.56,b	0.00
0.8	10	95.27,g	99.15,b	97.21,b	96.87,b	96.55,b	10.75
0.8	15	95.28,g	99.09,b	97.72,b	97.31,b	96.96,b	3.44
0.8	20	95.58,b	99.16,b	97.95,b	97.65,b	97.40,b	1.18
0.8	30	95.21,g	99.15,b	97.95,b	97.73,b	97.63,b	0.12
0.9	10	94.58,g	98.62,b	94.60,g	95.97,b	92.90,b	34.55
0.9	15	94.27,b	98.32,b	95.16,g	95.48,b	93.58,b	20.53
0.9	20	94.64,g	98.33,b	95.72,b	95.78,b	94.42,b	12.28
0.9	30	94.63,g	97.98,b	96.03,b	95.94,b	94.98,g	4.14

Table 13.11. Coverage Probabilities for the Gamma Censoring Distribution and the Weibull Censoring Model.

Weibull Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.94,g	98.93,b	97.66,b	97.80,b	97.68,b	0.01
0.5	15	95.14,g	99.26,b	98.17,b	98.24,b	98.18,b	0.01
0.5	20	95.23,g	99.23,b	98.28,b	98.35,b	98.31,b	0.00
0.5	30	95.18,g	99.52,b	98.96,b	98.98,b	98.96,b	0.00
0.6	10	95.82,b	99.21,b	98.37,b	98.36,b	98.31,b	0.64
0.6	15	95.37,g	99.45,b	98.64,b	98.57,b	98.55,b	0.05
0.6	20	95.60,b	99.45,b	98.92,b	98.80,b	98.79,b	0.01
0.6	30	95.49,b	99.56,b	99.21,b	99.12,b	99.11,b	0.00
0.7	10	95.61,b	99.48,b	98.61,b	98.34,b	98.27,b	2.73
0.7	15	95.17,g	99.54,b	98.87,b	98.71,b	98.67,b	0.42
0.7	20	95.14,g	99.66,b	99.18,b	99.11,b	99.07,b	0.04
0.7	30	95.23,g	99.71,b	99.40,b	99.31,b	99.31,b	0.00
0.8	10	95.85,b	99.48,b	98.09,b	97.82,b	97.52,b	10.75
0.8	15	95.82,b	99.39,b	98.38,b	98.13,b	97.88,b	3.44
0.8	20	95.96,b	99.46,b	98.59,b	98.41,b	98.23,b	1.18
0.8	30	95.68,b	99.46,b	98.70,b	98.41,b	98.31,b	0.12
0.9	10	95.29,g	98.17,b	95.06,g	94.60,g	93.50,b	34.55
0.9	15	95.86,b	97.70,b	92.26,b	92.57,b	90.08,b	20.53
0.9	20	96.36,b	96.34,b	92.66,b	92.36,b	90.92,b	12.28
0.9	30	96.66,b	94.65,g	89.32,b	90.44,b	86.85,b	4.14

Table 13.12. Coverage Probabilities for the Gamma Censoring Distribution and the Kaplan-Meier Exponential Censoring Model.

KM Exponential Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	99.34,b	98.30,b	98.29,b	98.19,b	0.01	49.76
0.5	15	95.07,g	99.48,b	98.78,b	98.81,b	98.74,b	0.01	49.97
0.5	20	95.12,g	99.26,b	98.59,b	98.64,b	98.60,b	0.00	49.70
0.5	30	95.17,g	99.48,b	98.93,b	98.92,b	98.92,b	0.00	49.87
0.6	10	95.68,b	99.22,b	98.24,b	98.12,b	98.10,b	0.64	46.50
0.6	15	95.31,g	99.40,b	98.46,b	98.40,b	98.38,b	0.05	46.26
0.6	20	95.59,b	99.28,b	98.50,b	98.36,b	98.36,b	0.01	46.70
0.6	30	95.41,g	99.35,b	98.70,b	98.60,b	98.60,b	0.00	47.47
0.7	10	95.50,b	99.05,b	97.84,b	97.46,b	97.42,b	2.73	42.67
0.7	15	95.14,g	99.07,b	98.00,b	97.73,b	97.69,b	0.42	43.86
0.7	20	95.11,g	99.14,b	98.14,b	97.80,b	97.76,b	0.04	44.18
0.7	30	95.16,g	99.19,b	98.44,b	98.26,b	98.24,b	0.00	45.17
0.8	10	95.66,b	98.96,b	97.08,b	96.60,b	96.22,b	10.75	38.27
0.8	15	95.52,b	99.11,b	97.56,b	97.20,b	96.90,b	3.44	39.34
0.8	20	95.77,b	99.22,b	98.02,b	97.69,b	97.45,b	1.18	40.72
0.8	30	95.41,g	99.37,b	98.38,b	98.12,b	98.04,b	0.12	41.31
0.9	10	95.66,b	98.74,b	95.21,g	96.52,b	93.61,b	34.55	32.21
0.9	15	95.77,b	98.69,b	96.36,b	96.73,b	94.98,g	20.53	34.42
0.9	20	95.99,b	98.84,b	96.95,b	97.11,b	95.96,b	12.28	35.79
0.9	30	95.61,b	98.97,b	97.51,b	97.48,b	96.76,b	4.14	37.61

Table 13.13. Coverage Probabilities for the Gamma Censoring Distribution and the Kaplan-Meier Efron Censoring Model.

KM Efron Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	95.06,g	97.59,b	95.03,g	95.24,g	94.95,g	0.01	49.76
0.5	15	95.20,g	98.09,b	96.08,b	96.21,b	96.09,b	0.01	49.97
0.5	20	95.20,g	98.10,b	96.68,b	96.77,b	96.68,b	0.00	49.70
0.5	30	95.28,g	98.70,b	97.71,b	97.80,b	97.74,b	0.00	49.87
0.6	10	95.82,b	97.91,b	95.45,b	95.15,g	95.10,g	0.64	46.50
0.6	15	95.39,g	98.22,b	96.38,b	96.20,b	96.15,b	0.05	46.26
0.6	20	95.68,b	98.48,b	97.11,b	96.87,b	96.84,b	0.01	46.70
0.6	30	95.46,b	98.65,b	97.65,b	97.54,b	97.52,b	0.00	47.47
0.7	10	95.59,b	97.91,b	95.41,g	94.78,g	94.69,g	2.73	42.67
0.7	15	95.25,g	98.08,b	95.96,b	95.60,b	95.55,b	0.42	43.86
0.7	20	95.22,g	98.27,b	96.67,b	96.25,b	96.16,b	0.04	44.18
0.7	30	95.25,g	98.60,b	97.50,b	97.26,b	97.24,b	0.00	45.17
0.8	10	95.72,b	97.83,b	94.85,g	94.25,b	93.70,b	10.75	38.27
0.8	15	95.61,b	98.10,b	95.90,b	95.37,g	94.95,g	3.44	39.34
0.8	20	95.84,b	98.50,b	96.89,b	96.48,b	96.13,b	1.18	40.72
0.8	30	95.53,b	98.63,b	97.34,b	97.07,b	96.92,b	0.12	41.31
0.9	10	95.70,b	97.84,b	92.88,b	94.34,b	90.90,b	34.55	32.21
0.9	15	95.87,b	97.78,b	94.53,b	94.89,g	92.62,b	20.53	34.42
0.9	20	96.08,b	98.11,b	95.68,b	95.74,b	94.24,b	12.28	35.79
0.9	30	95.72,b	98.32,b	96.48,b	96.39,b	95.42,b	4.14	37.61

Table 13.14. Coverage Probabilities for the Gamma Censoring Distribution and the Kaplan-Meier Weibull Censoring Model.

KM Weibull Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	99.00,b	97.95,b	98.03,b	97.87,b	0.01	49.76
0.5	15	95.07,g	99.34,b	98.62,b	98.62,b	98.59,b	0.01	49.97
0.5	20	95.12,g	99.20,b	98.51,b	98.50,b	98.47,b	0.00	49.70
0.5	30	95.17,g	99.48,b	98.91,b	98.91,b	98.90,b	0.00	49.87
0.6	10	95.68,b	99.17,b	98.18,b	98.08,b	98.05,b	0.64	46.50
0.6	15	95.32,g	99.38,b	98.53,b	98.44,b	98.42,b	0.05	46.26
0.6	20	95.59,b	99.30,b	98.58,b	98.47,b	98.47,b	0.01	46.70
0.6	30	95.41,g	99.38,b	98.72,b	98.65,b	98.65,b	0.00	47.47
0.7	10	95.50,b	99.21,b	98.09,b	97.79,b	97.75,b	2.73	42.67
0.7	15	95.14,g	99.21,b	98.28,b	98.05,b	98.04,b	0.42	43.86
0.7	20	95.09,g	99.33,b	98.45,b	98.13,b	98.10,b	0.04	44.18
0.7	30	95.15,g	99.34,g	98.65,b	98.49,b	98.48,b	0.00	45.17
0.8	10	95.66,b	99.34,b	97.87,b	97.54,b	97.19,b	10.75	38.27
0.8	15	95.53,b	99.33,b	98.22,b	97.99,b	97.75,b	3.44	39.34
0.8	20	95.76,b	99.41,b	98.49,b	98.23,b	98.02,b	1.18	40.72
0.8	30	95.40,g	99.47,b	98.79,b	98.61,b	98.53,b	0.12	41.31
0.9	10	95.75,b	99.25,b	96.52,b	97.58,b	95.29,g	34.55	32.21
0.9	15	95.79,b	99.18,b	97.26,b	97.66,b	96.38,b	20.53	34.42
0.9	20	96.02,b	99.17,b	97.79,b	97.86,b	96.91,b	12.28	35.79
0.9	30	95.62,b	99.24,b	98.16,b	98.13,b	97.53,b	4.14	37.61

Table 13.15. Coverage Probabilities for the Gamma Censoring Distribution and the Kaplan-Meier with Expected Order Statistics Censoring Model.

KM EOS Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.91,g	99.04,b	97.91,b	97.90,b	97.78,b	0.01	49.76
0.5	15	95.06,g	99.34,b	98.62,b	98.63,b	98.58,b	0.01	49.97
0.5	20	95.12,g	99.23,b	98.52,b	98.56,b	98.53,b	0.00	49.70
0.5	30	95.16,g	99.51,b	98.93,b	98.97,b	98.96,b	0.00	49.87
0.6	10	95.64,b	99.21,b	98.26,b	98.16,b	98.12,b	0.64	46.50
0.6	15	95.27,g	99.43,b	98.59,b	98.52,b	98.49,b	0.05	46.26
0.6	20	95.57,b	99.33,b	98.66,b	98.54,b	98.54,b	0.01	46.70
0.6	30	95.37,g	99.46,b	98.92,b	98.82,b	98.82,b	0.00	47.47
0.7	10	95.48,b	99.28,b	98.21,b	97.92,b	97.88,b	2.73	42.67
0.7	15	95.11,g	99.30,b	98.42,b	98.22,b	98.21,b	0.42	43.86
0.7	20	95.03,g	99.42,b	98.67,b	98.43,b	98.39,b	0.04	44.18
0.7	30	95.14,g	99.43,b	98.82,b	98.66,b	98.65,b	0.00	45.17
0.8	10	95.62,b	99.39,b	98.14,b	97.85,b	97.45,b	10.75	38.27
0.8	15	95.51,b	99.37,b	98.32,b	98.12,b	97.88,b	3.44	39.34
0.8	20	95.72,b	99.47,b	98.64,b	98.38,b	98.19,b	1.18	40.72
0.8	30	95.38,g	99.54,b	98.89,b	98.75,b	98.67,b	0.12	41.31
0.9	10	95.74,b	99.24,b	96.38,b	97.37,b	95.21,g	34.55	32.21
0.9	15	95.73,b	99.15,b	97.14,b	97.52,b	96.03,b	20.53	34.42
0.9	20	95.92,b	99.07,b	97.60,b	97.67,b	96.75,b	12.28	35.79
0.9	30	95.56,b	99.20,b	98.06,b	97.98,b	97.26,b	4.14	37.61

13.2.3. Weibull Censoring Times. Tables 13.16 , 13.17, 13.18, 13.19, 13.20, 13.21 and 13.22 present results for simulations where the censoring distribution is Weibull. Once again the SPBB method basically outperforms the Bartholomew methods and the Wilson and Agresti methods are sometimes competitive with the SPBB method when $cf = 0.9$ and sometimes perform very well for smaller cf values which probably correspond to "Lucky n, Lucky p" cases. As one might expect, all of the methods perform very well when the assumed censoring model is Weibull. It is surprising to see that all of the other censoring models, with exception of the Exponential model, yield results for all of the confidence interval methods which are quite similar to those obtained with the Weibull censoring model. Furthermore, the unexpected strong performance of the SPBB method with the Exponential censoring model for $cf = 0.9$ maybe due to the fact that the associated Weibull censoring distribution has a thick exponential-like tail. Here again we see that overall, in terms of computation simplicity and accuracy, one would probably prefer the KM Exponential method over the other KM tail completed methods.

Table 13.16. Coverage Probabilities for the Weibull Censoring Distribution and the Exponential Censoring Model.

Exponential Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	95.52,b	99.92,b	99.85,b	99.89,b	99.88,b	0.17
0.5	15	95.22,g	99.94,b	99.82,b	99.87,b	99.86,b	0.02
0.5	20	95.42,b	99.98,b	99.76,b	99.81,b	99.80,b	0.00
0.5	30	95.31,g	99.89,b	99.62,b	99.68,b	99.63,b	0.00
0.6	10	95.44,b	99.93,b	99.55,b	99.61,b	99.55,b	0.69
0.6	15	95.39,g	99.87,b	99.34,b	99.38,b	99.36,b	0.04
0.6	20	95.48,b	99.75,b	99.15,b	99.15,b	99.13,b	0.01
0.6	30	95.44,b	99.50,b	98.89,b	98.91,b	98.89,b	0.00
0.7	10	95.64,b	99.72,b	98.67,b	98.55,b	98.51,b	3.19
0.7	15	95.47,b	99.43,b	98.47,b	98.23,b	98.21,b	0.50
0.7	20	95.28,g	99.22,b	98.07,b	97.90,b	97.89,b	0.06
0.7	30	95.26,g	98.91,b	98.12,b	97.93,b	97.93,b	0.00
0.8	10	95.71,b	99.08,b	97.31,b	96.72,b	96.68,b	11.20
0.8	15	95.60,b	98.91,b	97.29,b	96.86,b	96.80,b	3.57
0.8	20	95.08,g	98.47,b	96.78,b	96.41,b	96.32,b	1.09
0.8	30	95.42,b	98.48,b	97.21,b	96.87,b	96.79,b	0.13
0.9	10	94.59,g	98.63,b	95.02,g	93.97,b	93.52,b	35.29
0.9	15	94.82,g	98.22,b	95.24,g	94.76,g	94.51,b	20.03
0.9	20	95.39,g	97.98,b	95.63,b	95.41,g	95.02,g	12.18
0.9	30	95.08,g	97.68,b	95.83,b	95.37,g	94.77,g	4.26

Table 13.17. Coverage Probabilities for the Weibull Censoring Distribution and the Gamma Censoring Model.

Gamma Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.91,g	98.29,b	95.94,b	96.58,b	96.18,b	0.17
0.5	15	94.88,g	98.05,b	96.22,b	96.60,b	96.43,b	0.02
0.5	20	95.03,g	97.91,b	96.23,b	96.39,b	96.32,b	0.00
0.5	30	94.99,g	97.71,b	96.30,b	96.49,b	96.43,b	0.00
0.6	10	94.88,g	98.17,b	95.54,b	95.55,b	95.41,g	0.69
0.6	15	94.88,g	97.86,b	95.69,b	95.57,b	95.52,b	0.04
0.6	20	95.07,g	97.45,b	95.63,b	95.52,b	95.47,b	0.01
0.6	30	95.06,g	97.27,b	95.77,b	95.70,b	95.69,b	0.00
0.7	10	94.92,g	97.73,b	95.30,g	94.86,g	94.85,g	3.19
0.7	15	95.08,g	97.56,b	95.47,b	95.09,g	95.09,g	0.50
0.7	20	94.89,g	97.22,b	95.25,g	94.91,g	94.88,g	0.06
0.7	30	94.99,g	97.08,b	95.49,b	95.11,g	95.11,g	0.00
0.8	10	94.96,g	97.52,b	95.21,g	94.34,b	94.26,b	11.20
0.8	15	95.04,g	97.60,b	95.21,g	94.45,b	94.32,b	3.57
0.8	20	94.65,g	96.98,b	94.80,g	94.29,b	94.18,b	1.09
0.8	30	95.09,g	97.22,b	95.22,g	94.74,g	94.56,g	0.13
0.9	10	93.56,b	98.36,b	93.65,b	93.06,b	92.84,b	35.29
0.9	15	94.47,b	97.74,b	94.75,g	94.38,b	94.21,b	20.03
0.9	20	95.05,g	97.15,b	95.25,g	95.05,g	94.55,b	12.18
0.9	30	94.70,g	97.36,b	95.23,g	94.80,g	94.10,b	4.26

Table 13.18. Coverage Probabilities for the Weibull Censoring Distribution and the Weibull Censoring Model.

Weibull Censoring Model							
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$
0.5	10	94.90,g	98.00,b	95.79,b	96.36,b	96.02,b	0.17
0.5	15	94.92,g	97.88,b	96.05,b	96.33,b	96.15,b	0.02
0.5	20	95.03,g	97.69,b	96.01,b	96.14,b	96.08,b	0.00
0.5	30	94.98,g	97.49,b	96.07,b	96.18,b	96.15,b	0.00
0.6	10	94.87,g	97.97,b	95.34,g	95.39,g	95.23,g	0.69
0.6	15	94.87,g	97.65,b	95.51,b	95.37,g	95.32,g	0.04
0.6	20	95.03,g	97.25,b	95.46,b	95.37,g	95.35,g	0.01
0.6	30	95.04,g	97.08,b	95.63,b	95.45,b	95.45,b	0.00
0.7	10	94.94,g	97.57,b	95.14,g	94.69,g	94.63,g	3.19
0.7	15	95.08,g	97.45,b	95.38,g	94.88,g	94.88,g	0.50
0.7	20	94.87,g	97.17,b	95.15,g	94.83,g	94.83,g	0.06
0.7	30	94.99,g	96.91,b	95.33,g	95.03,g	95.02,g	0.00
0.8	10	95.00,g	97.36,b	95.00,g	94.07,b	94.02,b	11.20
0.8	15	95.02,g	97.56,b	95.08,g	94.33,b	94.21,b	3.57
0.8	20	94.66,g	96.91,b	94.71,g	94.19,b	94.06,b	1.09
0.8	30	95.09,g	97.17,b	95.08,g	94.70,g	94.54,b	0.13
0.9	10	93.70,b	98.03,b	93.17,b	92.75,b	92.60,b	35.29
0.9	15	94.53,b	97.43,b	94.42,b	94.30,b	94.07,b	20.03
0.9	20	95.12,g	96.78,b	95.03,g	94.78,g	94.02,b	12.18
0.9	30	94.74,g	97.05,b	94.74,g	94.22,b	93.54,b	4.26

Table 13.19. Coverage Probabilities for the Weibull Censoring Distribution and the Kaplan-Meier Exponential Censoring Model.

KM Exponential Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.88,g	98.80,b	97.03,b	97.65,b	97.29,b	0.17	16.82
0.5	15	94.81,g	98.45,b	96.71,b	97.12,b	96.94,b	0.02	15.37
0.5	20	95.03,g	98.11,b	96.41,b	96.60,b	96.51,b	0.00	14.23
0.5	30	94.97,g	97.63,b	96.33,b	96.48,b	96.42,b	0.00	13.06
0.6	10	94.78,g	98.39,b	95.97,b	96.03,b	95.87,b	0.69	11.66
0.6	15	94.86,g	97.98,b	95.87,b	95.82,b	95.70,b	0.04	10.26
0.6	20	95.05,g	97.47,b	95.59,b	95.50,b	95.46,b	0.01	9.37
0.6	30	95.06,g	97.14,b	95.65,b	95.57,b	95.56,b	0.00	8.57
0.7	10	94.87,g	97.94,b	95.44,b	94.94,g	94.90,g	3.19	8.18
0.7	15	95.05,g	97.59,b	95.58,b	95.16,g	95.16,g	0.50	7.03
0.7	20	94.86,g	97.16,b	95.28,g	94.95,g	94.95,g	0.06	6.60
0.7	30	94.98,g	96.99,b	95.46,b	95.11,g	95.10,g	0.00	5.75
0.8	10	94.97,g	97.50,b	95.07,g	94.17,b	94.14,b	11.20	5.00
0.8	15	95.02,g	97.63,b	95.17,g	94.48,b	94.37,b	3.57	4.45
0.8	20	94.63,g	96.93,b	94.72,g	94.22,b	94.12,b	1.09	4.16
0.8	30	95.09,g	97.16,b	95.17,g	94.66,g	94.49,b	0.13	3.61
0.9	10	93.70,b	98.07,b	93.24,b	92.76,b	92.61,b	35.29	1.92
0.9	15	94.52,b	97.42,b	94.43,b	94.29,b	94.03,b	20.03	1.81
0.9	20	95.11,g	96.80,b	95.03,g	94.82,g	94.04,b	12.18	1.54
0.9	30	94.73,g	97.05,b	94.73,g	94.17,b	93.55,b	4.26	1.45

Table 13.20. Coverage Probabilities for the Weibull Censoring Distribution and the Kaplan-Meier Efron Censoring Model.

KM Efron Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = 0$	**Adjust**
0.5	10	94.93,g	97.36,b	94.97,g	95.54,b	95.19,g	0.17	16.82
0.5	15	94.91,g	97.51,b	95.51,b	95.86,b	95.68,b	0.02	15.37
0.5	20	95.08,g	97.54,b	95.70,b	95.81,b	95.76,b	0.00	14.23
0.5	30	95.02,g	97.26,b	95.87,b	96.04,b	95.94,b	0.00	13.06
0.6	10	94.85,g	97.64,b	95.04,g	95.07,g	94.90,g	0.69	11.66
0.6	15	94.89,g	97.42,b	95.24,g	95.19,g	95.07,g	0.04	10.26
0.6	20	95.08,g	97.23,b	95.28,g	95.13,g	95.11,g	0.01	9.37
0.6	30	95.09,g	96.96,b	95.51,b	95.41,g	95.41,g	0.00	8.57
0.7	10	94.93,g	97.46,b	95.02,g	94.52,b	94.49,b	3.19	8.18
0.7	15	95.05,g	97.43,b	95.28,g	94.83,g	94.82,g	0.50	7.03
0.7	20	94.89,g	97.04,b	95.10,g	94.71,g	94.71,g	0.06	6.60
0.7	30	94.99,g	96.88,b	95.34,g	94.96,g	94.95,g	0.00	5.75
0.8	10	94.98,g	97.35,b	94.91,g	93.87,b	93.84,b	11.20	5.00
0.8	15	95.02,g	97.55,b	95.07,g	94.35,b	94.25,b	3.57	4.45
0.8	20	94.66,g	96.89,b	94.68,g	94.14,b	94.03,b	1.09	4.16
0.8	30	95.11,g	97.11,b	95.09,g	94.59,g	94.42,b	0.13	3.61
0.9	10	93.75,b	98.01,b	93.16,b	92.72,b	92.60,b	35.29	1.92
0.9	15	94.54,b	97.38,b	94.42,b	94.28,b	94.03,b	20.03	1.81
0.9	20	95.11,g	96.77,b	95.01,g	94.80,g	94.03,b	12.18	1.54
0.9	30	94.74,g	97.04,b	94.70,g	94.17,b	93.52,b	4.26	1.45

Table 13.21. Coverage Probabilities for the Weibull Censoring Distribution and the Kaplan-Meier Weibull Censoring Model.

KM Weibull Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum\delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	98.10,b	95.90,b	96.51,b	96.19,b	0.17	16.82
0.5	15	94.88,g	97.93,b	96.14,b	96.48,b	96.27,b	0.02	15.37
0.5	20	95.03,g	97.81,b	96.06,b	96.22,b	96.12,b	0.00	14.23
0.5	30	94.99,g	97.43,b	96.06,b	96.22,b	96.15,b	0.00	13.06
0.6	10	94.82,g	97.92,b	95.40,g	95.44,b	95.29,g	0.69	11.66
0.6	15	94.87,g	97.69,b	95.42,b	95.37,g	95.25,g	0.04	10.26
0.6	20	95.08,g	97.31,b	95.44,b	95.31,g	95.29,g	0.01	9.37
0.6	30	95.08,g	97.02,b	95.55,b	95.48,b	95.48,b	0.00	8.57
0.7	10	94.90,g	97.61,b	95.18,g	94.65,g	94.61,g	3.19	8.18
0.7	15	95.05,g	97.47,b	95.38,g	94.93,g	94.93,g	0.50	7.03
0.7	20	94.88,g	97.09,b	95.17,g	94.78,g	94.78,g	0.06	6.60
0.7	30	94.99,g	96.94,b	95.38,g	95.01,g	95.00,g	0.00	5.75
0.8	10	94.97,g	97.41,b	94.97,g	94.03,b	94.00,b	11.20	5.00
0.8	15	95.02,g	97.56,b	95.12,g	94.39,b	94.30,b	3.57	4.45
0.8	20	94.65,g	96.91,b	94.70,g	94.15,b	94.05,b	1.09	4.16
0.8	30	95.10,g	97.12,b	95.11,g	94.61,g	94.45,b	0.13	3.61
0.9	10	93.73,b	98.03,b	93.16,b	92.73,b	92.60,b	35.29	1.92
0.9	15	94.53,b	97.40,b	94.42,b	94.28,b	94.03,b	20.03	1.81
0.9	20	95.11,g	96.78,b	95.02,g	94.80,g	94.03,b	12.18	1.54
0.9	30	94.74,g	97.04,b	94.70,g	94.17,b	93.52,b	4.26	1.45

Table 13.22. Coverage Probabilities for the Weibull Censoring Distribution and the Kaplan-Meier with Expected Order Statistics Censoring Model.

KM EOS Censoring Model								
cf	**n**	**SPA**	**Clopper**	**Jeffreys**	**Agresti**	**Wilson**	$\sum \delta_i = \mathbf{0}$	**Adjust**
0.5	10	94.90,g	98.05,b	95.81,b	96.39,b	96.06,b	0.17	16.82
0.5	15	94.87,g	97.92,b	96.10,b	96.40,b	96.23,b	0.02	15.37
0.5	20	95.03,g	97.79,b	96.04,b	96.19,b	96.10,b	0.00	14.23
0.5	30	94.99,g	97.42,b	96.03,b	96.20,b	96.11,b	0.00	13.06
0.6	10	94.81,g	97.90,b	95.38,g	95.40,g	95.27,g	0.69	11.66
0.6	15	94.87,g	97.67,b	95.43,b	95.39,g	95.26,g	0.04	10.26
0.6	20	95.08,g	97.30,b	95.43,b	95.31,g	95.30,g	0.01	9.37
0.6	30	95.08,g	97.01,b	95.56,b	95.50,b	95.50,b	0.00	8.57
0.7	10	94.90,g	97.60,b	95.15,g	94.60,g	94.55,b	3.19	8.18
0.7	15	95.05,g	97.47,b	95.39,g	94.91,g	94.91,g	0.50	7.03
0.7	20	94.89,g	97.07,b	95.16,g	94.77,g	94.77,g	0.06	6.60
0.7	30	94.99,g	96.92,b	95.38,g	95.00,g	94.99,g	0.00	5.75
0.8	10	94.97,g	97.40,b	94.97,g	94.02,b	93.97,b	11.20	5.00
0.8	15	95.02,g	97.56,b	95.12,g	94.39,b	94.30,b	3.57	4.45
0.8	20	94.65,g	96.90,b	94.67,g	94.16,b	94.05,b	1.09	4.16
0.8	30	95.10,g	97.12,b	95.11,g	94.61,g	94.45,b	0.13	3.61
0.9	10	93.73,b	98.03,b	93.17,b	92.73,b	92.60,b	35.29	1.92
0.9	15	94.54,b	97.39,b	94.42,b	94.28,b	94.03,b	20.03	1.81
0.9	20	95.11,g	96.77,b	95.02,g	94.80,g	94.03,b	12.18	1.54
0.9	30	94.73,g	97.04,b	94.70,g	94.17,b	93.52,b	4.26	1.45

14. CONCLUSIONS

We have developed novel methods of unconditional confidence interval construction for the rate parameter of exponential survival times in the presence of heavy right censoring. These methods consist of a saddlepoint method and four generalized Bartholomew methods. In addition, we have proven some large sample results for these methods including weak convergence of our saddlepoint approximations and consistency of the Bartholomew methods. Simulation results show that the SPBB method is clearly superior to the Bartholomew methods and the latter should only be given serious consideration when the censoring fraction is at least 90%. Furthermore, these studies suggest that when one does not have a parametric model for the censoring time distribution one would probably prefer the KM Exponential censoring model over the KM Efron, KM Weibull, KM EOS methods due to its computational simplicity and accuracy. Heavy censoring often occurs because of poor study design and as a result may provide little information about survival rates. In such settings, the saddlepoint method could be used to glean more information about the survival rate, than one might normally obtain, which could then be used for the design of future studies. Related future work involves the use of the joint MGF of $(\sum \Delta_i, \sum Z_i)$ to make small-sample inference about the parameters in the exponential regression model and for making inference about exponential rates in a multiple-sample setting. In addition to that, work is under way on the approximation of PDFs which are Meijer G-functions.

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APPENDIX A

EXPONENTIAL CENSORING TIMES

Cumulative distribution function $P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum\Delta_i > 0\right)$ is available in closed-form for IID exponential censoring times, with hazard rate θ. To see this first note that

$$\begin{aligned}
P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \textstyle\sum\Delta_i > 0\right) &= P\left(\hat{\lambda}^{-1} \geq c^{-1}|\lambda_0, \theta_0, \textstyle\sum\Delta_i > 0\right) \\
&= P\left([\textstyle\sum\Delta_i]^{-1}\sum Z_i \geq c^{-1}|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \\
&= \sum_{k=1}^{n} P\left([\textstyle\sum\Delta_i]^{-1}\sum Z_i \geq c^{-1}|\lambda_0, \theta_0, \sum\Delta_i = k\right) \\
&\times P\left(\textstyle\sum\Delta_i = k|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \\
&= \sum_{k=1}^{n} P\left(\textstyle\sum Z_i \geq kc^{-1}|\lambda_0, \theta_0, \sum\Delta_i = k\right) \\
&\times P\left(\textstyle\sum\Delta_i = k|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \\
&= \sum_{k=1}^{n} P\left(\textstyle\sum Z_i \geq kc^{-1}|\lambda_0, \theta_0\right) \\
&\times P\left(\textstyle\sum\Delta_i = k|\lambda_0, \theta_0, \sum\Delta_i > 0\right).
\end{aligned}$$

The last expression follows from the proportionality of the failure and censoring time hazard functions which implies that the time on study variables, $Z_1, Z_2, \ldots, Z_n$, are independent of the associated survival indicators, $\Delta_1, \Delta_2, \ldots, \Delta_n$. As a result, $\sum Z_i$ has a gamma distribution with rate parameter $\lambda_0 + \theta_0$ and shape parameter n, and

$$P\left(\textstyle\sum Z_i \geq kc^{-1}|\lambda_0, \theta_0\right) = \sum_{i=1}^{n-1} \frac{\left[(\lambda_0 + \theta_0) kc^{-1}\right]^i}{i!} \exp\left\{-(\lambda_0 + \theta_0) kc^{-1}\right\}.$$

Also, since $\Delta_1, \Delta_2, \ldots, \Delta_n$ are IID Bernoulli random variables with probability of success $\lambda_0/(\lambda_0 + \theta_0)$ then

$$P\left(\textstyle\sum\Delta_i = k|\lambda_0, \theta_0, \sum\Delta_i > 0\right) = \frac{\binom{n}{k}\left(\frac{\lambda_0}{\lambda_0+\theta_0}\right)^k\left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^{n-k}}{1 - \left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^n}.$$

Finally, we obtain, after a bit of simplification,

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \textstyle\sum\Delta_i > 0\right) = \sum_{k=1}^{n}\left[\sum_{i=1}^{n-1} \frac{\left[(\lambda_0 + \theta_0) kc^{-1}\right]^i}{i!} \exp\left\{-(\lambda_0 + \theta_0) kc^{-1}\right\}\right]$$

$$\times \frac{\binom{n}{k}\left(\frac{\lambda_0}{\lambda_0+\theta_0}\right)^k\left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^{n-k}}{1-\left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^n}$$

$$= \sum_{i=1}^{n-1}\sum_{k=1}^{n}\frac{\left[\left(\lambda_0+\theta_0\right)kc^{-1}\right]^i}{i!}\exp\left\{-\left(\lambda_0+\theta_0\right)kc^{-1}\right\}$$

$$\times \frac{\binom{n}{k}\left(\frac{\lambda_0}{\lambda_0+\theta_0}\right)^k\left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^{n-k}}{1-\left(\frac{\theta_0}{\lambda_0+\theta_0}\right)^n}$$

$$= \sum_{i=1}^{n-1}\frac{\left[\left(\lambda_0+\theta_0\right)c^{-1}\right]^i}{i!}M^{(i)}_{\sum\Delta_i|\sum\Delta_i>0}\left[-\left(\lambda_0+\theta_0\right)c^{-1}\right]$$

where $M^{(i)}_{\sum\Delta_i|\sum\Delta_i>0}(s)$ is the ith derivative of the MGF for a zero-truncated binomial random variable with n trials and probability of success $\lambda_0/\left(\lambda_0+\theta_0\right)$. Notice that in fact this CDF does not approach one as $\lambda_0 \to 0$ which is contrary to what one would normally expect to happen. Application of L'Hôpital's rule shows that

$$P\left(\hat{\lambda} \leq c|\lambda_0, \theta_0, \sum\Delta_i > 0\right) \to P\left(\sum Z_i \geq c^{-1}|\lambda_0, \theta_0\right),$$

where $\sum Z_i$ has a gamma distribution with rate parameter θ_0, as $\lambda_0 \to 0$. This provides further evidence that the unconditional approach is the proper approach for making inference about λ_0.

APPENDIX B

BIAS OF THE SCORE FUNCTION

The MGF of random score function $U(\lambda)$, when one does not condition upon the event $\sum \Delta_i > 0$, is given as

$$M_{U(\lambda)}(s) = [M(s/\lambda, -s)]^n$$

where the joint MGF of (Δ_i, Z_i) is given in (4.1). From this we obtain

$$M(s/\lambda, -s) = M_C(-s-\lambda_0) + \frac{\lambda_0 e^{s/\lambda}}{\lambda_0 + s}\{1 - M_C(-s-\lambda_0)\}$$

and

$$\frac{dM(s/\lambda, -s)}{ds} = -M_C'(-s-\lambda_0) + \left(\frac{\lambda_0 e^{s/\lambda}/\lambda}{\lambda_0 + s} - \frac{\lambda_0 e^{s/\lambda}}{(\lambda_0+s)^2}\right)\{1 - M_C(-s-\lambda_0)\} + \frac{\lambda_0 e^{s/\lambda}}{\lambda_0 + s} M_C'(-s-\lambda_0)$$

which yields

$$E[U(\lambda)] = \left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)\{1 - M_C(-\lambda_0)\}.$$

This means that $E[U(\lambda_0)] = 0$ so that $U(\lambda)$ is an unbiased estimating equation when one does not condition upon the event $\sum \Delta_i > 0$.

In contrast, when one does condition upon $\sum \Delta_i > 0$, the MGF of $U(\lambda)$ is

$$\begin{aligned} M_{U(\lambda)}(s|\textstyle\sum \Delta_i > 0) &= M(s/\lambda, -s|\textstyle\sum \Delta_i > 0) \\ &= \frac{[M(s/\lambda, -s)]^n - [M_C(-s-\lambda_0)]^n}{1 - [M_C(-\lambda_0)]^n} \end{aligned}$$

and as a result

$$\frac{dM_{U(\lambda)}(s|\sum \Delta_i > 0)}{ds} = \frac{n[M(s/\lambda, -s)]^{n-1} M'(s/\lambda, -s) + n[M_C(-s-\lambda_0)]^{n-1} M_C'(-s-\lambda_0)}{1 - [M_C(-\lambda_0)]^n}$$

so that

$$E\left[U\left(\lambda_0\right)\left|\sum\Delta_i > 0\right.\right] = \frac{n\left[M_C(-\lambda_0)\right]^{n-1} M_C'(-\lambda_0)}{1-\left[M_C(-\lambda_0)\right]^n}.$$

As a result, $U\left(\lambda\right)$ is a biased estimating equation when one conditions upon the event $\sum\Delta_i > 0$.

VITA

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