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ABSTRACT

The invariance properties of the solutions of those functional equations naturally occurring in the construction of Mielnik probability spaces are studied, and in turn are related to one another. In particular, the possibilities for fixed points of these solutions are found, and the relationships between these results are discussed. The two functional equations studied include a representation of the generalized parallelogram law and an equation used in the modeling of polarization phenomena. The main result of the paper lies in the extention of previous research on Mielnik probability spaces to a higher dimension, as well as a discussion of their applicability in characterizing inner product spaces.

INTRODUCTION

In [1], G. Birkhoff and J. von Neumann confronted the problem of finding a fundamental set of axioms which would ensure the use of orthogonal projectors in Hilbert space as a representation of quantum logic propositions. C. Piron [2], assuming weak semimodularity, showed that it is possible to represent yes-no measurements by orthogonal projectors in some unitary spaces. The insufficiency of this approach was later noted by Mielnik [3], who instead proposed a geometric foundation to quantum mechanics. To do so, Mielnik introduced an abstract space of states in which to construct his geometric theory. Let S be a nonempty set and p a real-valued function defined on SxS such that the following three axioms hold.

Axiom A: $0 \le p(a,b) \le 1$ and $p(a,b)=1$ if and only if $a=b$.

Axiom B: $p(a,b) = p(b,a)$.

The third axiom deals with the property of orthogonality. Two elements $a \in S$ and $b \in S$ are said to be orthogonal if $p(a,b) = 0$. An orthogonal system of S is defined as a subset $R \subseteq S$ such that any two distinct elements of R are orthogonal. A maximal orthogonal system, or basis, is an orthogonal system which cannot be contained within any larger orthogonal system in S. The existence of at least one basis in S is a consequence of Zorn's lemma. In [3], Mielnik proved the uniqueness of the number of elements contained in a basis. Thus, the usual definition of dimension is applicable to the set S. With this concept of basis, Axiom C may be formulated as follows.

Axiom C: For each basis $R \subseteq S$ and for each $x \in S$, $\sum_{r \in R} p(x,r)=1$.

These three axioms form the foundation of a Mielnik probability space, as indicated in the following definition.

Definition: A Mielnik probability space (S,p) consists of a nonempty set S and a real-valued function p defined on $S \times S$ such that axioms A, B, and C hold.

FUNCTIONAL EQUATIONS IN MIELNIK PROBABILITY SPACES

The Generalized Parallelogram Law.

Let $[0,2]$ be the domain of f, and let $[0,1]$ be the range of f. Further, let f be continuous and strictly increasing, and let $f(0)=0$ and $f(2)=1$. We denote the class of all such functions F.
Let $[0,2]$ be both the domain and range of g. Further, let g be continuous and strictly decreasing with $g(0)=2$ and $g(2)=0$. We denote the class of all such function G. In [4], Stanojević examined those $f \in F$ and $g \in G$ that satisfied the functional equation

$$f+f\circ g=1 \tag{1}$$

where $f\circ g(t) = f(g(t))$. Stanojević was able to generalize the parallelogram law as a test for an inner product space using the above functional equation, and then went on to characterize these inner product spaces in terms of Mielnik probability spaces. The following result from [4] guarantees the existence of solutions to (1).

Lemma 1. (C. V. Stanojević) Let $f \in F$ Then there exists a $g \in G$ such that (1) holds.

We now proceed to examine the invariance properties of those $f \in F$ and $g \in G$ that together satisfy (1).

Lemma 2. Let $f \in F$ and $g \in G$ satisfy (1). Every such g has a unique fixed point.

Proof. First note that Lemma 1 guarantees the existence of solutions to (1). Now, the continuity of f guarantees that f takes on all values in $[0,1]$. Further, the monotonicity of f results in f being a one-to-one function, and so f attains each value in $[0,1]$ exactly once. Thus, consider $x \in [0,2]$ such that $f(x)=1/2$. Then

$$f(x) + f\circ g(x) = 1 \text{ implies } f\circ g(x) = 1/2.$$

But,

$$f(x) = 1/2 \text{ and } f\circ g(x) = 1/2 \text{ implies that } g(x) = x$$

since f is one-to-one. Thus, g has at least one fixed point. We now show the uniqueness of this fixed point. Suppose that x and y are fixed by g. Then (1) immediately reduces to

$$f(x) = 1/2 \text{ and } f(y) = 1/2.$$

Again, due to the one-to-one nature of f, we have x = y. Hence, the fixed point of g is unique, and the proof is complete.
From the proof of Lemma 2, we have established the following result.

Lemma 3. Let $f \in F$ and $g \in G$ satisfy (1). Then $g(x) = x$ if and only if $f(x) = 1/2$.

An interesting consequence of these lemmas is the following theorem.

Theorem 1. Let $f \in F$ and $g \in G$ satisfy (1). Then the unique common fixed point for all such f and g is $x=1/2$.

Proof. From Lemma 3, if g fixes x then $f(x) = 1/2$. Thus, if f is also to fix x, we must have $x = 1/2$. In no other instance can both f and g fix a value simultaneously. Thus the unique common fixed point is $x = 1/2$. It appears from this development that $x = 1/2$ represents some center point of symmetry.

A Functional Equation Related to Polarization Phenomena.

In [5], Stanojević and Guccione introduced the following functional equation in an effort to offer a very general class of probability functions to be applied to modeling polarization phenomena. This equation, given by

$$\phi(t) + \phi(1-t) = 1, \tag{2}$$

is applied by setting the probability function p(x,y) to by

$$p(x,y) = \phi o f(\|x+y\|),$$

where $f \in F$. From here on, we denote the class of all such functions ϕ from [0,1] onto [0,1] satisfying (2) by Φ. Further, in our consideration of applying this functional equation to polarization phenomena, we will denote the subclass $\Phi_1 \subseteq \Phi$ where Φ_1 is the set of all strictly increasing, continuous functions satisfying (2). Noting the obvious existence of a solution to (2) given by $\phi(t)=t$, we now proceed with an investigation of the invariance properties of this functional equation.

Theorem 2. Every $\phi \subseteq \Phi$ has at least one fixed point, namely $t = 1/2$.

Proof. Direct substitution of $t = 1/2$ into (2) verifies $\phi(1/2) = 1/2$.

Theorem 3. Let $\phi \in \Phi$ and suppose t is fixed by ϕ. Then (1-t) is also fixed by ϕ.

Proof. Since $\phi(t) = t$, we have from (2) that $t + \phi(1-t) = 1$. Thus, $\phi(1-t) = 1-t$, and the proof is complete.

It is to be noted that a more complete characterization of Φ was given in [5], where it was proposed that every $\phi \in \Phi$ was of the form

$$\phi(t) = h(t-1/2) + 1/2,$$

where h is an odd function defined on $[-1/2,1/2]$. These same fixed point theorems follow readily from this observation as well.
An interesting result is obtained by considering the application of this functional equation to polarization phenomena. Letting $p(x,y) = \phi o f(\|x+y\|)$, we have the following theorem, which links the invariance properties of the two functional equations.

Theorem 4. Let N be a real inner product space, S its unit sphere, $f \in F$, $g \in G$, and $\phi \in \Phi_1$. Further, let $p(x,y) = \phi o f(\|x+y\|)$ define a probability function for $x,y \in S$. Then ϕ attains the fixed point at $t = 1/2$ when and only when g takes on its unique fixed point.

Proof. The verification that (S,p) results in a Mielnik probability space of dimension 2 was accomplished by Stanojević and Guccione in [5]. Thus, the probability space axioms hold. It is immediate that the bases are of the form $\{x,-x\}$. Hence, from Axiom C, we have

$$p(x,y) + p(x,-y) = 1 \Rightarrow$$

$$\phi o f(\|x+y\|) + \phi o f(\|x-y\|) = 1.$$

Now, ϕ satisfies (2), so $f(\|x-y\|) = 1 - f(\|x+y\|)$. From Lemma 1, there exists a $g \in G$ such that $f(\|x-y\|) = f o g(\|x+y\|)$. Let $z=\|x+y\|$. Clearly, z takes on all the values in the interval $[0,2]$. Thus, we may now write

$$\phi(f(z)) + \phi(f o g(z)) = 1, \text{ or}$$

$$\phi(f(z)) + \phi(1-f(z)) = 1. \qquad (3)$$

Suppose ϕ takes on the central fixed point at z. Then $f(z) = 1/2$, and from Lemma 3, $g(z) = z$.
Conversely, suppose $g(z) = z$. Again from Lemma 3, $f(z) = 1/2$, and (3) reduces to show that ϕ takes on the central fixed point.

A FOUR-DIMENSIONAL PROBABILITY SPACE

Let N be a complex normed linear space, and let S be its unit sphere. Further, let $f \in F$ and for all $x,y \in S$ let

$$D_1 = D_1(x,y) = f(\|x-y\|)f(\|x-iy\|)f(\|x+iy\|),$$

$$D_2 = D_2(x,y) = f(\|x-y\|)f(\|x+y\|)f(\|x-iy\|),$$

$$D_3 = D_3(x,y) = f(\|x-y\|)f(\|x+y\|)f(\|x+iy\|),$$

$$D_4 = D_4(x,y) = f(\|x+y\|)f(\|x-iy\|)f(\|x+iy\|).$$

Theorem 4. For all $x,y \in S$, let

$$p(x,y) = D_4 / (D_1 + D_2 + D_3 + D_4).$$

Then (S,p) is a probability space of dimension 4.

Proof. We divide the proof into sections verifying the probability space axioms.

Axiom A. Notice that $D_1 + D_2 + D_3 + D_4$ is never zero. For, assuming the contrary, each D_i would have to be zero since the range of f contains no negative values. But, $f(t) = 0$ if and only if $t = 0$. Thus, each D_i must contain the common term $f(0)$. However, the D_i do not all share a common term, so it is impossible for each D_i to be zero simultaneously. Thus, $p(x,y)$ is well defined.

Now, since the range of f can take on only nonnegative values, we have $D_i \geq 0$ for each $i \in \{1,2,3,4\}$ and $D_4 \leq D_1 + D_2 + D_3 + D_4$. Thus, $0 \leq p(x,y) \leq 1$.

Consider $p(x,x)$. $D_1 = D_2 = D_3 = 0$ since $f(\|x-x\|) = f(0) = 0$, and so

$$p(x,x) = D_4 / D_4 = 1.$$

Conversely, suppose $p(x,y) = 1$. Then $D_4 = D_1 + D_2 + D_3 + D_4$. Hence, $D_1 = D_2 = D_3 = 0$. Again, this can only occur if each of D_1, D_2, and D_3 contain the common term $f(0)$. The only term common to D_1, D_2, and D_3 is $f(\|x-y\|)$. Thus, $\|x-y\| = 0$, and so $x=y$.

We have thus shown that $0 \leq p(x,y) \leq 1$ and $p(x,y) = 1$ if and only if $x=y$. Thus, Axiom A holds.

Axiom B. We must show $p(x,y) = p(y,x)$. We start by showing that D_4 is invariant on interchanging x and y. Clearly, $f(\|x+y\|)$ remains unchanged due to the commutativity of addition in N. Now, $|i| = 1$, so $\|y-ix\| = \|iy-i^2x\| = \|iy + x\| = \|x + iy\|$. Similarly, $\|y+ix\| = \|x-iy\|$. Thus, upon interchanging x and y, we have $f(\|y-ix\|) = f(\|x+iy\|)$ and $f(\|y+ix\|) = f(\|x-iy\|)$. Hence, D_4 is invariant under the interchanging of x and y.

Using the same procedure (multiplying through by i when necessary), it is found that D_1 remains invariant while D_2 becomes

D_3 and vice versa when x and y are interchanged. Thus, ΣD_i is itself invariant. Thus, $p(x,y) = p(y,x)$, and Axiom B holds.

Axiom C. The bases in this instance are of the form

$$B = \{b, -b, ib, -ib\}.$$

To see this, note that since each two distinct elements of an orthogonal system have a zero transitional probability, we must have $D_4=0$ for any two orthogonal elements. It follows, since $D_4=0$ if and only if it contains a term of the form f(0), that if r and s are orthogonal, then $r = -s$ or $r = \pm is$. The form of the bases B immediately follows. We now formulate the sum required for Axiom C. Upon direct substitution into p(x,y) and summing, we find that $\sum_{b \in B} p(x,b) = 1$, and so Axiom C holds also.

Thus, the axioms of a probability space are satisfied, and so (S,p) is a probability space. From the verification of Axiom C, we have that (S,p) is of dimension 4. This proof is complete.

Future Research.

The extension of a probability space structure to n dimensions has already been accomplished in a manner similar to the specific four dimensional case presented here. The significance of these results is that it can be shown that these higher dimensional probability spaces can not be used as a characterization of inner product spaces as the two dimensional spaces were by Stanojević. The extension to the n dimensional case and the inability of probability spaces of higher dimension to characterize inner product spaces will be presented in an upcoming paper. The applicability of Mielnik probability spaces to physical phenomena as well as the significance of the invariance properties of the functional equations used in their construction offer interesting avenues of future research to follow up on.

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