

01 Aug 2019

Chapter 2: 2D/3D Finite Element Spaces

Xiaoming He

Missouri University of Science and Technology, hex@mst.edu

Follow this and additional works at: <https://scholarsmine.mst.edu/oer-course-materials>

 Part of the [Mathematics Commons](#)

Recommended Citation

He, Xiaoming, "Chapter 2: 2D/3D Finite Element Spaces" (2019). *Course Materials*. 5.
<https://scholarsmine.mst.edu/oer-course-materials/5>



This work is licensed under a [Creative Commons Attribution 4.0 License](#).

This Course materials is brought to you for free and open access by Scholars' Mine. It has been accepted for inclusion in Course Materials by an authorized administrator of Scholars' Mine. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact scholarsmine@mst.edu.

Mathematical Foundation of Finite Element Methods

Chapter 2: 2D/3D Finite Element Spaces

Xiaoming He

Department of Mathematics & Statistics
Missouri University of Science & Technology

Outline

- 1 2D uniform Mesh
- 2 Triangular elements
- 3 Rectangular elements
- 4 3D elements
- 5 More discussion

Outline

- 1 2D uniform Mesh
- 2 Triangular elements
- 3 Rectangular elements
- 4 3D elements
- 5 More discussion

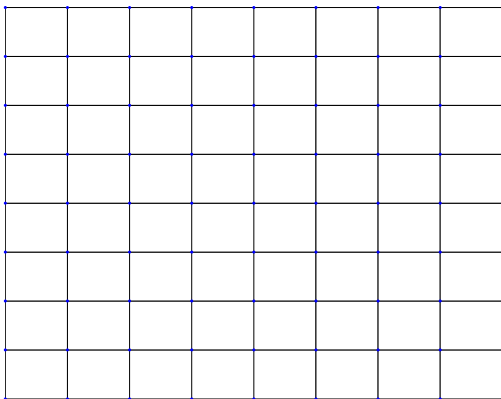
Triangular mesh: uniform partition

- Consider $\Omega = [left, right] \times [bottom, top]$.
- First, we form a uniform rectangular partition of Ω into N_1 elements in x – axis and N_2 elements in y – axis with mesh size

$$h = [h_1, h_2] = \left[\frac{right - left}{N_1}, \frac{top - bottom}{N_2} \right].$$

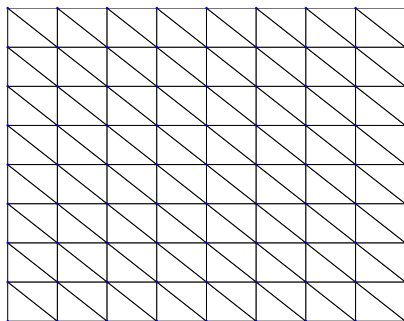
Triangular mesh: global indices

- For example, when $N_1 = N_2 = 8$, we have



Triangular mesh: global indices

- Then we divide each rectangular element into two triangular elements by connecting the left-top corner and the right-lower corner of the rectangular element.
- For example, when $N_1 = N_2 = 8$, we have

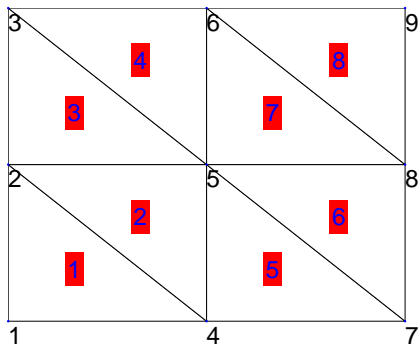


Triangular mesh: global indices

- This would give an uniform triangular partition.
- There are $N = 2N_1N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.

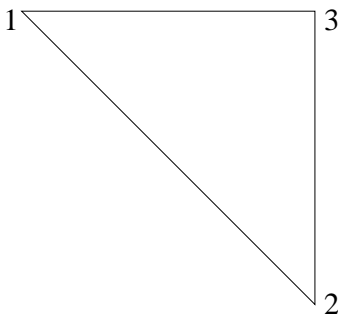
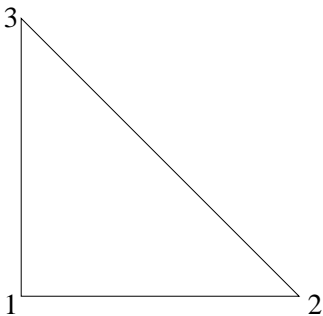
Triangular mesh: global indices

- Define your global indices for all the mesh elements E_n ($n = 1, \dots, N$) and mesh nodes Z_k ($k = 1, \dots, N_m$).
- For example, when $N_1 = N_2 = 2$, we have



Triangular mesh: local node index

- Let N_i denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.



Triangular mesh: information matrices

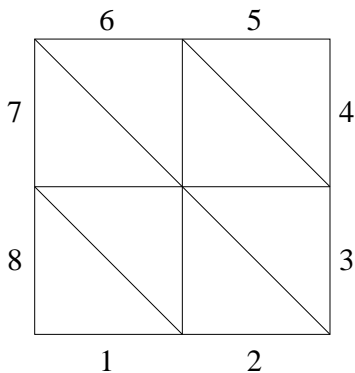
- Define matrix P to be an information matrix consisting of the **coordinates of all mesh nodes**.
- Define matrix T to be an information matrix consisting of the **global node indices of the mesh nodes of all the mesh elements**.
- We can use the j^{th} column of the matrix P to store the coordinates of the j^{th} mesh node and the n^{th} column of the matrix T to store the global node indices of the mesh nodes of the n^{th} mesh element. For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$

Triangular mesh: boundary edge index

- Define your index for the boundary edges.
- For example, when $N_1 = N_2 = 2$, we have



Triangular mesh: boundary edge information matrix

- Matrix *boundaryedges*:
- $boundaryedges(1, k)$ is the type of the k^{th} boundary edge e_k : Dirichlet (-1), Neumann (-2), Robin (-3).....
- $boundaryedges(2, k)$ is the index of the element which contains the k^{th} boundary edge e_k .
- Each boundary edge has two end nodes. We index them as the first and the second counterclock wise along the boundary.
- $boundaryedges(3, k)$ is the global node index of the first end node of the k^{th} boundary edge e_k .
- $boundaryedges(4, k)$ is the global node index of the second end node of the k^{th} boundary edge e_k .
- Set $nbe = size(boundaryedges, 2)$ to be the number of boundary edges;

Triangular mesh: boundary edge information matrix

- For the mesh with $N_1 = N_2 = 2$ and all Dirichlet boundary condition, we have:

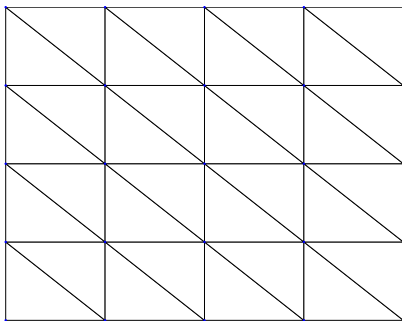
$$\mathit{boundaryedges} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 5 & 6 & 8 & 8 & 4 & 3 & 1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\ 4 & 7 & 8 & 9 & 6 & 3 & 2 & 1 \end{pmatrix}.$$

Triangular mesh

- What are the information matrices

P , T , *boundaryedges*

for the following mesh?



Triangular mesh

- What are the information matrices

$$P, T, \text{ boundaryedges}$$

for a general uniform **triangular** mesh with the mesh size

$$h = [h_1, h_2] = \left[\frac{\text{right} - \text{left}}{N_1}, \frac{\text{top} - \text{bottom}}{N_2} \right]$$

in the domain

$$\Omega = [\text{left}, \text{right}] \times [\text{bottom}, \text{top}]?$$

Rectangular mesh: uniform partition

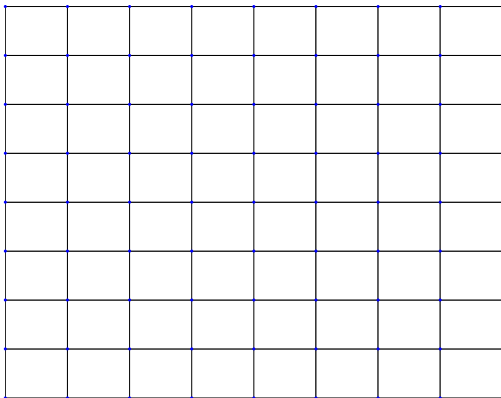
- Consider $\Omega = [left, right] \times [bottom, top]$.
- Consider a uniform rectangular partition of Ω into N_1 elements in x – axis and N_2 elements in y – axis with mesh size

$$h = [h_1, h_2] = \left[\frac{right - left}{N_1}, \frac{top - bottom}{N_2} \right].$$

- There are $N = N_1 N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.

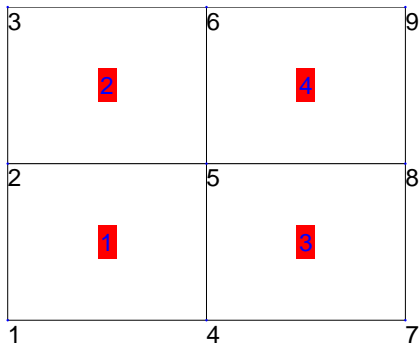
Rectangular mesh: uniform partition

- For example, when $N_1 = N_2 = 8$, we have



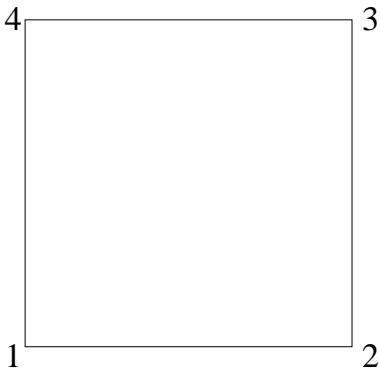
Rectangular mesh: global indices

- Define your global indices for all the mesh elements E_n ($n = 1, \dots, N$) and mesh nodes Z_k ($k = 1, \dots, N_m$).
- For example, when $N_1 = N_2 = 2$, we have



Rectangular mesh: local node index

- Let N_i denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.



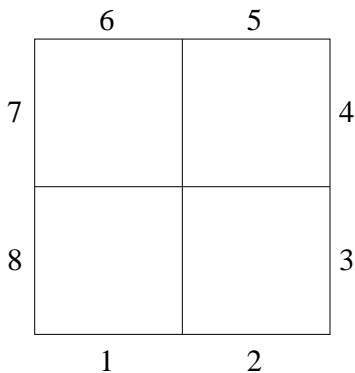
Rectangular mesh: information matrices

- Define matrix P to be an information matrix consisting of the coordinates of all mesh nodes.
- Define matrix T to be an information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.
- For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}.$$

Rectangular mesh: boundary edge index

- Define your index for the boundary edges.
- For example, when $N_1 = N_2 = 2$, we have



Rectangular mesh: boundary edge information matrix

- Matrix *boundaryedges*:
- $boundaryedges(1, k)$ is the type of the k^{th} boundary edge e_k : Dirichlet (-1), Neumann (-2), Robin (-3).....
- $boundaryedges(2, k)$ is the index of the element which contains the k^{th} boundary edge e_k .
- Each boundary edge has two end nodes. We index them as the first and the second counterclock wise along the boundary.
- $boundaryedges(3, k)$ is the global node index of the first end node of the k^{th} boundary edge e_k .
- $boundaryedges(4, k)$ is the global node index of the second end node of the k^{th} boundary edge e_k .
- Set $nbe = size(boundaryedges, 2)$ to be the number of boundary edges;

Rectangular mesh: boundary edge information matrix

- For example, when $N_1 = N_2 = 2$ and all the boundary are Dirichlet type, we have:

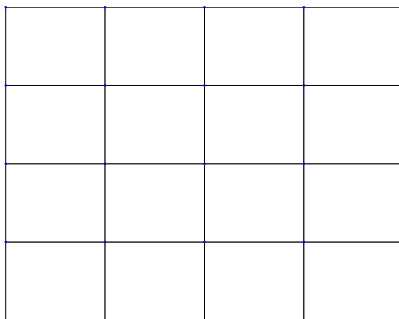
$$\mathit{boundaryedges} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 3 & 3 & 4 & 4 & 2 & 2 & 1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\ 4 & 7 & 8 & 9 & 6 & 3 & 2 & 1 \end{pmatrix}.$$

Rectangular mesh

- What are the information matrices

P , T , *boundaryedges*

for the following mesh?



Rectangular mesh

- What are the information matrices

$$P, T, \text{ boundaryedges}$$

for a general uniform **rectangular** mesh with the mesh size

$$h = [h_1, h_2] = \left[\frac{\text{right} - \text{left}}{N_1}, \frac{\text{top} - \text{bottom}}{N_2} \right]$$

in the domain

$$\Omega = [\text{left}, \text{right}] \times [\text{bottom}, \text{top}]?$$

Outline

- 1 2D uniform Mesh
- 2 Triangular elements**
- 3 Rectangular elements
- 4 3D elements
- 5 More discussion

2D linear finite element: reference basis functions

- The “reference \rightarrow local \rightarrow global” framework will be used to construct the finite element spaces.
- We only consider the nodal basis functions (Lagrange type) in this course.
- We first consider the reference 2D linear basis functions on the reference triangular element $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ where $\hat{A}_1 = (0, 0)$, $\hat{A}_2 = (1, 0)$, and $\hat{A}_3 = (0, 1)$.
- Define **three reference 2D linear basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}) = a_j \hat{x} + b_j \hat{y} + c_j, \quad j = 1, 2, 3,$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3$.

2D linear finite element: reference basis functions

- Then it's easy to obtain

$$\hat{\psi}_1(\hat{A}_1) = 1 \Rightarrow c_1 = 1,$$

$$\hat{\psi}_1(\hat{A}_2) = 0 \Rightarrow a_1 + c_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_3) = 0 \Rightarrow b_1 + c_1 = 0,$$

$$\hat{\psi}_2(\hat{A}_1) = 0 \Rightarrow c_2 = 0,$$

$$\hat{\psi}_2(\hat{A}_2) = 1 \Rightarrow a_2 + c_2 = 1,$$

$$\hat{\psi}_2(\hat{A}_3) = 0 \Rightarrow b_2 + c_2 = 0,$$

$$\hat{\psi}_3(\hat{A}_1) = 0 \Rightarrow c_3 = 0,$$

$$\hat{\psi}_3(\hat{A}_2) = 0 \Rightarrow a_3 + c_3 = 0,$$

$$\hat{\psi}_3(\hat{A}_3) = 1 \Rightarrow b_3 + c_3 = 1.$$

2D linear finite element: reference basis functions

- Hence

$$a_1 = -1, b_1 = -1, c_1 = 1,$$

$$a_2 = 1, b_2 = 0, c_2 = 0,$$

$$a_3 = 0, b_3 = 1, c_3 = 0.$$

- Then the three reference 2D linear basis functions are

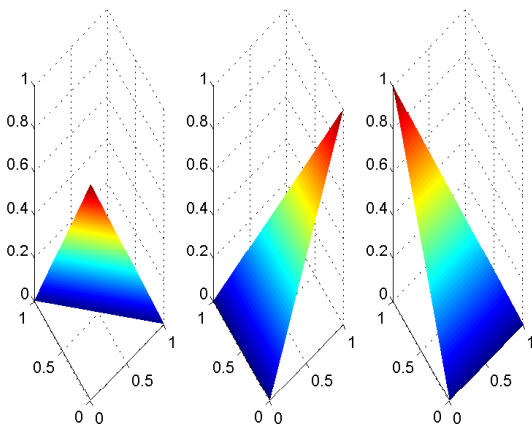
$$\hat{\psi}_1(\hat{x}, \hat{y}) = -\hat{x} - \hat{y} + 1,$$

$$\hat{\psi}_2(\hat{x}, \hat{y}) = \hat{x},$$

$$\hat{\psi}_3(\hat{x}, \hat{y}) = \hat{y}.$$

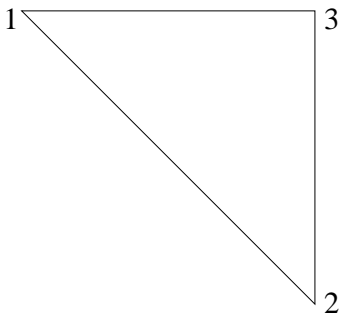
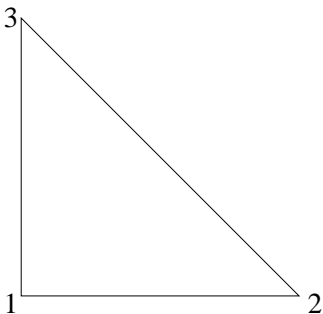
2D linear finite element: reference basis functions

- Plots of the three linear basis functions on the reference triangle:



2D linear finite element: local node index

- Let N_{lb} denote the number of local finite element nodes (local finite element basis functions) in a mesh element. Here $N_{lb} = 3$. Define your index for the local finite element nodes in a mesh element.

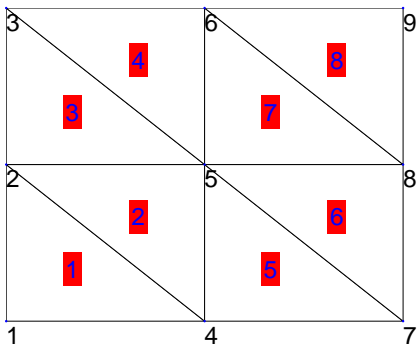


2D linear finite element: information matrices

- The mesh information matrices P and T are for the mesh nodes.
- We also need similar finite element information matrices P_b and T_b for the finite elements nodes, which are the nodes corresponding to the finite element basis functions.
- **Note:** For the nodal finite element basis functions, the correspondence between the finite elements nodes and the finite element basis functions is one-to-one in a straightforward way. But it could be more complicated for other types of finite element basis functions in the future.
- Let N_b denote the total number of the finite element basis functions (= the number of unknowns = the total number of the finite element nodes). Here $N_b = N_m = (N_1 + 1)(N_2 + 1)$.

2D linear finite element: information matrices

- Define your global indices for all the mesh elements E_n ($n = 1, \dots, N$) and finite element nodes X_j ($j = 1, \dots, N_b$) (or the finite element basis functions).
- For example, when $N_1 = N_2 = 2$, we have



2D linear finite element: information matrices

- Define matrix P_b to be an information matrix consisting of the **coordinates of all finite element nodes**.
- Define matrix T_b to be an information matrix consisting of the **global node indices of the finite element nodes of all the mesh elements**.

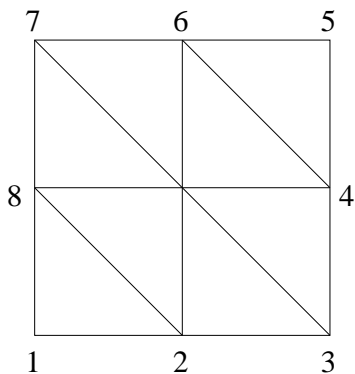
2D linear finite element: information matrices

- For the 2D linear finite elements, P_b and T_b are the same as the P and T of the triangular mesh since the nodes of the 2D linear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$
$$T_b = T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$

2D linear finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have,



2D linear finite element: boundary node information matrix

- Matrix *boundarynodes*:
- $boundarynodes(1, k)$ is the type of the k^{th} boundary finite element node: Dirichlet (-1), Neumann (-2), Robin (-3).....
- The intersection nodes of Dirichlet boundary condition and other boundary conditions usually need to be treated as Dirichlet boundary nodes.
- $boundarynodes(2, k)$ is the global node index of the k^{th} boundary finite element node.
- Set $nb_n = size(boundarynodes, 2)$ to be the number of boundary finite element nodes;
- For the above example with all Dirichlet boundary condition, we have:

$$boundarynodes = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \end{pmatrix}.$$

2D linear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary triangle $E = \triangle A_1 A_2 A_3$ and the reference triangle $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ to construct the local basis functions from the reference ones.
- Assume

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2, 3.$$

- Consider the affine mapping

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= (A_2 - A_1, A_3 - A_1) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + A_1 \\ &= \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \end{aligned}$$

2D linear finite element: affine mapping

- The affine mapping actually maps

$$\hat{A}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A_1,$$

$$\hat{A}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A_2,$$

$$\hat{A}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = A_3.$$

- Hence the affine mapping maps $\triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ to $\triangle A_1 A_2 A_3$.
- Also,

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},$$
$$\hat{y} = \frac{(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}.$$

2D linear finite element: affine mapping

- Define the Jacobi matrix:

$$J = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}.$$

- Then

$$|J| = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1),$$

and

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|},$$
$$\hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}.$$

2D linear finite element: affine mapping

- For a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \triangle A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|},$$
$$\hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}.$$

2D linear finite element: affine mapping

- Then by chain rule, we get

$$\begin{aligned}\frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \\ &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|}, \\ \frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\ &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.\end{aligned}$$

2D linear finite element: local basis functions

- Consider the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix} \quad (i = 1, 2, 3).$$

- The three local 2D linear basis functions are

$$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, 2, 3,$$

where

$$\hat{x} = \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|},$$

$$\hat{y} = \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|},$$

$$|J_n| = (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}).$$

2D linear finite element: local basis functions

- And for $i = 1, 2, 3$,

$$\frac{\partial \psi_{ni}}{\partial x} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{y_{n3} - y_{n1}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{y_{n1} - y_{n2}}{|J_n|},$$

$$\frac{\partial \psi_{ni}}{\partial y} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{x_{n1} - x_{n3}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{x_{n2} - x_{n1}}{|J_n|}.$$

- The reference and local basis functions defined in this section are what you need to input into the code in order to use the “reference \rightarrow local” framework to define the local basis functions.

2D linear finite element: local basis functions

- In more details, we have

$$\begin{aligned}\psi_{n1}(x, y) &= \hat{\psi}_1(\hat{x}, \hat{y}) = -\hat{x} - \hat{y} + 1 \\ &= \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|} \\ &\quad - \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|} + 1,\end{aligned}$$

$$\begin{aligned}\psi_{n2}(x, y) &= \hat{\psi}_2(\hat{x}, \hat{y}) = \hat{x} \\ &= \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|},\end{aligned}$$

$$\begin{aligned}\psi_{n3}(x, y) &= \hat{\psi}_3(\hat{x}, \hat{y}) = \hat{y} \\ &= \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}.\end{aligned}$$

2D linear finite element: local basis functions

- And

$$\frac{\partial \psi_{n1}}{\partial x} = -\frac{y_{n3} - y_{n1}}{|J_n|} + \frac{y_{n2} - y_{n1}}{|J_n|} = \frac{y_{n2} - y_{n3}}{|J_n|},$$

$$\frac{\partial \psi_{n2}}{\partial x} = \frac{y_{n3} - y_{n1}}{|J_n|},$$

$$\frac{\partial \psi_{n3}}{\partial x} = -\frac{y_{n2} - y_{n1}}{|J_n|},$$

$$\frac{\partial \psi_{n1}}{\partial y} = \frac{x_{n3} - x_{n1}}{|J_n|} - \frac{x_{n2} - x_{n1}}{|J_n|} = \frac{x_{n3} - x_{n2}}{|J_n|},$$

$$\frac{\partial \psi_{n2}}{\partial y} = -\frac{x_{n3} - x_{n1}}{|J_n|},$$

$$\frac{\partial \psi_{n3}}{\partial y} = \frac{x_{n2} - x_{n1}}{|J_n|}.$$

- You can also directly input these local basis functions and their derivatives into your code.

2D linear finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ as follows:

$$\psi_{nj}(x, y) = a_{nj}x + b_{nj}y + c_{nj}, \quad j = 1, 2, 3,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3$.

- Obtain the local basis functions in the above way and compare them with the ψ_{n1} , ψ_{n2} , and ψ_{n3} obtained before.
- They are the same!

2D linear finite element: global basis functions

“local \rightarrow global” framework:

- Define the **local finite element space**

$$S_h(E_n) = \text{span}\{\psi_{n1}, \psi_{n2}, \psi_{n3}\}.$$

- At each finite element node X_j ($j = 1, \dots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, N_b$.

- Then define the **global finite element space** to be

$$U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}.$$

2D linear finite element: global basis functions

- Hence

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1, n), \\ \psi_{n2}, & \text{if } j = T_b(2, n), \\ \psi_{n3}, & \text{if } j = T_b(3, n), \\ 0, & \text{otherwise.} \end{cases}$$

for $j = 1, \dots, N_b$ and $n = 1, \dots, N$.

2D quadratic finite element: reference basis functions

- We first consider the reference 2D quadratic basis functions on the reference triangular element $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ where $\hat{A}_1 = (0, 0)$, $\hat{A}_2 = (1, 0)$, and $\hat{A}_3 = (0, 1)$. Define $\hat{A}_4 = (0.5, 0)$, $\hat{A}_5 = (0.5, 0.5)$, and $\hat{A}_6 = (0, 0.5)$.
- Define **six reference 2D quadratic basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}) = a_j \hat{x}^2 + b_j \hat{y}^2 + c_j \hat{x} \hat{y} + d_j \hat{y} + e_j \hat{x} + f_j, \quad j = 1, \dots, 6,$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, 6$.

2D quadratic finite element: reference basis functions

- For $\hat{\psi}_1$, it's easy to obtain

$$\hat{\psi}_1(\hat{A}_1) = 1 \Rightarrow f_1 = 1,$$

$$\hat{\psi}_1(\hat{A}_2) = 0 \Rightarrow a_1 + e_1 + f_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_3) = 0 \Rightarrow b_1 + d_1 + f_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_4) = 0 \Rightarrow 0.25a_1 + 0.5e_1 + f_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_5) = 0 \Rightarrow 0.25a_1 + 0.25b_1 + 0.25c_1 + 0.5d_1 + 0.5e_1 + f_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_6) = 0 \Rightarrow 0.25b_1 + 0.5d_1 + f_1 = 0.$$

- Hence

$$a_1 = 2, b_1 = 2, c_1 = 4, d_1 = -3, e_1 = -3, f_1 = 1.$$

- Then

$$\hat{\psi}_1(\hat{x}, \hat{y}) = 2\hat{x}^2 + 2\hat{y}^2 + 4\hat{x}\hat{y} - 3\hat{y} - 3\hat{x} + 1.$$

2D quadratic finite element: reference basis functions

- Similarly, we can obtain all the six reference 2D quadratic basis functions

$$\hat{\psi}_1(\hat{x}, \hat{y}) = 2\hat{x}^2 + 2\hat{y}^2 + 4\hat{x}\hat{y} - 3\hat{y} - 3\hat{x} + 1,$$

$$\hat{\psi}_2(\hat{x}, \hat{y}) = 2\hat{x}^2 - \hat{x},$$

$$\hat{\psi}_3(\hat{x}, \hat{y}) = 2\hat{y}^2 - \hat{y},$$

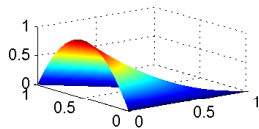
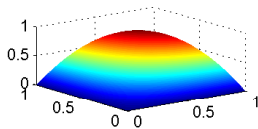
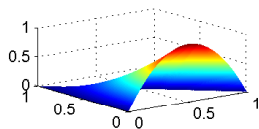
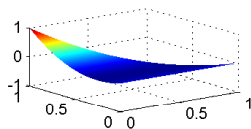
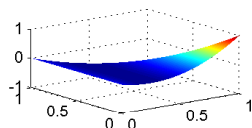
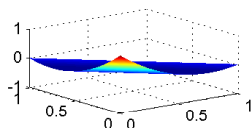
$$\hat{\psi}_4(\hat{x}, \hat{y}) = -4\hat{x}^2 - 4\hat{x}\hat{y} + 4\hat{x},$$

$$\hat{\psi}_5(\hat{x}, \hat{y}) = 4\hat{x}\hat{y},$$

$$\hat{\psi}_6(\hat{x}, \hat{y}) = -4\hat{y}^2 - 4\hat{x}\hat{y} + 4\hat{y}.$$

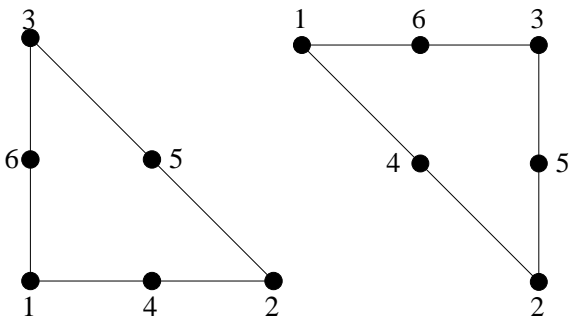
2D quadratic finite element: reference basis functions

- Plots of the six quadratic basis functions on the reference triangle:



2D quadratic finite element: local node index

- Define your index for the local finite element nodes in a mesh element with $N_{lb} = 6$.

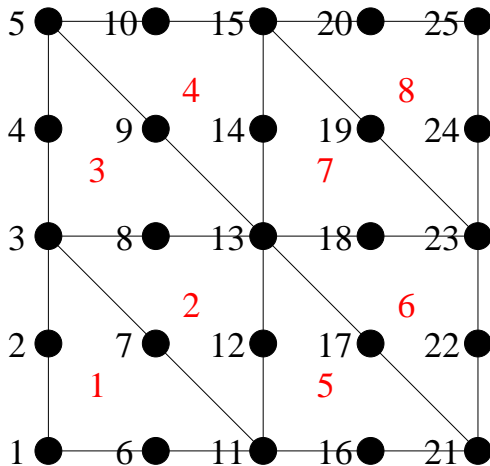


2D quadratic finite element: information matrices

- Define your global indices for all the mesh elements
 E_n ($n = 1, \dots, N$) and finite element nodes
 X_j ($j = 1, \dots, N_b$) (or the finite element basis functions) with
 $N_b = (2N_1 + 1)(2N_2 + 1) \neq N_m$.

2D quadratic finite element: information matrices

- For example, when $N_1 = N_2 = 2$, we have



2D quadratic finite element: information matrices

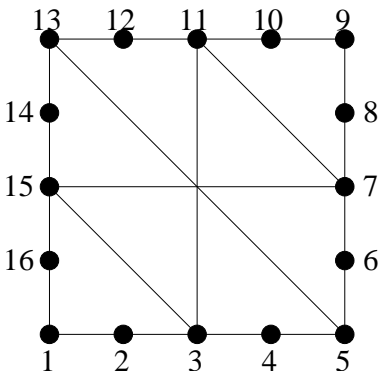
- The P_b and T_b for 2D quadratic finite element are different from the P and T for the triangular mesh. For the above example we have

$$P_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \dots & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix},$$

$$T_b = \begin{pmatrix} 1 & 3 & 3 & 5 & 11 & 13 & 13 & 15 \\ 11 & 11 & 13 & 13 & 21 & 21 & 23 & 23 \\ 3 & 13 & 5 & 15 & 13 & 23 & 15 & 25 \\ 6 & 7 & 8 & 9 & 16 & 17 & 18 & 19 \\ 7 & 12 & 9 & 14 & 17 & 22 & 19 & 24 \\ 2 & 8 & 4 & 10 & 12 & 18 & 14 & 20 \end{pmatrix}.$$

2D quadratic finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have,



2D quadratic finite element: boundary node information matrix

- Matrix *boundarynodes*:
- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

$$\text{boundarynodes} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & \cdots & -1 \\ 1 & 6 & 11 & 16 & 21 & \cdots & 25 & \cdots & 5 & \cdots & 2 \end{pmatrix}.$$

2D quadratic finite element: affine mapping

- The affine mapping we use here is exactly the same as the previous one!
- Recall: for a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \Delta \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \Delta A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|},$$
$$\hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}.$$

2D quadratic finite element: affine mapping

- Recall: by chain rule, we get

$$\begin{aligned}
 \frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \\
 &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|}, \\
 \frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\
 &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.
 \end{aligned}$$

2D quadratic finite element: affine mapping

- By chain rule again, we get

$$\begin{aligned}
 \frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial x} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial x} \frac{y_1 - y_2}{|J|} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial x} \frac{y_1 - y_2}{|J|} \\
 &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(y_3 - y_1)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(y_1 - y_2)^2}{|J|^2}.
 \end{aligned}$$

2D quadratic finite element: affine mapping

- And

$$\begin{aligned}
 \frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial y} \frac{x_2 - x_1}{|J|} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial y} \frac{x_2 - x_1}{|J|} \\
 &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(x_1 - x_3)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(x_2 - x_1)^2}{|J|^2}.
 \end{aligned}$$

2D quadratic finite element: affine mapping

- And

$$\begin{aligned}
 \frac{\partial^2 \psi}{\partial x \partial y} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial y} \frac{y_1 - y_2}{|J|} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial y} \frac{y_1 - y_2}{|J|} \\
 &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} \\
 &\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_2 - x_1)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}.
 \end{aligned}$$

2D quadratic finite element: local basis functions

- Consider the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}, \quad i = 1, 2, 3.$$

- Define

$$A_{n4} = \frac{A_{n1} + A_{n2}}{2}, \quad A_{n5} = \frac{A_{n2} + A_{n3}}{2}, \quad A_{n6} = \frac{A_{n3} + A_{n1}}{2}.$$

2D quadratic finite element: local basis functions

- The six local 2D linear basis functions are

$$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, \dots, 6,$$

where

$$\hat{x} = \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|},$$

$$\hat{y} = \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|},$$

$$|J_n| = (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}).$$

2D quadratic finite element: local basis functions

- And for $i = 1, \dots, 6$,

$$\frac{\partial \psi_{ni}}{\partial x} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{y_{n3} - y_{n1}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{y_{n1} - y_{n2}}{|J_n|},$$

$$\frac{\partial \psi_{ni}}{\partial y} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{x_{n1} - x_{n3}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{x_{n2} - x_{n1}}{|J_n|},$$

$$\frac{\partial^2 \psi_{ni}}{\partial x^2} = \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(y_3 - y_1)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(y_1 - y_2)^2}{|J|^2},$$

$$\frac{\partial^2 \psi_{ni}}{\partial y^2} = \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)^2}{|J|^2},$$

$$\begin{aligned} \frac{\partial^2 \psi_{ni}}{\partial x \partial y} &= \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} \\ &+ \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_2 - x_1)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}. \end{aligned}$$

2D quadratic finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ with edge middle points A_{n4} , A_{n5} , and A_{n6} : Define

$$\psi_{nj}(x, y) = a_{nj}x^2 + b_{nj}y^2 + c_{nj}xy + d_{nj}y + e_{nj}x + f_{nj},$$
$$j = 1, \dots, 6,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, 6$.

2D quadratic finite element: global basis functions

“local \rightarrow global” framework:

- Define the **local finite element space**

$$S_h(E_n) = \text{span}\{\psi_{n1}, \dots, \psi_{n6}\}.$$

- At each finite element node X_j ($j = 1, \dots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, N_b$.

- Then define the **global finite element space** to be

$$U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}.$$

2D quadratic finite element: global basis functions

- Hence

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1, n), \\ \psi_{n2}, & \text{if } j = T_b(2, n), \\ \psi_{n3}, & \text{if } j = T_b(3, n), \\ \psi_{n4}, & \text{if } j = T_b(4, n), \\ \psi_{n5}, & \text{if } j = T_b(5, n), \\ \psi_{n6}, & \text{if } j = T_b(6, n), \\ 0, & \text{otherwise.} \end{cases}$$

for $j = 1, \dots, N_b$ and $n = 1, \dots, N$.

Outline

- 1 2D uniform Mesh
- 2 Triangular elements
- 3 Rectangular elements**
- 4 3D elements
- 5 More discussion

Bilinear finite element: reference basis functions

- If we consider the reference bilinear basis functions on the reference rectangular element $\hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ where $\hat{A}_1 = (0, 0)$, $\hat{A}_2 = (1, 0)$, $\hat{A}_3 = (1, 1)$, and $\hat{A}_4 = (0, 1)$, then the formation of these basis functions is very similar that of the reference 2D linear basis functions.
- Also, the affine mapping between $\hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ and $e = \square A_1 A_2 A_3 A_4$ is very similar to the one we use for the triangular mesh. The only change is to use \hat{A}_4 and A_4 to replace \hat{A}_3 and A_3 respectively. Think about why!
- Hence the formation of the local and global bilinear basis functions is also very similar to that of the local and global 2D linear basis functions.
- Derive the reference, local and global bilinear basis functions in the above way by yourself.

Bilinear finite element: reference basis functions

- In this section, we consider the reference bilinear basis functions on another reference rectangular element $\hat{E} = \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ where $\hat{A}_1 = (-1, -1)$, $\hat{A}_2 = (1, -1)$, $\hat{A}_3 = (1, 1)$, and $\hat{A}_4 = (-1, 1)$. We will also take a look at a different affine mapping.
- Define **four reference bilinear basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}) = a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{x}\hat{y}, \quad j = 1, 2, 3, 4$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$.

Bilinear finite element: reference basis functions

- Then the four reference bilinear basis functions are

$$\hat{\psi}_1(\hat{x}, \hat{y}) = \frac{1 - \hat{x} - \hat{y} + \hat{x}\hat{y}}{4},$$

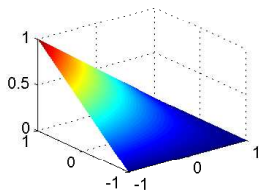
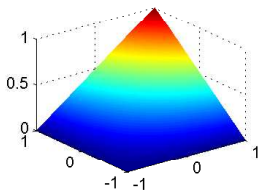
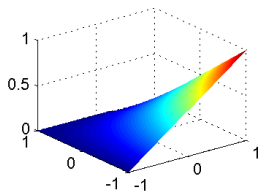
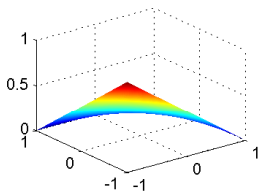
$$\hat{\psi}_2(\hat{x}, \hat{y}) = \frac{1 + \hat{x} - \hat{y} - \hat{x}\hat{y}}{4},$$

$$\hat{\psi}_3(\hat{x}, \hat{y}) = \frac{1 + \hat{x} + \hat{y} + \hat{x}\hat{y}}{4},$$

$$\hat{\psi}_4(\hat{x}, \hat{y}) = \frac{1 - \hat{x} + \hat{y} - \hat{x}\hat{y}}{4}.$$

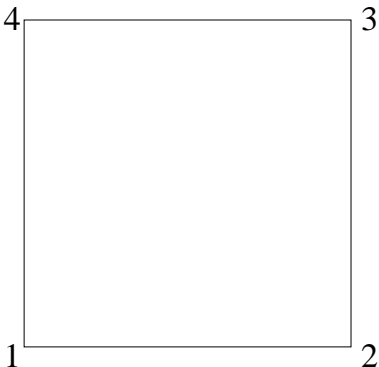
Bilinear finite element: reference basis functions

- Plots of the four bilinear basis functions on the reference triangle:



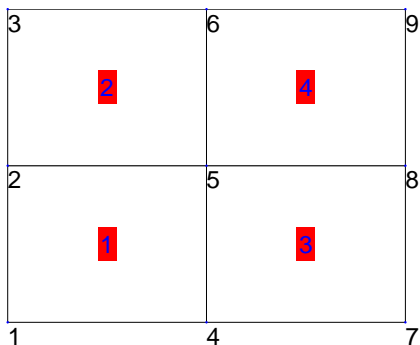
Bilinear finite element: local node index

- Define your index for the local finite element nodes in a mesh element with $N_{lb} = 4$.



Bilinear finite element: information matrices

- Define your global indices for all the mesh elements E_n ($n = 1, \dots, N$) and finite element nodes X_j ($j = 1, \dots, N_b$) (or the finite element basis functions) with $N_b = N_m = (N_1 + 1)(N_2 + 1)$.
- For example, when $N_1 = N_2 = 2$, we have



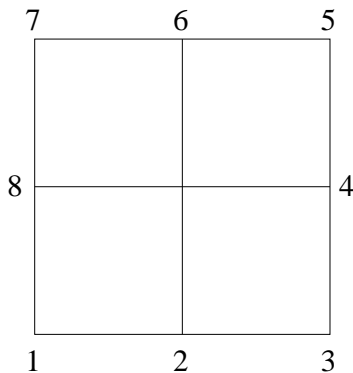
Bilinear finite element: information matrices

- For the bilinear finite elements, P_b and T_b are the same as the P and T of the rectangular mesh since the nodes of the bilinear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$
$$T_b = T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}.$$

Bilinear finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have



Bilinear finite element: boundary node information matrix

- Matrix *boundarynodes*:
- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

$$\mathit{boundarynodes} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \end{pmatrix}.$$

Bilinear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary rectangle $E = \square A_1 A_2 A_3 A_4$ and the reference rectangle $\hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ to construct the local basis functions from the reference ones.
- Assume A_1 , A_2 , A_3 , and A_4 are the left-lower, right-lower, right-upper, and left-upper vertices respectively.

- Assume

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad (i = 1, 2, 3, 4), \quad h_1 = x_2 - x_1, \quad h_2 = y_4 - y_1.$$

- Consider the affine mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}h_1 & 0 \\ 0 & \frac{1}{2}h_2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 + \frac{1}{2}h_1 \\ y_1 + \frac{1}{2}h_2 \end{pmatrix}.$$

Bilinear finite element: affine mapping

- The affine mapping actually maps

$$\hat{A}_i \rightarrow A_i, \quad i = 1, 2, 3, 4.$$

- Hence the affine mapping maps $\square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ to $\square A_1A_2A_3A_4$.
- Also,

$$\hat{x} = \frac{2x - 2x_1 - h_1}{h_1},$$
$$\hat{y} = \frac{2y - 2y_1 - h_2}{h_2}.$$

Bilinear finite element: affine mapping

- For a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$, we can define the corresponding function for $(x, y) \in \square A_1A_2A_3A_4$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{2x - 2x_1 - h_1}{h_1},$$
$$\hat{y} = \frac{2y - 2y_1 - h_2}{h_2}.$$

Bilinear finite element: affine mapping

- Then by chain rule, we get

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \\ &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{2}{h_1}, \end{aligned}$$

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\ &= \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{2}{h_2}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x \partial y} &= \frac{2}{h_1} \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} + \frac{2}{h_1} \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\ &= \frac{4}{h_1 h_2} \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}}. \end{aligned}$$

Bilinear finite element: local basis functions

- Consider the n^{th} element $E_n = \square A_{n1}A_{n2}A_{n3}A_{n4}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}.$$

Recall that the mesh size $h = (h_1, h_2)$.

- The four local bilinear basis functions are

$$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, 2, 3, 4$$

where

$$\hat{x} = \frac{2x - 2x_{n1} - h_1}{h_1},$$
$$\hat{y} = \frac{2y - 2y_{n1} - h_2}{h_2}.$$

Bilinear finite element: local basis functions

- And for $i = 1, 2, 3, 4$,

$$\frac{\partial \psi_{ni}}{\partial x} = \frac{2}{h_1} \frac{\partial \hat{\psi}_i}{\partial \hat{x}},$$

$$\frac{\partial \psi_{ni}}{\partial y} = \frac{2}{h_2} \frac{\partial \hat{\psi}_i}{\partial \hat{y}},$$

$$\frac{\partial^2 \psi_{ni}}{\partial x \partial y} = \frac{4}{h_1 h_2} \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}}.$$

- The reference and local functions defined in this section are what you will need to input into the code!

Bilinear finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \square A_{n1}A_{n2}A_{n3}A_{n4}$ as follows:

$$\psi_{nj}(x, y) = a_{nj} + b_{nj}x + c_{nj}y + d_{nj}xy, \quad j = 1, 2, 3, 4,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$.

Bilinear finite element: global basis functions

“local \rightarrow global” framework:

- Define the **local finite element space**

$$S_h(E_n) = \text{span}\{\psi_{n1}, \psi_{n2}, \psi_{n3}, \psi_{n4}\}.$$

- At each finite element node X_j ($j = 1, \dots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, N_b$.

- Then define the **global finite element space** to be

$$U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}.$$

Bilinear finite element: global basis functions

- Hence

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1, n), \\ \psi_{n2}, & \text{if } j = T_b(2, n), \\ \psi_{n3}, & \text{if } j = T_b(3, n), \\ \psi_{n4}, & \text{if } j = T_b(4, n), \\ 0, & \text{otherwise.} \end{cases}$$

for $j = 1, \dots, N_b$ and $n = 1, \dots, N$.

Biquadratic finite element: reference basis functions

- We consider the reference biquadratic basis functions on the reference rectangular element $\hat{E} = \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ where $\hat{A}_1 = (-1, -1)$, $\hat{A}_2 = (1, -1)$, $\hat{A}_3 = (1, 1)$, and $\hat{A}_4 = (-1, 1)$. Define $\hat{A}_5 = (0, -1)$, $\hat{A}_6 = (1, 0)$, $\hat{A}_7 = (0, 1)$, $\hat{A}_8 = (-1, 0)$, and $\hat{A}_9 = (0, 0)$.
- Define **nine reference biquadratic basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}) = a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{x}\hat{y} + e_j\hat{x}^2 + f_j\hat{y}^2 + g_j\hat{x}^2\hat{y} + h_j\hat{x}\hat{y}^2 + k_j\hat{x}^2\hat{y}^2, \quad j = 1, \dots, 9$$

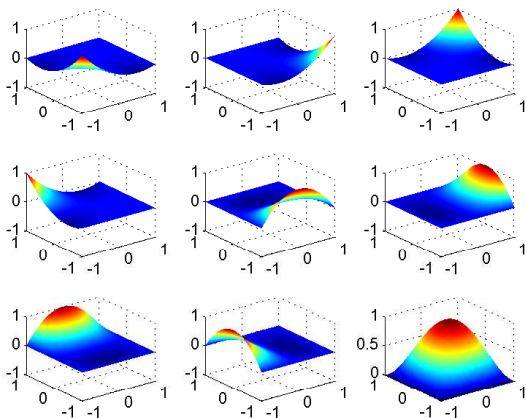
such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, 9$.

Biquadratic finite element: reference basis functions

- Plots of the nine biquadratic basis functions on the reference triangle:



Outline

- 1 2D uniform Mesh
- 2 Triangular elements
- 3 Rectangular elements
- 4 3D elements**
- 5 More discussion

3D linear finite element: reference basis functions

- We consider the reference 3D linear basis functions on the reference tetrahedron element $E = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ where $\hat{A}_1 = (0, 0, 0)$, $\hat{A}_2 = (1, 0, 0)$, $\hat{A}_3 = (0, 1, 0)$, and $\hat{A}_4 = (0, 0, 1)$.
- Define **four reference 3D linear basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}, \hat{z}) = a_j \hat{x} + b_j \hat{y} + c_j \hat{z} + d_j, \quad j = 1, 2, 3, 4$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$.

3D linear finite element: reference basis functions

- Then it's easy to obtain

$$\hat{\psi}_1(\hat{A}_1) = 1 \Rightarrow d_1 = 1,$$

$$\hat{\psi}_1(\hat{A}_2) = 0 \Rightarrow a_1 + d_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_3) = 0 \Rightarrow b_1 + d_1 = 0,$$

$$\hat{\psi}_1(\hat{A}_4) = 0 \Rightarrow c_1 + d_1 = 0,$$

$$\hat{\psi}_2(\hat{A}_1) = 0 \Rightarrow d_2 = 0,$$

$$\hat{\psi}_2(\hat{A}_2) = 1 \Rightarrow a_2 + d_2 = 1,$$

$$\hat{\psi}_2(\hat{A}_3) = 0 \Rightarrow b_2 + d_2 = 0,$$

$$\hat{\psi}_2(\hat{A}_4) = 0 \Rightarrow c_2 + d_2 = 0,$$

3D linear finite element: reference basis functions

- and

$$\hat{\psi}_3(\hat{A}_1) = 0 \Rightarrow d_3 = 0,$$

$$\hat{\psi}_3(\hat{A}_2) = 0 \Rightarrow a_3 + d_3 = 0,$$

$$\hat{\psi}_3(\hat{A}_3) = 0 \Rightarrow b_3 + d_3 = 1,$$

$$\hat{\psi}_3(\hat{A}_4) = 1 \Rightarrow c_3 + d_3 = 0,$$

$$\hat{\psi}_4(\hat{A}_1) = 0 \Rightarrow d_4 = 0,$$

$$\hat{\psi}_4(\hat{A}_2) = 0 \Rightarrow a_4 + d_4 = 0,$$

$$\hat{\psi}_4(\hat{A}_3) = 0 \Rightarrow b_4 + d_4 = 0,$$

$$\hat{\psi}_4(\hat{A}_4) = 1 \Rightarrow c_4 + d_4 = 1.$$

3D linear finite element: reference basis functions

- Hence

$$a_1 = -1, b_1 = -1, c_1 = -1, d_1 = 1,$$

$$a_2 = 1, b_2 = 0, c_2 = 0, d_2 = 0,$$

$$a_3 = 0, b_3 = 1, c_3 = 0, d_3 = 0,$$

$$a_4 = 0, b_4 = 0, c_4 = 1, d_4 = 0.$$

- Then the four reference 3D linear basis functions are

$$\hat{\psi}_1(\hat{x}, \hat{y}, \hat{z}) = -\hat{x} - \hat{y} - \hat{z} + 1,$$

$$\hat{\psi}_2(\hat{x}, \hat{y}, \hat{z}) = \hat{x},$$

$$\hat{\psi}_3(\hat{x}, \hat{y}, \hat{z}) = \hat{y},$$

$$\hat{\psi}_4(\hat{x}, \hat{y}, \hat{z}) = \hat{z}.$$

Trilinear finite element: reference basis functions

- We consider the reference trilinear basis functions on the reference cube element $E = \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \hat{A}_5 \hat{A}_6 \hat{A}_7 \hat{A}_8$ where $\hat{A}_1 = (0, 0, 0)$, $\hat{A}_2 = (1, 0, 0)$, $\hat{A}_3 = (1, 1, 0)$, $\hat{A}_4 = (0, 1, 0)$, $\hat{A}_5 = (0, 0, 1)$, $\hat{A}_6 = (1, 0, 1)$, $\hat{A}_7 = (1, 1, 1)$, and $\hat{A}_8 = (0, 1, 1)$.
- Define **eight reference 3D trilinear basis functions**

$$\hat{\psi}_j(\hat{x}, \hat{y}, \hat{z}) = a_j + b_j \hat{x} + c_j \hat{y} + d_j \hat{z} + e_j \hat{x} \hat{y} + f_j \hat{x} \hat{z} + g_j \hat{y} \hat{z} + h_j \hat{x} \hat{y} \hat{z}, \quad j = 1, \dots, 8$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \dots, 8$.

Outline

- 1 2D uniform Mesh
- 2 Triangular elements
- 3 Rectangular elements
- 4 3D elements
- 5 More discussion**

More topics for finite elements

- Higher degree finite elements.....
- Mixed finite elements: Raviart-Thomas elements, Taylor-Hood elements, Mini elements.....
- Hermitian types of finite elements
- Nonconforming finite elements
- Another way to construct the basis functions: use the product of 1D basis functions to form the corresponding basis functions on rectangle or cube elements.

Approximation capability of the finite element spaces

- Question: Given a function u and a finite element space $U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}$ with finite element nodes X_j ($j = 1, \dots, N_b$), how small is $\inf_{w \in U_h} \|u - w\|$?

- Finite element interpolation

$$u_I = \sum_{j=1}^{N_b} u(X_j) \phi_j.$$

- Since $u_I \in U_h$, then

$$\inf_{w \in U_h} \|u - w\| \leq \|u - u_I\|.$$

- The finite element interpolation error $\|u - u_I\|$ is a traditional tool to evaluate the approximation capability of a finite element space. Here the norm $\|\cdot\|$ needs to be chosen properly according to the interpolated basis function u . For example, if $u \in H^1(\Omega)$, then $\|\cdot\|$ can be chosen as the L^2 norm $\|\cdot\|_0$ or H^1 norm $\|\cdot\|_1$.