Laminar heat transfer for small Prandtl number fluids including step change in wall temperature

Shiny Ting

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LAMINAR HEAT TRANSFER FOR SMALL PRANDTL NUMBER FLUIDS
INCLUDING STEP CHANGE IN WALL TEMPERATURE

BY

SHINY TING, 1946-

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ABSTRACT

The object of this investigation is to provide a simple means of determining the heat transfer from a flat plate which has an unheated leading length \( x_0 \). The case of steady, two-dimensional, laminar flow of an incompressible fluid with a Prandtl number of less than one is considered. Since there exists an unheated length and the Prandtl number of the fluid is less than one, the thermal boundary layer intersects the flow boundary layer. Thus, the solution of the problem divides into two regions.

In this study, the problem is solved analytically by employing the energy integral equation. The temperature profile and velocity profile of the fluid are expressed using both linear and cubic polynomials. When the thermal boundary layer is outside the hydrodynamic boundary layer, the integration is performed in two operations.

The result obtained from the energy integral equation by substituting both the temperature and velocity profiles is a first order, non-linear differential equation. The solution of this differential equation for the cubic profile case is rather complicated and tedious. However, curves are provided which give the appropriate parameters of the final results. The local and average convective coefficients can then be deduced based on the solution of this differential equation. The results from this investigation are compared with those of other investigators for several special cases. An excellent agreement is obtained between the two sets of results. It is found that significant errors may occur if previous solutions for special cases are employed for the general situation.
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NOMENCLATURE

\(a\) - constant defined by equation (8), \(2C_1/PrC_2\)

\(A, B\) - constants in equation (25)

\(C_1\) - constant in equation (3)

\(C_2\) - constant in equation (6)

\(C_3, C_4, C_5, C_6\) - integral constants in equations (16), (18), (21), and (24) respectively

\(C, D, E, F, G\) - constants in equation (26)

\(f(C)\) - function of \(C\), equation (4)

\(f'(C)\) - first derivative of \(f\) with respect to \(C\)

\(h_x\) - local convection coefficient, equation (14)

\(\overline{h}\) - average convection coefficient, equation (14a)

\(\overline{h}_2\) - average convection coefficient in region 2, equation (30)

\(\overline{h}_3\) - average convection coefficient in region 3, equation (35)

\(\overline{h}|_{x_1}\) - average convection coefficient for \(L = x_1\)

\(k\) - thermal conductivity

\(L\) - length of the plate

\(\text{Nu}_x\) - local Nusselt number, \(h_xx/k\)

\(\overline{\text{Nu}}\) - average Nusselt number, \(\overline{h}(L-x_0)/k\)

\(\overline{\text{Nu}}_2\) - average Nusselt number in region 2, \(\overline{h}_2(L-x_0)/k\)

\(\overline{\text{Nu}}_3\) - average Nusselt number in region 3, \(\overline{h}_3(L-x_0)/k\)

\(\overline{\text{Nu}}|_{x_1}\) - average Nusselt number for \(L = x_1\)

\(\text{Pr}\) - Prandtl number, \(\nu/\alpha\)

\(\text{Re}_L\) - average Reynolds number, \(\omega L/\nu\)
\( \text{Re}_x \)  
- local Reynolds number, \( u_\infty x/v \)

\( T(x,y) \)  
- temperature distribution within the thermal boundary layer

\( T_\infty \)  
- constant free stream temperature

\( u(x,y) \)  
- velocity distribution within the hydrodynamic boundary layer

\( u_\infty \)  
- constant free stream velocity

\( x \)  
- axial coordinate of the plate

\( x_0 \)  
- unheated leading length of the plate

\( x_1 \)  
- location where the thermal boundary layer intersects the hydrodynamic boundary layer

\( y \)  
- transverse coordinate of the plate

\( \alpha \)  
- thermal diffusivity

\( \zeta \)  
- ratio of thermal boundary layer thickness to hydrodynamic boundary layer thickness

\( \zeta_2, \zeta_3 \)  
- two real roots of equation (25)

\( \delta \)  
- hydrodynamic boundary layer thickness

\( \delta_t \)  
- thermal boundary layer thickness

\( \theta(x,y) \)  
- temperature difference, \( T(x,y) - T_\infty \)

\( \theta_\infty \)  
- temperature difference, \( T_\infty - T_w \)

\( \nu \)  
- kinematic viscosity

\( \phi(\zeta) \)  
- function of \( \zeta \) defined in equation (9)
I. INTRODUCTION

The need for accurate values of convective heat transfer coefficients finds many industrial applications, such as in the problem of cooling gas turbine blades, heat transfer of liquid metals in nuclear power plants, etc. The exact mathematical solution of the boundary layer equations and the approximate analysis of the boundary layer by integral method are two techniques which are often employed to estimate the convective heat transfer coefficient. Although this investigation is primarily concerned with the latter method, some background related to the exact solution of the boundary layer equations is appropriate.

Exact analysis of the boundary layer requires the simultaneous solution of the partial differential equations describing the fluid motion and the transfer of energy in a moving fluid. In some cases, it is difficult to solve the partial differential equation mathematically. Furthermore, many exact solutions require considerable numerical effort to obtain the answer after the formal analysis is completed. For example, consider the convective heat transfer problem for a flat plate with uniform wall temperature in laminar flow and with zero angle of attack. This classic problem was solved by E. Pohlhausen (1) using a similarity transformation which yields the exact solution. The solution of this basic problem is difficult to evaluate in both the range of high and low Prandtl numbers. Fisher and Kundsen (2) overcame this difficulty by evaluating the integral associated with Pohlhausen's solution by dividing the integration into two parts.
They worked out the second part of their computation by means of graphical integration.

Scesa and Levy (3) used the boundary layer equations to approach the problem of wedge flow over a plane wall with stepwise wall temperature. They assumed exponential functions for the temperature and velocity distributions. The solution of the boundary layer equations for the case of a flat plate with an unheated leading length was obtained by simplifying their results for the wedge flow case. The accuracy of their solution for the flat plate with an unheated leading length diminishes considerably for Prandtl numbers less than 0.7.

Morgen, Pipkin, and Warner (4) for the case of very low Prandtl number utilized the free stream velocity as the velocity distribution. Their solution of the boundary layer equations included the viscous effect. The flat plate with uniform wall temperature is a special case of their solution. The basis of their analysis is the assumption that the hydrodynamic boundary layer is much thinner than the thermal boundary layer. Naturally, the precision of their solution increases as the Prandtl number decreases.

As previously mentioned, an alternative to the exact solution of the boundary layer equations is the approximate analysis of the boundary layer by the integral method. This method of approximating the velocity and temperature distributions yields a solution which is usually quicker and simpler than the exact solution of the boundary layer equations. The technique in general is not only applicable to laminar flow but also to turbulent flow.

This approximate method was first devised by K. Pohlhausen (1) in 1921. He obtained the hydrodynamic boundary layer thickness by
substituting a fourth order polynomial for the velocity profile into the momentum integral equation. Squire (1) employed Pohlhausen's approximate method by also postulating a fourth order polynomial for the temperature distribution inside the thermal boundary layer. For convenience of computation, he introduced the parameter \( \zeta \) which is defined as the ratio of the thermal boundary layer thickness \( \delta_t \) to the hydrodynamic boundary layer thickness \( \delta \) and which can be found from the energy integral equation. As soon as \( \zeta \) is found, the temperature is known. This method is not only suitable for flat plate flow but also for wedge flow at arbitrary values of Prandtl number. Squire's solution is restricted to the case of constant wall temperature.

The case of a flat plate with an unheated leading length for Prandtl number greater than unity in laminar flow was investigated by Eckert (5). He assumed a cubic parabola for the velocity and temperature profiles. The convective heat transfer coefficient which results from the energy integral equation is in a very simple form. The case of uniform wall temperature for flat plate flow is a special case of his solution. This approach has an error of about 8 per cent as compared with the exact solution. Beyond doubt, the integral method gives results which are within the accuracy necessary for most engineering applications.

Other researchers have concentrated their efforts on the case of very low Prandtl numbers \( (\text{Pr} \ll 1) \), as in the range of liquid metals \( (0.002 < \text{Pr} < 0.03) \). The uniform wall temperature case with the Prandtl number of the fluid far below unity was investigated by Sparrow (6). He made use of the Karman-Pohlhausen type of integral equation plus cubic polynomial profiles. Then, he took \( \delta_t/\delta \) as a constant, i.e.,
\( \delta_t/\delta \) independent of location, and divided the integration of the energy integral equation into two parts.

Rubesin (7) considered a flat plate in which the thermal boundary layer started in the rear of the hydrodynamic boundary layer. He hypothesized a linear velocity and temperature profile and gave the solutions for both laminar and turbulent flow using the energy integral equation. He discussed laminar flow for both the cases of Prandtl numbers greater than 1 and less than 1. In the case of Prandtl numbers less than 1, he set the minimum Prandtl number no less than 0.7, which is in the range of gases. He analyzed the influence due to that part of the thermal boundary layer which is outside the hydrodynamic boundary layer. Since the ratio \( \delta_t/\delta \) is less than 1.14 (for \( Pr = 0.7 \), \( \delta_t/\delta = 1.14 \)), he did not separate the integration of the energy integral equation into two portions. That is, he did not perform the first portion of the integration from 0 to \( \delta \) using the linear velocity profile and the second portion from \( \delta \) to \( \delta_t \) using the free stream velocity. Consequently, his solution can not apply to fluids which have Prandtl numbers lower than 0.7.

In summary, the previous investigations concerning laminar flow over a flat plate with an unheated leading length, such as the solutions of Scesa and Levy (3), Eckert (5) and Rubesin (7), are not applicable if the Prandtl number is less than 0.7. The analyses of Pohlhausen (1), Morgan, Pipkin and Warner (4), Squire (1) and Sparrow (6) are applicable for very low Prandtl numbers, but do not include an unheated leading length.

The present investigation will consider the case of laminar flow with an unheated leading length for Prandtl numbers less than unity,
without the restriction of very low values of Pr. It appears that the integral method has not been previously applied to this situation. The case of zero angle of incident will be considered. The energy integral equation is adopted instead of the energy boundary layer equation in order to avoid solving high order nonlinear differential equations. It is anticipated that this method will reveal a simple solution which can be evaluated readily for engineering application. Both linear and cubic polynomials are selected for the velocity and temperature profiles. The properties of the fluid are postulated as constants.
II. FORMULATION OF THE PROBLEM

A flat plate with zero angle of incidence in steady, two-dimensional laminar flow is shown in Figure 1.

A constant property Newtonian fluid with Prandtl number less than 1 approaches the plate with a free stream velocity $u_\infty$ and a free stream temperature $T_\infty$. The length $x_0$ from the leading edge of the plate remains unheated. Starting from $x_0$, the plate is maintained at a constant wall temperature $T_w$ which is different from $T_\infty$. The hydrodynamic boundary layer thickness and thermal boundary layer thickness are represented by $\delta$ and $\delta_t$ respectively. Because the Prandtl number of the fluid is less than 1, the thermal boundary layer thickness grows faster than the hydrodynamic boundary layer thickness. At a distance $x_1$ from the leading edge of the plate, the thermal boundary
layer meets the hydrodynamic boundary layer. That is, the thickness of the two boundary layers is equal at $x_1$. The symbols $T$ and $u$ stand for the temperature and axial velocity distributions inside the thermal and hydrodynamic boundary layers, respectively. The ratio of the thermal boundary layer thickness to the hydrodynamic boundary layer thickness $\delta_t/\delta$ is not a constant but is a function of $x$. To simplify the mathematical expressions which follow, it is convenient to introduce a new parameter $\zeta = \delta_t/\delta$. Since the thermal boundary layer starts at the rear of the hydrodynamic boundary layer and the rate of growth of thermal boundary layer is faster than that of the hydrodynamic boundary layer, three regions of interest can be differentiated:

Region 1, $T(x,0) = T_\infty$, $0 < x < x_0$. Since the temperature of the surface of the plate equals the free stream temperature, there is no heat transfer and the thermal boundary layer has zero thickness.

Region 2, $T(x,0) = T_w$, $x_0 < x < x_1$. In this region the temperature of the surface of the plate is equal to $T_w$ and $\delta_t/\delta$ is less than unity.

Region 3, $T(x,0) = T_w$, $x \geq x_1$. This region is similar to region 2 except $\delta_t/\delta$ is greater than unity.

In region 2, the integration of the energy integral equation is carried out directly from 0 to $\delta_t$. In region 3 the integration must be divided into two parts. The first integration employs the varied flow velocity and is executed from 0 to $\delta$; the second integration employs the constant free stream velocity and is performed over the range from $\delta$ to $\delta_t$. 

It is observed that in Figure 1 there is a discontinuity in the thermal boundary condition at $x_0$; that is, the temperature of the plate jumps from $T_\infty$ to $T_w$. Any problems associated with this discontinuity may be avoided by employing the boundary condition

$$\delta_t = 0 \quad \text{or} \quad \zeta = 0, \text{ at } x = x_0.$$ 

This boundary condition will be used in the solution for $\zeta$ in region 2.

The momentum and energy integral equations for the case of interest are (5):

$$\frac{d}{dx} \int_0^\delta u(u_\infty - u) dy = \nu \frac{\partial u}{\partial y} \bigg|_{y=0},$$

and

$$\frac{d}{dx} \int_0^{\delta_t} u(T_\infty - T) dy = \alpha \frac{\partial T}{\partial y} \bigg|_{y=0},$$

where $\nu$ is the kinematic viscosity and $\alpha$ is the thermal diffusivity.

The limitations to the use of equation (2) are that the energy dissipation due to friction and the effects of compressibility of the flow are both negligible.

To render the boundary conditions homogeneous and to simplify the temperature profiles, it is helpful to define $\theta$ and $\theta_\infty$ as,

$$\theta = T - T_w$$

and

$$\theta_\infty = T_\infty - T_w.$$
After replacing $T$ by $\theta$, equation (2) can be written as

$$\frac{d}{dx} \int_0^{\delta_t} (u/u_\infty)(1-\theta/\theta_\infty)dy = \left(\frac{\alpha}{u_\infty \theta_\infty}\right) \frac{\partial \theta}{\partial y}\bigg|_{y=0} \cdot (2a)$$

The velocity and temperature profiles are derived by assuming a polynomial and solving for the coefficients of the polynomial using the appropriate boundary conditions. If linear profiles are postulated, the result is

$$u/u_\infty = y/\delta,$$

and

$$\theta/\theta_\infty = y/\delta_t.$$ 

For the cubic profile postulate, the profiles are

$$u/u_\infty = (3/2)(y/\delta) - (1/2)(y/\delta)^3,$$

and

$$\theta/\theta_\infty = (3/2)(y/\delta_t) - (1/2)(y/\delta_t)^3.$$

The details of the development of these profiles are provided in reference (5). The next section treats the solution of the problem.
III. METHOD OF SOLUTION

In this section, the method of solution of the problem by means of the energy integral equation will be presented. The procedures of the mathematical operations are similar for both the linear and cubic profiles. For this reason, a general method of solution will be described first. This general method will be specialized for both the linear and cubic profile cases to obtain the solution.

A. General Analysis of the Solution

The hydrodynamic boundary layer thickness $\delta$ can be obtained as a function of $x$ by employing the momentum integral equation. The result for $\delta$ can be presented in the form

$$\delta^2 = C_1 (\nu x / u_\infty),$$

where $C_1$ is a constant which depends upon the velocity profile employed. The values of $C_1$ for the linear profile case and the cubic profile case are given, respectively, in Tables I and II at the end of this analysis.

The energy integral formulation can be considered after the determination of $\delta$. The integration of the left-hand side of equation (2a) yields a function of $\delta$ and $\delta_t$. Since $\delta$ is known, $\delta_t$ can be replaced by a product of $\zeta$ and $\delta$ to make mathematical operations easier to handle. The integration of the left-hand side of equation (2a) can be expressed in the form

$$\delta f(\zeta) = \int_0^{\delta_t} (u/u_\infty)(1-\theta/\theta_\infty)dy,$$

where $f(\zeta)$ is a function of $\zeta$ which is dependent upon the region of...
interest (Region 2 or Region 3) as well as the type of profile which
is assumed. The right-hand side of equation (2a) may be expressed as

\[ \frac{a}{u_\infty} \frac{\partial}{\partial y} \left( \frac{\theta}{u_\infty} \right) \bigg|_{y=0} = \frac{a}{u_\infty} \frac{c_2}{C_\delta} . \]  

The constant \( C_2 \) changes its values when different profiles are
employed. Both \( C_2 \) and \( f \) are listed in Tables I and II.

By employing equations (4) and (5), equation (2a) reduces to

\[ \zeta \delta \frac{d}{dx} (\delta f) = C_2 a/u_\infty , \]

which, on substitution for \( \delta \), becomes

\[ \zeta (C_1 v x / u_\infty)^{1/2} \frac{d}{dx} [(C_1 v x / u_\infty)^{1/2} f] = C_2 a / u_\infty . \]  

After rearranging and replacing \( v/\alpha \) by the Prandtl number, \( Pr \),
equation (6) assumes the form

\[ \frac{\zeta df}{\zeta f = (2C_1/(PrC_1))} + \frac{dx}{2x} = 0 . \]  

In order to simplify equation (7), it is convenient to define

\[ a = (2C_2)/(PrC_1) . \]

and

\[ f' = \frac{df}{d\zeta} , \]

where \( a \) is a function of the Prandtl number and the type of profile.
The notation \( f' \) is the derivative of \( f \) with respect to \( \zeta \). Substitu-
tion of \( a \) and \( f' \) into equation (7) yields

\[ \frac{\zeta f'd\zeta}{\zeta f = a} = - \frac{dx}{2x} . \]  

Equation (8) is a first order non-linear ordinary differential equa-
tion. The result of integrating the right-hand side of equation (8)
is a simple logarithm function of \( x \). This leads us to assume that the
left-hand side of equation (8) is also a simple logarithm function. The function \( \Phi(\zeta) \) is defined by the equation
\[
d[\ln \Phi(\zeta)] = \frac{\zeta f\,d\zeta}{\zeta^2-a}.
\] (9)
Introducing equation (9) into equation (8) yields
\[
d[\ln \Phi(\zeta)] = - \frac{1}{2} \, d(\ln x).
\] (10)
This first order non-linear differential equation which results from the successive simplifications outlined above, can be easily solved.
In different regions, there exist different functions \( \Phi(\zeta) \) and different integral limits which are applicable to equation (10). Therefore, it is necessary to discuss the integration of equation (10) for region 2 and for region 3 separately.

In region 2, the integral limits of the right-hand side of equation (10) are from \( x_0 \) to \( x \) where \( x \) is greater than \( x_0 \) and less than or equal to \( x_1 \). Correspondingly, the integral limits of the left-hand side of equation (10) are from 0 to \( \zeta \) since \( \zeta \) equals zero when \( x = x_0 \). Thus,
\[
\int_0^\zeta d(\ln \Phi) = - \frac{1}{2} \int_{x_0}^{x} d(\ln x),
\]
or
\[
\Phi(\zeta)/\Phi(0) = (x_0/x)^{1/2}, \quad x_0 < x < x_1.
\] (11)
This is the relationship between \( \zeta \) and \( x \) in region 2, i.e., the solution of equation (8) in region 2. The value of \( \zeta \) in equation (11) corresponding to a location \( x \) is always less than or equal to unity. This is because the thermal boundary layer in this region is thinner than the hydrodynamic boundary layer. When \( \zeta \) equals unity, the
corresponding location \( x = x_1 \) is the junction of the two boundary layers. Letting \( \zeta \) in equation (11) equal 1 gives

\[
x_1 = x_0 \left( \frac{\phi(0)}{\phi(1)} \right)^2.
\]

The expressions for \( x_1/x_0 \) for both the linear and cubic profile cases are given in the Appendix.

The boundary condition of the solution of equation (8) in region 3 is \( \zeta = 1 \) at \( x = x_1 \). The value of \( x_1 \) can be obtained from equation (12). This is employed as the lower limit of the integration of equation (10). Thus,

\[
\int_{1}^{\zeta} d(\ln \phi) = - (1/2) \int_{x_1}^{x} d(\ln x),
\]

which results in

\[
x_1/x = \left( \frac{\phi(\zeta)}{\phi(1)} \right)^2, \quad x > x_1.
\]

The value of \( x \) in equation (13) is greater than or equal to \( x_1 \) and \( \zeta \) is always greater than or equal to 1 because the thermal boundary layer overtakes the hydrodynamic boundary layer at \( x_1 \). Equation (13) is the solution of equation (8) in region 3. After the determination of equation (11) or (13), the value of \( \zeta \) can be expressed analytically in terms of \( x \) and the Prandtl number. As soon as \( \zeta \) is known, the temperature distribution inside the thermal boundary layer can be found. The local convective heat transfer coefficient \( h_x \) can be calculated by using the equation

\[
h_x = \left( \frac{k}{\theta_{\omega}} \right) \left( \frac{\partial \theta}{\partial y} \right) \bigg|_{y=0}.
\]

Following the determination of the local convection coefficient, the
average convective heat transfer coefficient, $\bar{h}$, can be defined as

$$\bar{h} = \frac{1}{(L-x_0)} \int_{x_0}^{L} h_x \, dx,$$

(14a)

where the symbol $L$ is the total length of the plate.

Tables I and II list the constants and the functions which appear in the previous equations for the regions 2 and 3.

**Table I**

<table>
<thead>
<tr>
<th>Constants and Functions Applicable to Region 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>linear profile</strong></td>
</tr>
<tr>
<td>$C_1$</td>
</tr>
<tr>
<td>$C_2$</td>
</tr>
<tr>
<td>$f(\zeta)$</td>
</tr>
<tr>
<td>$a$</td>
</tr>
</tbody>
</table>

**Table II**

<table>
<thead>
<tr>
<th>Constants and Functions Applicable to Region 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>linear profile</strong></td>
</tr>
<tr>
<td>$C_1$</td>
</tr>
<tr>
<td>$C_2$</td>
</tr>
<tr>
<td>$f(\zeta)$</td>
</tr>
<tr>
<td>$a$</td>
</tr>
</tbody>
</table>
B. The Derivation of $\phi(\zeta)$

In this part of section III, the functions $\phi(\zeta)$ for region 2 and region 3 will be derived for both the linear and cubic profiles. The functions $\phi(\zeta)$ for the linear profile case can be derived exactly with no approximations or simplifications. The cubic profile case presents more mathematical difficulties.

1. Linear Profile

   a. Region 2. The integration of equation (9) is

   \[ \ln \phi(\zeta) = \int \frac{\zeta' d\zeta}{\zeta-a} + \text{Constant} \]  

   (15)

   The terms $a$ and $f(\zeta)$ can be obtained from Table I and $f'(\zeta)$ is the first derivative of $f(\zeta)$ with respect to $\zeta$. After introducing $a$, $f$ and $f'$ into equation (15), the expression becomes

   \[ \ln \phi(\zeta) = \int \frac{2\zeta^2 d\zeta}{\zeta^3 - 1/Pr} + \ln C_3, \]

   (16)

   where $\ln C_3$ is the constant of integration. This equation can be integrated directly; the result for the right-hand side of equation (16) is $\ln C_3 (1/Pr - \zeta^3)^{2/3}$. Thus,

   \[ \phi(\zeta) = C_3 (1/Pr - \zeta^3)^{2/3}. \]  

   (17)

   Note that $C_3$ will be cancelled when equation (17) is inserted into equation (11).

   b. Region 3. In a similar manner, one can utilize equation (9) and insert $a$, $f$ and $f'$ from Table II. The result is,

   \[ \ln \phi(\zeta) = \int \frac{(3\zeta-1/\zeta) d\zeta}{3\zeta^2 - 3\zeta - (1-1/Pr)} + \ln C_4. \]  

   (18)
The integrand of the right-hand side of equation (18) can be reduced by the method of partial fractions to

\[
\frac{3\zeta-1/\zeta}{3\zeta^2-3\zeta+3P} = \frac{1}{3P\zeta} + \frac{1/P+3}{3\zeta^2-3\zeta+3P} - \frac{1/P}{3\zeta^2-3\zeta+3P},
\]

where

\[ P = 1/Pr - 1. \]  \hspace{1cm} (19a)

Substituting equation (19) into equation (18) and carrying out the integration, one can find an expression for \( \Phi(\zeta) \) with Prandtl number, \( Pr \), as a parameter. Thus,

\[
\Phi(\zeta) = C_4^{Pr/(1-Pr)(3\zeta^2-3\zeta-(Pr-1)/Pr)}(2Pr-1)/(2Pr-2) \\
\times \left[ \frac{2\zeta-1-\sqrt{(4-Pr)/(3Pr)}}{2\zeta-1+\sqrt{(4-Pr)/(3Pr)}} \right]^{\sqrt{3P}/(4-Pr)/(2-2Pr)}. \]  \hspace{1cm} (20)

2. Cubic Profile

a. Region 2. Applying equation (9) and using Table I gives

\[
\ln\Phi(\zeta) = \int \frac{(4\zeta^4-28\zeta^2)\,d\zeta}{\zeta^6-14\zeta^3+13/Pr} + \ln C_5. \]  \hspace{1cm} (21)

Thus,

\[
\ln\Phi(\zeta) = (2/3) \int \frac{(6\zeta^4-42\zeta^2)\,d\zeta}{\zeta^6-14\zeta^3+13/Pr} + \ln C_5. \]  \hspace{1cm} (22)

Since \( \zeta \leq 1 \) in region 2, the numerator can be approximated by \( 6\zeta^4-42\zeta^2 = 5\zeta^4-42\zeta^2 \) with little loss of accuracy. The maximum error in this approximation occurs at \( \zeta = 1 \) and is about 2.8 per cent. When \( \zeta \) is less than 1, the per cent error is less than this amount.

Employing this approximation, equation (22) can be integrated directly to give

\[
\Phi(\zeta) = C_5(\zeta^5-14\zeta^3+13/Pr)^{2/3}. \]  \hspace{1cm} (23)
b. Region 3. Employing equation (9) and following Table II gives

\[
\ln\phi(\zeta) = -(1/2) \int \frac{(2\zeta^4 - 4\zeta^2/5 + 6/35) d\zeta}{-\zeta^5 + \zeta^4 + [13/(35\Pr) - 2/5] \zeta^3 + \zeta/35} + \ln C_6 , \quad (24)
\]

If \( \Pr < 13/14 \), the value of \( 13/(35\Pr) - 2/5 \) is always positive. According to Descartes' rule (8), the polynomial

\[-\zeta^4 + \zeta^3 + [13/(35\Pr) - 2/5] \zeta^2 + 1/35\]

has one positive and one negative root. This also can be demonstrated by numerical methods. The two roots are given in Figure 2. The denominator of the right-hand side of equation (24) can then be factored as

\[
\zeta(-\zeta^4 + \zeta^3 + [13/(35\Pr) - 2/5] \zeta^2 + 1/35) = \zeta(-\zeta^2 + A\zeta + B)(\zeta - \zeta_2)(\zeta - \zeta_3) , \quad (25)
\]

where \( \zeta_2 \) and \( \zeta_3 \) are the two real roots of the polynomial

\[-\zeta^4 + \zeta^3 + [13/(35\Pr) - 2/5] \zeta^2 + 1/35 . \]

The two constants \( A \) and \( B \) can be solved from equation (24) by comparing corresponding terms on both sides of the equation. This results in

\[
A = 1 - (\zeta_2 + \zeta_3) ,
\]

and

\[
B = 1/(35\zeta_3 \zeta_2) .
\]

The integrand of equation (24) can now be reduced by the method of partial fractions using equation (25). This operation yields

\[
\frac{2\zeta^4 - 4\zeta^2/5 + 6/35}{-\zeta^5 + \zeta^4 + [13/(35\Pr) - 2/5] \zeta^3 + \zeta/35} = \frac{C}{\zeta} + \frac{D\zeta + E}{-\zeta^2 + A\zeta + B} + \frac{F}{\zeta - \zeta_2} + \frac{G}{\zeta - \zeta_3} . \quad (26)
\]
Figure 2: Variation of $\zeta_2, \zeta_3$ with $Pr$
A set of five equations in terms of C, D, E, F, and G can be obtained by expanding equation (25) in the usual manner. The set of equations which result is expressed below in matrix form.

\[
\begin{bmatrix}
-1 & 1 & 0 & -1 & -1 \\
(\zeta_2+\zeta_3+A) & (-\zeta_2-\zeta_3) & 1 & (\zeta_3+A) & (\zeta_2+A) \\
[-\zeta_2\zeta_3-A(\zeta_2+\zeta_3)+B] & \zeta_2\zeta_3 & (-\zeta_2-\zeta_3) & (-A\zeta_3+B) & (-A\zeta_2+B) \\
[A(\zeta_2\zeta_3-B(\zeta_2+\zeta_3)] & 0 & \zeta_2\zeta_3 & -B\zeta_3 & -B\zeta_2 \\
B\zeta_2\zeta_3 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The five equations represented above were solved by elimination to give the five constants C, D, E, F, and G in terms of the Prandtl number. The result is

\[
\begin{align*}
C &= \frac{6}{(35B\zeta_2\zeta_3)} = 6, \\
D &= \frac{A(2A-1)(6\zeta_2\zeta_3-2B)+(B+\zeta_2\zeta_3)(6\zeta_2\zeta_3-4/5+2B+2A^2)}{(2A-1)(A\zeta_2\zeta_3+B-AB)+(B+\zeta_2\zeta_3)^2}, \\
E &= \frac{(6\zeta_2\zeta_3-4/5+2B+2A^2)-(\zeta_2\zeta_3+B+2A^2-A)D}{2A-1}, \\
F &= \frac{(2\zeta_2^2-6\zeta_3+2A)-(A-\zeta_3)D-E}{\zeta_3-\zeta_2}, \\
G &= D-B-F.
\end{align*}
\]

Since \(\zeta_2, \zeta_3, A, \) and \(B\) are functions of Prandtl number only, the
constants $C, D, E, F,$ and $G$ are also functions of Prandtl number. The accuracy of the above equations from which $C, D, E, F,$ and $G$ are defined has been examined by the numerical scheme of Gaussian elimination (9) for the matrix. The maximum deviation for the two different methods of evaluating $C, D, E, F,$ and $G$ is less than 2 per cent. A graph of $C, D, E, F,$ and $G$ versus Prandtl number is presented in Figure 3.

Having obtained the appropriate constants, the integration may now be performed. Incorporating equations (24) and (26), one has

$$\ln \phi(\zeta) = -\frac{1}{2} \int \left( \frac{C}{\zeta} + \frac{D+E}{-\zeta^2 + A\zeta + B} + \frac{F}{\zeta^2 - \zeta_2} + \frac{G}{\zeta^2 - \zeta_3} \right) d\zeta + \ln C_6.$$  

Because $-\zeta^2 + A\zeta + B$ has two complex roots, the integration

$$\int \frac{(D\zeta + E) d\zeta}{-\zeta^2 + A\zeta + B}$$

always yields the $\tan^{-1}$ term. Performing the required integrations and combining gives

$$\phi(\zeta) =$$

$$C_6 \left[ \zeta (\zeta - \zeta_2)^F (\zeta - \zeta_3)^G \exp \left[ (2E + AD)/\sqrt{4B - A^2} \right] \tan^{-1} \left( (A - 2\zeta)/\sqrt{4B - A^2} \right) \right]^{1/2}.$$  

Now the equations (17), (20), (23), and (27) represent the different $\phi(\zeta)$ functions for the different cases. Those equations are ready to be introduced into equation (11) or (13) to proceed with this convective heat transfer problem.
C. The Method of Calculating the Average Convective Heat Transfer Coefficient

In this part of section III, the average convective heat transfer coefficient, \( \overline{h} \), defined by equation (14a) will be obtained. In the case of the linear profile in region 2, the parameter \( \zeta \) can be expressed in terms of \( x \) explicitly; therefore, \( \overline{h} \) can be integrated analytically. However, for region 3 of the linear profile case, \( \zeta \) is an implicit function of \( x \). Equation (14a) is thus difficult to execute analytically for the later cases. For those cases in which \( \overline{h} \) can not be derived analytically, a numerical method will be employed to carry out the integration.

1. Linear Profile

a. Region 2. \( x_0 < L \leq x_1 \). In this case the length of the plate \( L \) can not be greater than \( x_1 \) nor less than \( x_0 \). By employing equation (14) and incorporating the linear temperature profile, one has

\[
h_x = \frac{k}{\zeta \delta} . \tag{28}
\]

Next introduction of equations (11), (17), and (3) into equation (28) gives

\[
h_x = k \Pr^{1/3} \left( \frac{12\nu}{u_\infty} \right)^{1/2} x^{-1/2} \left[ 1 - \left( x_0 / x \right)^{3/4} \right]^{-1/3} . \tag{29}
\]

The convective heat transfer coefficient in region 2 is defined as

\[
\overline{h}_2 = \frac{1}{(L-x_0)} \int_{x_0}^{L} h_x \, dx , \quad x_0 < L \leq x_1 . \tag{30}
\]

After equation (29) is substituted into equation (30) and the
integration is performed, there results
\[ \bar{h}_2 = 0.577(u \omega L / \nu)^{1/2} \Pr^{1/3} (L-x_0)^{-1/2} [1-(x_0/L)^{3/4}]^{2/3} \, . \] (31)
The details of the integration of equation (30) are described in the Appendix.

It is convenient to express equation (31) in terms of dimensionless quantities. If one defines
\[ \bar{Nu}_2 = \bar{h}_2 (L-x_0) / k \] (32)
as the average Nusselt number for a flat plate with an unheated leading length \( x_0 \) in which the total length of the plate \( L \) is between \( x_0 \) and \( x_1 \), equation (31) can be written as
\[ \bar{Nu}_2 / \Re_L^{1/2} = 0.577 \Pr^{1/3} [1-(x_0/L)^{3/4}]^{2/3} , \quad x_0 < L \leq x_1 , \] (33)
where \( \Re_L = u_\omega L / \nu \) is the Reynolds number.

b. Region 3. \( L \geq x_1 \). In region 3, the length \( L \) of the plate is greater than or equal to \( x_1 \). From equation (14a), the convective heat transfer coefficient is given by
\[ \bar{h}_3 = \frac{1}{(L-x_0)} \int_{x_0}^{L} h_x \, dx , \quad L \geq x_1 , \] (34)
where \( \bar{h}_3 \) denoted the average convection coefficient when the length of the plate extends into region 3. Because part of the plate is in region 2, equation (34) can be written as
\[ \bar{h}_3 = \frac{1}{(L-x_0)} \int_{x_0}^{x_1} h_x \, dx + \frac{1}{(L-x_0)} \int_{x_0}^{x_1} h_x \, dx , \]
or
where \( \bar{h}_2 \) is the value of \( h_2 \) from equation (31) evaluated for a length of plate length equal to \( x_1 \). A convenient dimensionless group is

\[
\overline{\text{Nu}_3/\text{Re}_L^{1/2}} = \left[ \frac{h(L-x_0)}{k} \right] / \left[ \frac{u_\infty}{\nu} \right]^{1/2},
\]

where the notation \( \overline{\text{Nu}_3} \) represents the average Nusselt number for a length of plate which extends into region 3. By introducing equations (35), (3), and (28) into equation (36), one obtains

\[
\text{Nu}_3/\text{Re}_L^{1/2} = \left[ \text{Nu}_2 \right]_{x_1} / (\text{Re} \left| x_1 \right|)^{1/2} \cdot (x_1/L)^{1/2} \cdot \frac{1}{12L} \int_{x_1}^{L} \frac{dx}{\zeta \sqrt{\chi}}, \quad L > x_1,
\]

where

\[
\text{Nu}_2 \left| x_1 \right| = \bar{h}_2 \left| x_1 \right| \cdot (x_1 - x_0)/k.
\]

\[
\text{Re} \left| x_1 \right| = u_\infty x_1/\nu.
\]

The function in equation (37) can not be expressed in terms of \( x \). Instead of integrating \( \int_{x_1}^{L} dx/(\zeta \sqrt{\chi}) \) analytically, a numerical method using the trapzodial rule is employed to complete the integration.

2. Cubic Profile

a. Region 2. \( x_0 < x \leq x_1 \). From equation (14) and the cubic temperature profile one obtains

\[
h_x = 3k/(2\zeta \delta).
\]

In region 2, \( \bar{h}_2 \) is given by equation (30). After employing equation
(14) and substituting for \( \delta \), there results

\[
\overline{h}_2 = 0.323 \frac{k}{\mu} (u_\infty/v)^{1/2} \int_{x_0}^L \frac{dx}{\zeta \sqrt{x}}.
\]

The dimensionless group \( \overline{Nu}_2/Re_1^{1/2} \) becomes

\[
\overline{Nu}_2/Re_1^{1/2} = 0.323/L^{1/2} \int_{x_0}^L \frac{dx}{\zeta \sqrt{x}}, \quad x_0 < L \leq x_1. \tag{39}
\]

The trapezoidal rule of numerical integration is used to integrate equation (39).

b. Region 3. \( L \geq x_1 \). The equation which results by combining equations (35) and (38) is

\[
\overline{h}_3 = \frac{0.323k}{\mu} (u_\infty/v)^{1/2} \left[ \int_{x_0}^{x_1} \frac{dx}{\zeta \sqrt{x}} + \int_{x_1}^L \frac{dx}{\zeta \sqrt{x}} \right],
\]

or

\[
\overline{Nu}_3/Re_1^{1/2} = \frac{0.323}{L^{1/2}} \left[ \int_{x_2}^{x_1} \frac{dx}{\zeta \sqrt{x}} + \int_{x_1}^L \frac{dx}{\zeta \sqrt{x}} \right], \quad L \geq x_1. \tag{40}
\]

One can evaluate equation (40) by using numerical methods similar to that for equation (37).

D. The Special Case of \( x_0/x \) Approaching Zero

The special case of \( x_0/x \) approaching zero is discussed in this section. If \( x_0/x \) is very small, either region 2 is very small or \( x \) is very large. If region 2 is very small, one has the case of a plate with a uniform wall temperature; if \( x \) is very large, the value of \( \zeta \) becomes a constant for a given Prandtl number as \( x \) approaches infinity.
As \( x_0/x \) approaches zero, equation (13) will dominate the convective heat transfer problem. If \( x_0/x \) approaches zero, equation (12) implies that \( x_1/x \) approaches zero also. Since \( \phi(1) \) in equation (13) is finite for both the cubic and linear profiles, it is necessary that

\[
\phi(\zeta) \bigg|_3 = 0 ,
\]

where the notation \( \phi(\zeta) \bigg|_3 \) represents the function \( \phi(\zeta) \) in region 3.

1. Linear Profile. For the linear profile, equations (20) and (41) imply that

\[
3\zeta^2 - 3\zeta(1-Pr)/Pr = 0 , \tag{42}
\]

or

\[
2\zeta-1-[(4-Pr)/(3Pr)]^{1/2} = 0 . \tag{43}
\]

The roots of equation (42) are

\[
\zeta = 1/2\pm[(4-Pr)/(12Pr)]^{1/2} ,
\]

and the root of equation (43) is

\[
\zeta = 1/2+[(4-Pr)/(12Pr)]^{1/2} .
\]

The root \( \zeta=1/2-[(4-Pr)/(12Pr)]^{1/2} \) can be omitted because it has a negative value for sufficiently low Prandtl numbers and would violate physical meaning. The other root of equation (42) is identical to the root of equation (43). Thus,

\[
\zeta = 1/2+[(4-Pr)/(12Pr)]^{1/2} . \tag{44}
\]

2. Cubic Profile. For the cubic profile, the values \( F \) and \( G \) are always negative and \( D \) is always positive (see Figure 3). According to equations (27) and (41), there exists three possibilities. Either

\[
-\zeta^2 + AC + B = 0 ,
\]
or
\[
\zeta - \zeta_2 = 0 \quad \text{or}
\]
\[
\zeta - \zeta_3 = 0
\]
Since \( \zeta \) cannot be a complex number, the equation,
\[
-\zeta^2 + A\zeta + B = 0
\]
is not the correct relation to yield \( \zeta \), according to the analysis of equation (28). From Figure 2, it is known that \( \zeta_3 \) is negative and \( \zeta_2 \) is positive. Therefore, \( \zeta_3 \) is not the solution. The correct solution is
\[
\zeta = \zeta_2 \quad \quad (45)
\]
One can obtain the heat transfer coefficient from equation (44) or (45) for the case of \( x_0/x \) approaching zero. For the flat plate with a uniform wall temperature, it is beneficial to define
\[
Nu_x = \frac{h_x}{k}
\]
to compare with the results from other references. The notation \( Re_x \) represents \( u_\infty x/v \). For the linear profile case, one can combine equations (44), (28) and obtain
\[
Nu_x/Re_x^{1/2} = 1/[\sqrt{3+\sqrt{(4-Pr)/Pr}}] \quad (46)
\]
For the cubic profile case, one can combine equations (38) and (45) to give
\[
Nu_x/Re_x^{1/2} = 0.323/\zeta_2 \quad (47)
\]
where \( \zeta_2 \) is the root discussed in equation (45). A comparison of the results from equations (46) and (47) with those of other investigations will be made in the next section.
IV. RESULTS AND DISCUSSION

In this section, representative results are presented for the flat plate with an unheated leading length for a fluid with a Prandtl number less than 1. Curves are presented which show $\zeta$ versus $x_0/x$, the dimensionless group $Nu_x/Re_x^{1/2}$ versus $x_0/x$ and $Nu/Re_L^{1/2}$ versus $x_0/L$ for several Prandtl numbers. A graph for $Nu_x/Re_x^{1/2}$ versus Prandtl number for the case of a flat plate with constant wall temperature is also shown. Eckert's results (5) for both the uniform wall temperature case for very low Prandtl numbers and the unheated leading length case for Prandtl numbers greater than 1 are included for comparison. The present results for a plate with an unheated leading length are also compared with those of Rubesin (7) and Scesa and Levy (3).

The parameter $\zeta = \delta_c/\delta$ is related to $x$ by equations (11) and (13). The detailed equations which yield $\zeta$ for the different cases can be obtained by substituting the proper $\phi(\zeta)$ function into equation (11) or (13). After the proper $\phi(\zeta)$ is introduced, the numerical values for $x$ can be obtained from equations (11) and (13), for given $x_0$ and Prandtl number by the following method. First, a value of $\zeta$ is inserted in the proper equation and the corresponding value of $x$ is calculated. The range of $\zeta$ in equation (11) for the cubic profile case is between zero and unity and the range in equation (13) for the linear profile case is between unity and $1/2 + [(4-Pr)/(12Pr)]^{1/2}$. For the cubic profile case, the range in equation (13) is from unity to $\zeta_2$, which is given in Figure 2 (page 18).
Figures 4, 5, 6, and 7 show the variation of \( \zeta \) with \( x_0/x \). The case of Prandtl number of 0.7 is given so that the convective heat transfer coefficients from the present results can be compared with those of other workers. To include the range of Prandtl numbers for liquid metals, which is approximately from 0.002 to 0.03, it is suitable to have the convective heat transfer effect for Prandtl numbers down to 0.001. The dimensionless locations \( x_0/x_1 \) where the thicknesses of the boundary layers are equal are about 0.2, 0.865, 0.979 and 0.994 for Prandtl numbers of, respectively, 0.7, 0.1, 0.01, and 0.001, as shown in the figures. These \( x_0/x_1 \) values divide the region of interest into two parts. Note that the right-hand side to the line \( x = x_1 \) is region 2 and the left-hand side is region 3.

For Prandtl number 0.7, region 2 may play an important role in the heat transfer of the plate since region 2 may occupy a considerable portion of the plate (from \( x_0/x = 0.2 \) to 1.0). For Prandtl numbers between 0.001 and 0.01, it is possible that region 2 could be very small. Thus region 3 will dominate the heat transfer characteristics for Prandtl numbers below 0.01 if the plate is sufficiently long. In this case, the accuracy of the results will not be affected if region 2 is neglected. From Figures 4, 5, 6, and 7, it can be seen that from \( x_0/x = 1.0 \) to 0.65, \( \zeta \) increases rapidly with \( x_0/x \) and that the slope of this segment of the curve also changes rapidly. From \( x_0/x = 0.65 \) to zero, the rate of increase of \( \zeta \) becomes slower. The relation between \( \zeta \) and \( x_0/x \) is nearly linear when the value of \( x_0/x \) is less than 0.65. Thus, for \( x_0/x \) less than 0.65, the \( \zeta \) curves can be extended and extrapolated to give \( \zeta \) values for \( x_0/x \) equal to zero. The values of \( \zeta \) for \( x_0/x = 0 \) as obtained from Figures 4, 5, 6, and 7
Figure 4: Variation of $\zeta$ with $x_0/x$ at $Pr = 0.7$
Figure 5: Variation of $\zeta$ with $x_0/x$ at $Pr = 0.1$
Figure 6: Variation of $\zeta$ with $x_0/x$ at $Pr = 0.01$
Figure 7: Variation of $\zeta$ with $x_0/x$ at $Pr = 0.001$
can be compared with those obtained from equations (44) and (45). The deviations between the two sets of values for $\zeta$ are always found to be less 2 per cent.

The accuracy of the results as shown in Figure 4, 5, 6, and 7 was verified by a direct numerical integration of equation (8) using Runge-Kutta method. The results from the numerical integration are identical to those for Prandtl numbers of 0.7 and 0.1. Round-off errors in the numerical scheme prevented a direct comparison for Prandtl numbers of 0.01 and 0.001.

In general, the deviation of $\zeta$ between the cubic and linear profile case in region 2 is about 3 per cent based on the cubic profile case. In addition, since equation (11) as applied to the linear profile case is simpler than that applied to the cubic profile case, the former can be used in place of the latter in region 2 with good approximation. In region 3, the difference between the two cases is larger than that in region 2. The maximum difference in region 3 is about 5 per cent.

Figures 8, 9, 10, and 11 give the local Nusselt number divided by the square root of the Reynolds number, $\frac{N_u}{\sqrt{Re_x}}$, as a function of $x_0/x$. The curves include the results of Eckert (5), Rubesin (7), and Scesa and Levy (3) for a plate with an unheated length, as well as the results for the cubic and linear profile cases deduced in this investigation. For Prandtl number equal to 0.7, Figure 8, Scesa and Levy's results nearly coincide with those of the cubic profile case. The solution of Eckert gives $\frac{N_u}{\sqrt{Re_x}}$ results which are 5 per cent higher than those of the cubic profile case. Both Rubesin's solution and the linear profile case give results which are lower than those
Figure 8: Variation of $\frac{Nu_x}{Re_x^{1/2}}$ with $x_0/x$ at $Pr = 0.7$
Figure 9: Variation of $\frac{\text{Nu}_x}{\text{Re}_x^{1/2}}$ with $\frac{x_0}{x}$ at $Pr = 0.1$
Figure 10: Variation of $\frac{N_u}{Re}^{1/2}$ with $x_0/x$ at Pr = 0.01
Figure 11: Variation of $\frac{Nu_x}{Re_x^{1/2}}$ with $x_0/x$ at $Pr = 0.001$
of the cubic profile case by 6 and 7 per cent, respectively, based on the cubic profile case. It is to be noted that the curves in Figure 8 have almost the same slope. For Pr = 0.1, Figure 9, the slope of the curves for the cubic profile case in the present investigation is in general different from that of the other solutions. The curve for the cubic profile case in Figure 9 crosses the curves of Eckert's, Scesa and Levy's, and Rubesin's solutions, giving a maximum difference of 15, 10, and 10 per cent, respectively. Greater differences are observed for the Prandtl numbers of 0.01 and 0.001. Figures 10 and 11 give the results for these two values of Pr.

Figure 12 shows $\frac{Nu_x}{Re_x}^{1/2}$ as a function of Prandtl number for the case when $x_0/x$ approaches zero. The curve for the linear profile case is generated from equation (46) and the curve for the cubic profile case is generated from equation (47). Figure 12 shows that Eckert's result for the case of a plate with constant wall temperature at very low Prandtl number agrees well with the solution obtained in the present investigation using the cubic profile. The curve for the linear profile case has an 8 per cent deviation compared with the cubic profile case.

In Table III, the values of $\frac{Nu_x}{Re_x}^{1/2}$ from the present analysis are compared with those of the exact solution of Fisher and Kundsen (2) from the energy boundary layer equation for the flat plate at zero angle of incidence with constant wall temperature. The cubic profile case, equation (47), and linear profile case, equation (46), are employed to yield the values of $\frac{Nu_x}{Re_x}^{1/2}$ for Prandtl numbers 0.1, 0.01, and 0.001. The cubic profile case has an error of about 4 per
Figure 12: Variation of $\frac{Nu_x}{Re_x^{1/2}}$ with $Pr$. 

Note: The curves are for the case $x_0=0$. 

- Cubic Profile Case of the present work
- Linear Profile Case of the present work
- Eckert (5)
Table III

Comparison of $\frac{Nu_x}{Re_x^{1/2}}$ for the Cubic and Linear Profile Cases with the Exact Solution when $x_0=0$

<table>
<thead>
<tr>
<th>Pr</th>
<th>Exact</th>
<th>Cubic</th>
<th>Linear</th>
<th>% Error for Cubic Profile</th>
<th>% Error for Linear Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.14</td>
<td>0.136</td>
<td>0.12</td>
<td>2.8</td>
<td>14.3</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0516</td>
<td>0.0494</td>
<td>0.0461</td>
<td>4.2</td>
<td>10.7</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0173</td>
<td>0.0165</td>
<td>0.0154</td>
<td>4.6</td>
<td>11</td>
</tr>
</tbody>
</table>
cent, while the linear profile case has an error of about 13 per cent when compared to the exact solution.

In Figures 13, 14, 15, and 16 are shown the variation of the average Nusselt number divided by the square root of the Reynolds number versus $x_0/L$ for $Pr = 0.7, 0.1, 0.01, and 0.001$, respectively. Those curves are based on the results obtained by equations (33), (37), (39), and (40). The accuracy of the numerical results depends on the step size used in the integration. Use of a smaller step size gives rise to a better accuracy in the results. It was found that 30 to 60 steps gave sufficiently accurate results.

In region 2, the difference between the $\bar{Nu}/Re_L^{1/2}$ results for the two profile cases are usually less than 3 per cent. In region 3, the differences between the two profile cases are about 8 to 10 per cent based on the cubic profile case.
Figure 13: Variation of $\frac{\bar{N}_u}{Re_L^{1/2}}$ with $x_0/L$ at $Pr = 0.7$
Figure 14: Variation of $\bar{\text{Nu}}/\text{Re}_L^{1/2}$ with $x_0/L$ at $Pr = 0.1$
Figure 15: Variation of $\frac{\bar{Nu}}{Re_{L}^{1/2}}$ with $x_0/L$ at $Pr = 0.01$
Figure 16: Variation of $\frac{\bar{Nu}}{Re_L^{1/2}}$ with $x_0/L$ at Pr = 0.001
V. CONCLUSION

This investigation considers the laminar convective heat transfer from a flat plate to a constant property fluid. It is postulated that the fluid has a Prandtl number less than one and that the plate has an unheated leading edge of length \( x_0 \). Several formulations are available for Prandtl numbers greater than one for this problem. Results are also available for very low Prandtl numbers for the special case of \( x_0 = 0 \). This work closes the gap between the two extremes. The energy integral equation is employed in the solution.

Three regions of interest are formed as the fluid flows over the plate. The first region is the unheated leading edge which has a temperature equal to the fluid temperature. In the second region, the thermal boundary layer thickness \( \delta_t \) is less than the momentum boundary layer thickness \( \delta \). In the third region, the opposite is true. The integration of the energy equation in the region 3 was divided into two parts. The linear profile case employs a linear polynomial for both the velocity and temperature profiles while the cubic profile case employs a cubic polynomial for both profiles. The solutions of the ordinary differential equation are simpler for the linear profile case than for the cubic profile case. However, the accuracy of the cubic profile case is better than the linear profile case by 5 per cent in general.

The cubic profile case of this investigation provides heat transfer results which are in a good agreement with those results of previous investigators for the special case of \( x_0 = 0 \). The results also agree with those for the case with an unheated length and with a
Prandtl number of about one. From these comparisons, the accuracy of the cubic profile case of the present study is strongly supported. In addition, it was concluded that the simpler solution for the linear profile case will meet the need of many engineering applications, especially in the region in which the thermal boundary layer is thinner than the flow boundary layer.

The results show that significant errors may result if previous solutions are applied beyond their range of applicability.
VI. REFERENCES


VII. APPENDIX

The Determination of $\bar{h}_2$ and $x_1$

The substitution of equation (29) into equation (30) results in

$$\bar{h}_2 = \frac{kPr^{1/3}}{(L-x_0)} \left(12v/u_\infty\right)^{-1/2} \int_{x_0}^{L} \frac{dx}{\sqrt{x \left[1-\left(x_0/x\right)^{3/4}\right]^{1/3}}} . \quad (A-1)$$

The integral

$$I = \int_{x_0}^{L} \frac{dx}{\sqrt{x \left[1-\left(x_0/x\right)^{3/4}\right]^{1/3}}} \quad (A-2)$$

can be evaluated as follows. The square root may be eliminated by letting $x$ equal to $M^4$. Let $x_0$ correspond to $M^4_0$ and $L$ correspond to $M^4_L$. Inserting the new parameters into equation (A-2) and rearranging, one has

$$I = \int_{M^4_0}^{M^4_L} \frac{4M^2 dM}{(M^3-M^3_0)^{1/3}} . \quad (A-3)$$

The integration of equation (A-3) may be easily performed, giving

$$I = \frac{4/3}{(M^3-M^3_0)^{2/3}} \left|_{M^4_0}^{M^4_L} \right. . \quad (A-4)$$

Replacing $M_L$, $M_0$ by $L^{1/4}$ and $x_0^{1/4}$, respectively, gives

$$I = \frac{4/3}{L^{1/2}} \left[1-\left(x_0/L\right)^{3/4}\right]^{2/3} . \quad (A-5)$$

Substituting equation (A-5) into equation (A-1) results in

$$\bar{h}_2 = 0.577(u_\infty L/v)^{1/2} kPr^{1/3} \left(L-x_0\right)^{-1/2} \left[1-\left(x_0/L\right)^{3/4}\right]^{2/3} , \quad \text{which is equation (31) in the text.}$$
For the linear profile case, by introduction of equation (17) into equation (12), the location \( x_1 \) can be obtain

\[
x_1 = x_0/(1-Pr)^{4/3}.
\]

In the cubic profile case, inserting equation (23) into equation (12), gives the same result as for the linear profile case.
VIII. VITA

The author, Shiny Ting, was born on July 13, 1946, in Shang-Hai, China.

After he graduated from Taiwan Normal University Subordinate Middle School in Taipei, Taiwan, Republic of China, he entered the Taiwan Provincial Cheng-Kung University in 1964 and received the degree of Bachelor of Science in Mechanical Engineering in 1968. He served one year in the Chinese Army after graduation.

In January 1970, he enrolled at the University of Missouri-Rolla as a graduate student in Mechanical Engineering Department and began to work for his Master of Science degree.